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PART I

ARITHMETIC

RULES AND DEFINITIONS*

1. The name *quantity* is given to everything which may be expressed in numbers by comparing it with a quantity of the same sort taken as *unity*. *Lengths* which are expressed in *feet* or *meters; surfaces* in *square feet* or *square meters; volumes* in *cubic feet* or *cubic meters; weights* and *forces* in *pounds* or *kilograms; prices* in *dollars* and *cents; time* in *days; angles* in *degrees,* etc., are quantities.

Number, space, and *time* are quantities of which everyone has an idea and need not be defined.

2. *Mathematics* is the science of quantities.

3. *Arithmetic* is the science of numbers.

4. Numeration is that part of arithmetic which deals with the formation, the reading, and the writing of numbers. It is divided into *spoken numeration,* or *numeration* which deals with the formation and reading of the numbers, and *written numeration,* or *notation* which has for a purpose the expression of numbers by *figures* and *letters.*

5. The number *one* is the unit of numbers, to which the name *simple unit* or *unit of the first order* has been given; the number *ten,* which consists of ten simple units, is a number of the *second order;* one hundred is of the third; one thousand of the fourth; ten thousand of the fifth, and so on.

It may be noted that units of successive orders are each ten times that of the order immediately preceding.

6. The *simple unit,* the *thousand,* which is equal to one thousand simple units; the million, which is equal to one thousand thousands; the billion, which is equal to one thousand millions;

* A number placed in parenthesis ( ) indicates cross reference to the article bearing that number.

the trillion, which is equal to one thousand billions; the quadrillion; the quintillion, etc.; in a word, all the units, starting from simple units, which are one thousand times greater than the one immediately preceding, are called *principal units.*

7. The first nine numbers are represented respectively by the nine figures 1, 2, 3, 4, 5, 6, 7, 8, 9; with the aid of these, together with the tenth figure, 0, which has no value in itself, all possible numbers may be written.

To write a dictated number in figures, commencing at the left, write one after the other the figures which represent the number of hundreds, of tens and of units of each principal unit dictated, replacing the units which are lacking by ciphers. For example, the number *thirty million fifty thousand seven hundred eight* is written 30,050,708.

It is seen that *in a whole number any figure placed at the left of another expresses units ten times as great as that one.* It is this convention which permits the writing of all possible numbers with the aid of only ten figures.

8. All figures of a number have two values: one absolute, expressed by its form, the other relative, due to the position which it occupies; thus, in the number 508, the figure 5 has five for an absolute value, and five hundred for a relative value.

The 0 in a number has neither an absolute nor a relative value; it serves simply to place the other figures in the desired order, that is, to give them a determined relative value. It is for this reason that 0 is not called a *significative figure,* a designation given to the other nine figures.

9. *To pronounce a number written in figures,* commencing at the right, separate them, in thought, or by commas, into periods of three figures each, except the last period which may have one or two figures; then commencing at the left, pronounce successively the number of hundreds, tens and units of each period, giving the name of the principal units which they represent. Thus, the number 3,405,834,067 is pronounced *three billion four hundred five million eight hundred thirty-four thousand sixty-seven.*

Instead of saying one ten, two tens . . ., nine tens, usage has made it: *ten, twenty* . . ., ninety. The same instead of saying ten one, ten two . . ., ten nine, we say *eleven, twelve* . . ., *nineteen.*

10. *The base of a system of numeration* is a constant number

of any order, of which the unit of the immediately superior order (5) is composed. Thus, ten is the base of the system of numeration adopted; and for this reason it is called the *decimal system.* The number of figures employed in a system is equal to the base of the system.

11. *Roman Notation.* The Romans employed letters to represent the numbers. They are still used, especially on monumental inscriptions. The letters employed are:

I, V, X, L, C, D, M.

They represent respectively:

1, 5, 10, 50, 100, 500, 1,000.

The number I placed one, two, or three times at the right of the numbers I and V, increases these numbers by one, two, or three units; and if it is written at the left of V or X it decreases them by one unit; thus the first ten whole numbers:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10,

are respectively represented by:

I, II, III, IV, V, VI, VII, VIII, IX, X.

The number X written one, two, or three times at the right of the number X or L, increases these numbers by one, two, or three tens; and written at the left of L or C diminishes them by ten. Thus the numbers:

10, 20, 30, 40, 50, 60, 70, 80, 90, 100,

are written:

X, XX, XXX, XL, L, LX, LXX, LXXX, XC, C.

To write the whole numbers comprised between two consecutive whole numbers of tens, it suffices to write the first nine numbers at the right of each number of tens. Thus the numbers 13, 34, 56, 97 are written XIII, XXXIV, LVI, XCVII. The number C, placed after itself or the number D, or before D and M, permits the writing of the whole numbers of hundreds in the same manner as the whole numbers of tens were written. Thus the numbers:

100, 200, 300, 400, 500, 600, 700, 800, 900, 1000,

are written respectively:

C, CC, CCC, CD, D, DC, DCC, DCCC, CM, M.

The first hundred numbers written after each number of hundreds give all the whole numbers comprised between one and ten hundreds. The number M written one, two, or three times at the right of itself gives the numbers 2000, 3000, 4000.

To write the whole numbers comprised between two consecutive whole numbers of thousands, the first 999 numbers are written at the right of each number of thousands.

The above conventions permit the writing of all the numbers under 5000. Thus the numbers 1856 and 4584 are written

MDCCCLVI and MMMMDLXXXIV.

12. A number is *concrete* or *abstract*, according as it does or does not indicate the nature of the thing which it represents. Thus when we say seven o'clock, twelve dollars, 7 and 12 are concrete numbers; but when we say simply seven, twelve, they are abstract numbers.

13. An *operation* is a manner of transforming numbers. There are only four *fundamental operations* in arithmetic, because all the others are simply combinations of these four. They are: addition, subtraction, multiplication, and division.

14. A *calculation* is the sum and total of all the operations performed upon the numbers.

15. A *theorem* is a truth rendered evident by a course of reasoning called a *demonstration*.

16. An *axiom* is a self-evident truth which is accepted without demonstration.

17. A *problem* is a question to be solved.

18. The theorem, the axiom, and the problem come under the common name of proposition.

19. An *hypothesis* is a preliminary proposition established to fit the demonstration of a theorem or problem.

20. A *corollary* is the consequence of one or several propositions.

21. The *proof* of an operation is a second operation performed to verify the accuracy of the result obtained by the first; a proof establishes the probable but not the absolute correctness of a result.

22. *Axioms of Arithmetic* (16).

1st. Two quantities equal to a third quantity are equal to each other.

2d. When the same operation is performed upon two equal quantities the results are equal.

3d. The value of a whole is not altered by changing the order of its parts.

23. Sign abbreviations:

The sign	$=$	means equal to.
	$+$	plus.
	$-$	minus.
	$\pm$	plus or minus.
	$\times$ or $\cdot$	times.
	$\div$	divided by.
	$>$	greater than.
	$<$	less than.

Thus $$7 + 8 - 6 = 4 \times 3 - \frac{6}{2}$$

means 7 plus 8 minus 6 equals 4 times 3 minus 6 divided by 2.

The parenthesis ( ) expresses the result of the operations upon the quantities which it contains. Thus having

$$9 - 6 + 2 \times 4 = 3 + 8 = 11,$$

we have

$$18 - (9 - 6 + 2 \times 4) = 18 - 11 = 7,$$

and

$$5 \times (9 - 6 + 2 \times 4) \quad \text{or} \quad 5(9 - 6 + 2 \times 4) = 5 \times 11 = 55.$$

$18 - 9 - 6$ indicates that 9 is to be taken from 18 first, and then 6 from the remainder 9; which gives $18 - 9 - 6 = 3$; which is $18 - 9 - 6 = 18 - (9 + 6)$.

BOOK I

FUNDAMENTAL OPERATIONS ON WHOLE NUMBERS

ADDITION

24. *Addition* is an operation by which several quantities are united in a single one, called the sum or total.

25. To *add the whole numbers*, 4805, 27, 446, 9:

$$\begin{array}{r} 4805 \\ 27 \\ 446 \\ \underline{9} \\ 5287 \end{array}$$

In general, to add given numbers, write the numbers one below the other in such a manner that the figures which express units of the same order come in the same vertical column, and underline the last number, 9, to separate it from the result. Then commencing at the right add successively the figures of each column; place the units of that order in the result and carry the tens to the next column. Thus the sum of the figures in the first column being 27 units, we place 7 units in the result and carry 2 tens to the next column. The operation is commenced at the right because of the tens which have to be carried. In order to calculate rapidly, instead of saying, as ordinarily: 9 and 6 are 15, 15 and 7 are 22, 22 and 5 are 27, 7 in the result, and 2 to carry; 2 and 4 are 6 and 2 are 8, 8 in the result, etc., it is well to accustom oneself to saying: 9, 15, 22, 27 (write the 7 without pronouncing and pass to the column of tens); 6, 8 (write 8), etc. When there are many figures to be added, it is well, especially if one is not accustomed to it, to divide the operation into several partial additions, and afterwards add the partial results. It is also convenient, especially when one has long operations to make, to write the partial sums at one side in the order in which they are obtained. This permits one, in case of a distraction, to recommence the addition of the figures of a column, without

$$\begin{array}{lr} \text{Remainders} & \underline{45{,}433} \\ & 54{,}956 \\ & 97{,}864 \\ & 39{,}518 \\ & 58{,}763 \\ & 85{,}742 \\ & 46{,}434 \\ & \underline{39{,}358} \\ & 422{,}635 \end{array}$$

being obliged to repeat the whole operation. It permits also of the verification of the addition of any column without reference to the others. The scheme shown here is very convenient.

26. *To prove an addition,* recommence, making the partial additions in the opposite direction. Thus add from top to bottom, or from bottom to top, according as the first operation was made from bottom to top, or top to bottom (**97**).

SUBTRACTION

27. Subtraction is an operation by which the difference of two quantities is taken. These two quantities are the two *terms* of the difference. The larger one or the *first term* is called the *minuend,* the smaller or *second term,* the *subtrahend,* and the difference the *remainder.*

28. From these definitions it follows that:

1st. The first term is equal to the second term plus the remainder.

2d. When the first term is increased or decreased, the remainder is increased or decreased.

3d. When the second term is increased or decreased, the remainder is decreased or increased.

4th. The remainder is unchanged when both terms are increased or decreased by the same quantity.

5th. *To subtract a sum from a quantity,* subtract the first part of the sum from the quantity; the second part from this remainder, etc., until the last part has been subtracted.

6th. *To subtract a quantity from a sum,* subtract the quantity from one of the parts of the sum.

29. *To subtract two whole numbers,* 2935 and 372.

$$\begin{array}{r} 2935 \\ \underline{372} \\ 2563 \end{array}$$

In general, to find the difference between two whole numbers, write the smaller number below the larger in such a manner that the figures which express units of the same order come in the same column; underline the smaller number 372 to separate it from the remainder. Then commencing at the right, take each figure of the second term from the corresponding figure in the first and place the remainder below.

When a figure such as 7 in the second term is larger than the corresponding figure 3 of the first term, the subtraction is made possible by adding 10 units of that order to the first term, this being compensated by adding one unit to the following figure of the second term (**28**, 4th). This adding of one unit to the following figure of the second term is the reason for beginning at the right. In performing the operation one says, 2 from 5 leaves 3, 7 from 13 leaves 6, 4 from 9 leaves 5, 0 from 2 leaves 2, writing successively the *partial remainders* 3, 6, 5, 2 in the remainder.

30. *Proof of subtraction.* Adding the remainder 2563 to the second term 372, will give the first term 2935, if the work is correct (**28**, 1st). Another proof is to subtract the remainder from the first term which should give the second term.

31. When quantities are separated by the signs + or − (example: 3 + 4 − 5), 3 and 4 preceded by + are said to be *positive* and 5 preceded by − to be *negative.* When the first quantity is positive it is not necessary to write plus + before it, but if it is negative the sign − must precede it.

If 7 is to be taken from 4, the smaller is taken from the larger and the negative sign placed before the result, thus:

$$4 - 7 = -3.$$

$$\begin{array}{lr} \text{1st.} & 59{,}243 \\ & 87{,}564 \\ & -\ 32{,}932 \\ & 8{,}252 \\ & 29{,}848 \\ & -\ \ 3{,}624 \\ & -\ \ 2{,}808 \\ \hline & 184{,}907 \\ & -\ 39{,}364 \\ \hline & 145{,}543 \end{array}$$

The result −3 indicates that the quantity could not be subtracted.

To subtract the sum of several quantities from the sum of several other quantities, the sums are made separately and the difference of the results taken.

When all the quantities are written in a column, and one does not wish to rewrite them in order to separate them, the sign − is placed before all those to be subtracted, so as to avoid confusion in making the two sums. (See the operation at the left.) The last number 2808 is underlined and the two sums placed below, the sum to be subtracted coming last. Then the subtraction is made in the usual manner.

In place of this method, the rule of subtraction may be applied in a general way and the two partial sums be dispensed with. Commencing at the right the positive numbers are added and from each partial sum the negative numbers are successively subtracted. Thus one says (operation 2) 3 and 4, 7 and

2, 9 and 8, 17; 17 less 2, 15, less 4, 11, less 8, 3, and 3 is written in the result. The same operation is repeated for each column. It is seen that nothing is done except to follow the rule of subtraction (29) which is but a little extended in this case, since several figures are subtracted in succession, and it is possible to have several units to add to or to subtract from the next column (96 and 403, and the application of the preceding rule to the solution of any right triangles when logarithms are used, Part IV).

$$\begin{array}{lr} \text{2d.} & 59{,}243 \\ & 87{,}564 \\ & -\ 32{,}932 \\ & 8{,}352 \\ & 29{,}848 \\ & -\ 3{,}624 \\ & -\ 2{,}808 \\ \hline & 145{,}543 \end{array}$$

The preceding rule naturally applies in the case where there is but one number to be taken from a sum of several others (see operation 3), and also where the sum of several numbers is to be taken from a single number (see operation 4); in this last case it is better to operate in the following manner:

$$\begin{array}{lr} \text{3d.} & 59{,}243 \\ & 87{,}564 \\ & 8{,}252 \\ & 29{,}848 \\ & -\ 39{,}364 \\ \hline & 145{,}543 \end{array}$$

Commencing at the right, the negative figures of each column are added and the partial sum taken from the corresponding positive figure, the latter being increased by 1, 2, 3, . . . times 10 as the case may be and adding 1, 2, 3 . . . units to the next column for compensation. Thus one says: 8 and 4, 12 and 2, 14; from 17 leaves 3. 1 and 0, 1 and 2, 3 and 3, 6; from 10 leaves 4. 1 and 8, 9 and 6, 15 and 9, 24; from 29 leaves 5. 2 and 2, 4 and 3, 7 and 2, 9; from 14 leaves 5. 1 and 3, 4; from 8 leaves 4. 0 from 1 leaves 1.

$$\begin{array}{lr} \text{4th.} & 184{,}907 \\ & -\ 32{,}932 \\ & -\ 3{,}624 \\ & -\ 2{,}808 \\ \hline & 145{,}543 \end{array}$$

MULTIPLICATION

32. *Multiplication* is an operation by which a number called the *multiplicand* is repeated as many times as there are units in another called the *multiplier.* The result is called the *product.* The multiplicand and the multiplier are the factors of the product. Multiplication is an abbreviated method of adding as many numbers equal to the multiplicand as there are units in the multiplier.

From the definition of multiplication it follows:

1st. When one of the factors is 0, the product is 0, and when

one of the factors is unity 1, the product is equal to the other factor.

2d. In general the product is of the same sort as the multiplicand, and the multiplier an abstract number (12).

33. From the definition of multiplication and from axiom 2 (22), it follows:

1st. The product of the sum of several quantities and a number is equal to the sum of the products obtained by multiplying each part of the sum by the number:

given $$19 = 3 + 7 + 9,$$
we have
$$19 \times 5 \text{ or } 95 = (3 + 7 + 9)\,5 = 3 \times 5 + 7 \times 5 + 9 \times 5.$$

2d. The product of a quantity with the sum of several numbers is equal to the sum of the products obtained by multiplying the quantity by each part of the sum:
$$5 \times 19 \text{ or } 95 = 5 \times (3 + 7 + 9) = 5 \times 3 + 5 \times 7 + 5 \times 9.$$

34. When the two terms 25 and 8 of a difference are multiplied by the same number 4, the difference 17 is multiplied by that number 4:
$$25 \times 4 - 8 \times 4 = (25 - 8) \times 4 = 17 + 4 = 68.$$

35. The following table, constructed by Pythagoras, contains all the products of two numbers of a single figure each:

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

To find the product of two numbers of a single figure in the above table, 8×3, for instance, find the multiplicand 8 in the top horizontal row, and the multiplier 3 in the first vertical column; follow the vertical column which contains 8 down until it intersects the horizontal row, containing 3, and the block at this intersection will contain the product 24.

36. The result obtained by multiplying a series of numbers together in order of their positions; the first by the second, the product by the third, the new product by the fourth, and so on, is called the product, and the numbers the *factors*.

37. A number is said to contain all the factors of another number when it is equal to the product of several factors, among which are the factors of the other number.

Thus $2 \times 5 \times 3 \times 7 = 210$, contains all the factors of $5 \times 7 = 35$.

38. A quantity is a multiple of another when it is equal to the latter multiplied by a whole number. Thus $7 \times 3 = 21$ is a multiple of 7, also of 3.

Conversely, when one quantity is a multiple of another, the latter is an *under multiple* of the first.

39. The sum, $7\times4+7\times3+7\times5=7(4+3+5)=7\times12=84$ of several multiples of the same quantity; 7 is a multiple of that quantity (**33** and **38**).

40. The difference, $7 \times 9 - 7 \times 4 = 7\,(9 - 4) = 7 \times 5 = 35$ of two multiples of the same quantity; 7 is a multiple of that quantity (**34** and **38**).

41. *The product of any number of factors is not changed by any change in the order of the factors:*

$$3 \times 4 \times 7 \times 5 = 4 \times 5 \times 3 \times 7 = 420 \ (\mathbf{36}).$$

42. *To multiply any number* 9 *by a product* $3 \times 4 \times 7 = 84$, instead of multiplying the number by the product 84, it is possible to multiply it by the first factor 3, the product thus obtained by the second factor 4, and so on through until the last factor has been used as multiplier (**36**):

$$9 \times 84 = 9 \times (3 \times 4 \times 7) = 9 \times 3 \times 4 \times 7 = 756.$$

43. When a factor of a product, $5 \times 3 \times 4 = 60$, is multiplied by a number 7, the product is multiplied by the same number:

$$5 \times (3 \times 7) \times 4 = 5 \times 3 \times 4 \times 7 = 60 \times 7 = 420.$$

In multiplying several factors of a product by several numbers, the product is multiplied by the product of those numbers:

$$(5 \times 6) \times (3 \times 7) \times 4 = (5 \times 3 \times 4) \times (6 \times 7) = 60 \times 42 = 2520.$$

44. *To multiply a whole number by a unit followed by one or more ciphers,* it is only necessary to write as many ciphers after the number as there are at the right of the unit:

$$425 \times 100 = 42{,}500.$$

45. *To obtain the product of several numbers, all or part of which end with ciphers,* it suffices to obtain the product of the numbers neglecting the ciphers and write at the right of the product as many ciphers as have been neglected in the operation. Thus, in multiplying 400 by 6000, one multiplies 4 by 6, and writes five ciphers to the right of the product 24:

$$400 \times 6000 = 2{,}400{,}000.$$

46. *To multiply a number,* 458, *of several figures, by a number* 6, *of a single figure,*

$$\begin{array}{r} 458 \\ 6 \\ \hline 2748 \end{array}$$

Write the multiplier under the multiplicand, and underline it to separate it from the result. Then commencing at the right, multiply successively each figure of the multiplicand by the multiplier; write the units of each partial product under the corresponding figure of the multiplicand, and add the tens to the next product (the carrying of the tens is what obliges one to commence at the right).

Thus, one says: 6 times 8 are 48 (write 8, carry 4); 6 times 5 are 30, and 4 are 34 (write 4 and carry 3); and so on for all the figures of the multiplicand.

47. *To multiply a number,* 5736, *of several figures, by another number,* 743, *of several figures,*

$$\begin{array}{r} 5736 \\ 743 \\ \hline 17208 \\ 22944 \\ 40152 \\ \hline 4261848 \end{array}$$

Write as in the preceding case, the multiplier under the multiplicand, so that units of the same order correspond, and underline the multiplier. Then multiply the multiplicand successively by each figure of the multiplier, starting at the right (**46**); write each partial product below in such a manner that the first figure at the right comes under the figure of the multiplier which has been used; then add the partial products, which sum is the product desired.

If the multiplier contains ciphers between significative figures, as ciphers give 0 for a partial product, they are neglected, and the general rule is applied as before:

$$\begin{array}{r} 34256 \\ 3002 \\ \hline 68512 \\ 102768 \\ \hline 102836512 \end{array}$$

REMARK. It may be noted that the number of partial products is always equal to the number of significative figures in the multiplier.

48. *To prove a multiplication,* invert the order of the factors, that is, take the multiplier for the multiplicand and reciprocally, and if the operation is correct, the same result will be obtained (**41** and **99**).

REMARK. It will be shown farther on, after the operation of division, that by dividing the product by one of the factors, the quotient will give the other factor if the work is correct.

49. *The number of figures in the product is equal to the sum of the number of figures in the multiplicand and multiplier, or equal to this sum, less one.*

Thus the multiplicand containing 5 figures and the multiplier 3, the product contains 8 or 7.

50. *Short methods of multiplication* (**44** and **45**).

1st. The operation is sensibly shortened by taking the factor which contains the least number of significative figures (8) for multiplier, and above all, when there are figures which appear several times in the multiplier. The number of partial products is less, and the partial products which are equal have to be calculated only once.

2d. When the multiplier is 11 or 12, operate as if it were com-

posed of but one figure (46). Thus in multiplying 97,648 by 11, one says:

```
  97648          97648
     11            117
-------        -------
1074128         683536
               1074128
              --------
              11424816
```

11 times 8, 88 (write 8 and carry 8); 8 and 11 times 4, 44, 52 (write 2 and carry 5); 5 and 66, 71 (write 1); 7 and 77, 84; 8 and 99, 107.

With the multiplier 11, the product is equal to the sum of the multiplicand and itself, moved one place to the left. Thus in the preceding example, one says: 8 (write 8 in the result); 8 and 4 are 12 (write 2 and carry 1); 1 and 4, 5, and 6, 11; 1 and 6, 7, and 7, 14; 1 and 7, 8, and 9, 17; 1 and 9, 10.

When two adjacent figures of the multiplier form the number 11 or 12, as in the second example shown above, multiply the multiplicand by 11 or 12 as by a single figure; which gives one partial product less.

3d. When the multiplier contains only 9s, except the last figure at the right, which may be anything, to get the product, multiply the multiplicand by unity, followed by as many ciphers as there are figures in the multiplier, and from the result subtract the product of the multiplicand and the difference between 10 and the number at the right of the multiplier.

Having, for example, $9998 = 10{,}000 - (10 - 8) = 10{,}000 - 2$, to multiply with 65,873, we have $65{,}873 \times 9998 = 65{,}873 \times 10{,}000 - 65{,}873 \times 2 = 658{,}730{,}000 - 131{,}746 = 658{,}598{,}254$. In doing the operation, write simply

```
 658,730,000
   - 131,746
 -----------
 658,598,254
```

If instead of one figure at the right of the 9s there are 2, 3 . . ., figures, from the multiplicand, followed by as many ciphers as there are figures in the multiplier, subtract the product of the multiplicand and difference between 100, 1000 . . ., and the 2, 3 . . ., figures at the right of the multiplier.

4th. When a multiplier, such as 48,546, contains parts $54 = 6 \times 9$ and $48 = 6 \times 8$, which are multiples of one of its fig-

ures 6, after having multiplied by 6, multiply the partial product by 9, which gives the product of the multiplicand and 54; the same partial product by 8 gives the product of the multiplicand and 48.

$$\begin{array}{lr}
 & 58453 \\
 & 48546 \\
\hline
6 \quad \ldots\ldots & 350718 \\
54 = 6 \times 9 \ldots\ldots & 3156462 \\
48 = 6 \times 8 \ldots\ldots & 2805744 \\
\hline
 & 2837659338
\end{array}$$

5th. Having $5 = \frac{10}{2}$, $25 = \frac{100}{4}$ and $125 = \frac{1000}{8}$, to multiply a number by 5, 25, or 125 multiply by 10, 100, or 1000 and divide the product by 2, 4, or 8.

$$1479 \times 25 = \frac{147{,}900}{4} = 36{,}975, \quad 4729 \times 125 = \frac{4{,}729{,}000}{8} = 591{,}125.$$

When adjacent figures of the multiplier form the numbers 25 or 125, the multiplicand may be multiplied by these numbers as above:

$$\begin{array}{lr}
 & 1479 \\
 & 257 \\
\hline
7 \ldots\ldots & 10353 \\
25 \ldots\ldots & 36975 \\
\hline
 & 380103
\end{array}$$

6th. Since the product of several factors is not changed by changing the order of the factors (41), and since several of the factors can be replaced by their product (42) many times by suitable grouping of the factors, an operation may be materially shortened, which would be very long if carried out in the way indicated. Example:

$$25 \times 9 \times 5 \times 7 \times 2 \times 4 = 9 \times 7\ (25 \times 4) \times (5 \times 2) = 63 \times 100 \times 10 = 63 \times 1000 = 63{,}000.$$

DIVISION

51. *Division* is an operation by which a quantity called the *dividend* is separated into as many equal parts as there are units in a whole number called the *divisor;* one of these parts is the *quotient* of the division.

Division is a short method of performing a series of subtractions. In subtracting successively the divisor from the dividend

and from the remainder until a remainder is obtained which is smaller than the divisor, the number of subtractions performed is the quotient.

52. From the definition of division it follows that the dividend is equal to the product of the quotient and the divisor (32).

53. A number is said to be *divisible* by another, when the quotient obtained by the division of first by the second is a whole number. The second number is said to be a *divisor* of the first.

54. All numbers are divisible by themselves and unity. The quotient is equal to one in the first case and to the dividend in the second.

55. A number is *even* or *odd* according as it is or is not divisible by 2.

The numbers 2, 4, 6, 8, divisible by 2, are called even numbers, and 0 is also considered even. The other numbers, 1, 3, 5, 7, 9, are odd.

A number is odd or even according as its first figure at the right is odd or even (90).

56. When a number, 12, is a multiple of another, 4, the first is divisible by the second and conversely (52).

57. The product of several whole numbers is divisible by any one of its factors (38 and 56).

58. When a number contains all the factors of another number the first is divisible by the second (37, 38, and 56).

59. Any divisor, 4, common to several numbers 36, 12, 16, divides their sum, 64 (39 and 56).

60. Any divisor, 7, common to two numbers, 42 and 14, divides their difference, 28 (40 and 56).

61. Any divisor, 5, of a number, 35, will divide any multiple, $35 \times 3 = 105$, of that number (39 and 56).

62. *To divide a sum by a number*, divide each part of the sum by the number (33), thus:

$$\frac{32 + 12 + 16}{4} = \frac{32}{4} + \frac{12}{4} + \frac{16}{4} = 8 + 3 + 4 = 15.$$

63. *To divide a difference*, 32 — 12, *by a number*, 4, divide each of the terms by the number 4 (34), thus:

$$\frac{32 - 12}{4} = \frac{32}{4} - \frac{12}{4} = 8 - 3 = 5.$$

64. *To divide a whole number,* 4,145,824, *by another whole number,* 845.

1	845
2	1690
3	2535
4	3380
5	4225
6	5070
7	5915
8	6760
9	7605

$$\begin{array}{r|l} 4145824 & 845 \\ \underline{3380} & \overline{4906} \\ 7658 & \\ \underline{7605} & \\ 005324 & \\ \underline{5070} & \\ 254 & \end{array}$$

To divide one number by another, write the divisor at the right of the dividend, separate them by a vertical line, and underline the divisor. Then, from the left of the dividend, point off just enough figures so that the number 4145 which results will contain the divisor; look in the table of the first nine multiples of the divisor to find how many times the divisor is contained in the part of the dividend which has been pointed off and this gives the first figure 4 at left of the quotient; write this figure under the divisor; subtract from the first partial dividend 4145 the product 3380 of the divisor and the figure obtained in the quotient, which gives 765 as a remainder, at the right of this partial remainder bring down, that is, write, the next figure 8 of the dividend; find how many times the divisor is contained in the number 7658 which results, thus determining the second figure 9 of the quotient; subtract from the second partial dividend 7658 the product 7605 of the divisor and the second figure of the quotient, giving a remainder of 53, at the right of which write the following figure 2 of the dividend. Since the divisor is not contained in the third partial dividend 532, the third figure of the quotient is 0. At the right of 532, write the following figure 4 of the dividend; find how many times the divisor is contained in the fourth partial dividend 5324, and continue thus until all the figures of the dividend have been used. The last remainder obtained 254 is the *remainder of the division.*

Generally one does not take the trouble to write the first nine multiples of the divisor. Then to find the number of times that the divisor is contained in the partial dividend 4145, consider simply the first figure 8 at the left of the divisor; neglect as many figures at the right of the partial dividend as have been suppressed in the divisor, and find how many times 8 is contained

in the number 41 which results; 8 being contained 5 times in 41, it is natural to suppose that 5 is the number of times the divisor 845 is contained in the partial dividend 4145; but in multiplying 5 by the figure 4 of the divisor there will be 2 to carry to the product of 8 by 5, which will give 42, showing that 5 is too large. Trying 4 as we have just done with 5, we find it to be the first figure at the left of the quotient. The product of this figure and the divisor need not be written but may be subtracted as fast as the figures are obtained. The preceding division would be performed in the following manner:

$$\begin{array}{r|l} 4145824 & \underline{845} \\ 7658 & 4906 \\ 5324 & \\ 254 & \end{array}$$

and to perform the operation one says: How many times is 8 contained in 41? (trying 5, and saying 5 times 8 are 40, and 2, which results from 5 times 4, are 42, showing 5 to be too large) 4 times (write 4 in the quotient); 4 times 5, 20; 20 from 25, 5 remainder and 2 to carry; 4 times 4, 16, and 2, 18; 18 from 24, 6 and 2 to carry; 4 times 8, 32, and 2, 34; 34 from 41, 7. Bring down 8; how many times is 8 contained in 76? 9 times (write 9 in the quotient); 9 times 5, 45; 45 from 48, 3, and 4 to carry; 9 times 4, 36, and 4 are 40, from 45, 5; 9 times 8, 72, and 4, 76, from 76, 0 (not necessary to write 0). Bring down 2; how many times is 8 contained in 5? No times (write 0 in the quotient). Bring down 4; how many times is 8 contained in 53? 6 times, etc.

When the divisor is very large, and the quotient is to have a large number of figures, or when there are many numbers to be divided by the same divisor, it is advantageous to construct a table of the nine first multiples of the divisor. Because in this way the successive figures of the quotient are obtained immediately, and the multiplication of the divisor by the figures is avoided. The work can be shortened still more by not writing the multiples of the divisor under the partial dividends when subtracting.

When the divisor has only one figure, 7 for instance, write simply the dividend, and remember that to divide a number by 7 is simply to take one-seventh of it (162),

dividend 174,389
quotient 24,912 remainder 5,

one says: a seventh of 17 is 2 (write 2 in the quotient under the dividend and carry $17 - 7 \times 2 = 3$); a seventh of 34, 4 (write 4 and carry 6); the seventh of 63, 9; of 8, 1; of 19, 2; the remainder of the division is 5.

REMARK 1. The dividend, 4,145,824, and the divisor, 845, being given, the number of figures which the quotient is to contain may be found by pointing off at the left of the dividend just enough figures, 4145, to contain the divisor, then the number of figures left in dividend increased by one will equal the number of figures in the quotient, thus, in the example above, $3 + 1 = 4$ figures in the quotient.

REMARK 2. A figure in the quotient is too large when its product with the divisor is larger than the corresponding partial dividend, that is, when it can not be subtracted from the partial dividend.

If, however, the subtraction is possible and the remainder is larger than the divisor, then the figure in the quotient is too small.

65. *To prove a division,* multiply the divisor by the quotient and add the remainder, which is always smaller than the divisor, which will give the dividend if the work is correct (**52**); thus in the preceding example $4906 \times 845 + 254$ should equal 4,145,824 (**100**).

66. *To divide a number by one followed by any number of ciphers,* separate with a comma as many figures at the right of the dividend as there are ciphers in the divisor. The part at the left, expressing the simple units, is the quotient, and the part at the right is the remainder. Thus:

$$\frac{84735}{100} = 847.35$$

847 is the quotient, and 35 the remainder. In decimal numbers the quotient is 847.35, and the remainder 0 (**89** and **182**).

When ciphers at the right of a whole number are suppressed, it is the same as dividing the number by one followed by as many ciphers as have been suppressed (**44**):

$$\frac{8500}{100} = 85.$$

Having $5 = \frac{10}{2}$, $25 = \frac{100}{4}$ and $125 = \frac{1000}{8}$, it follows that when a number is to be divided by 5, 25, or 125 the operation may be shortened (**164**) by multiplying the number by 2, 4, or 8 and dividing the product by 10, 100, or 1000:

$$\frac{36{,}957}{25} = \frac{36{,}957 \times 4}{100} = 1478{,}28; \quad \frac{591{,}473}{125} = \frac{591{,}473 \times 8}{1000} = 4{,}731{,}784.$$

The decimal numbers obtained are the exact quotients (**91**).

67. *To divide a number*, 504, *by a product*, 42, *of several factors* 2, 3, 7, divide the number by the first factor, 2, of the product, the quotient, 252, obtained by the second, 3; and so on until the last factor, 7, has been used as divisor, which will give the quotient, 12, desired (**42**):

$$\frac{37{,}471}{700} = \frac{37{,}471}{100 \times 7} = \frac{374.71}{7} = 53.53 \ (\mathbf{182}).$$

68. When a factor, 8, of a product, $3 \times 8 \times 5 = 120$, is divided by a number, 4, the product is divided by that number (**43**), thus:

$$3 \times \frac{8}{4} \times 5 = \frac{3 \times 8 \times 5}{4} = \frac{120}{4} = 30.$$

69. *To divide a product by one of its factors*, suppress this factor in the product. Thus (**68**) :

$$\frac{3 \times 8 \times 5}{8} = 3 \times \frac{8}{8} \times 5 = 3 \times 1 \times 5 = 3 \times 5.$$

70. When a product contains all the factors of another product, the quotient of the first divided by the second may be obtained by suppressing in the first product all the factors of the second (**67** and **69**):

$$\frac{2 \times 3 \times 5 \times 7}{3 \times 7} = 2 \times 5.$$

71. When the dividend 54 is multiplied or divided by a number 3, without changing the divisor 6, the quotient 9 is multiplied or divided by that number:

$$\frac{54 \times 3}{6} = 9 \times 3 = 27, \quad \text{and} \quad \frac{54 : 3}{6} = \frac{9}{3} = 3.$$

72. When the divisor 6 is multiplied or divided by a number 3, without changing the dividend 54, the quotient 9 is divided or multiplied by that number:

$$\frac{54}{6 \times 3} = \frac{9}{3} = 3, \quad \text{and} \quad \frac{54}{6 \div 3} = 9 \times 3 = 27.$$

73. When the dividend 54 and the divisor 6 are multiplied or divided by the same number 3, the quotient 9 remains unchanged:

$$\frac{54 \times 3}{6 \times 3} = 9, \quad \text{and} \quad \frac{54 \div 3}{6 \div 3} = 9.$$

74. From (73) it follows that when the dividend and divisor have common factors, the operation may be shortened by eliminating those factors:

$$\frac{7 \times 324 \times 23}{7 \times 12 \times 23} = \frac{324}{12} = \frac{324 \div 4}{12 \div 4} = \frac{81}{3} = 27.$$

It follows also that when the dividend and divisor end with ciphers, the same number of ciphers may be suppressed at the right of each, without altering the quotient (66 and 73):

$$\frac{35{,}000}{700} = \frac{350}{7} = 50.$$

75. All common divisors, 6, of the dividend, 48, and divisor, 18, divide the remainder, 12, of the division, and all common divisors of the remainder, 12, and the divisor, 18, divide the dividend, 48.

76. When the dividend 48 and the divisor 18 are multiplied or divided by the same number 6, the quotient remains unchanged; but the remainder is multiplied or divided by that number.

77. When the dividend 48 is increased or diminished by a certain number of times the divisor 9, the quotient 5 is increased or diminished a certain number of times unity; but the remainder is unaltered.

Thus the sum $48 + 54 = 102$ of two numbers is not divisible by a third number 9, when only one of the numbers 54 is divisible by 9.

The sum 102 divided by 9 gives for a quotient the sum $5 + 6 = 11$ of the quotients of 48 and 54 by 9, and for a remainder. the remainder 3 of 48 by 9.

BOOK II

PROPERTIES OF WHOLE DIVISORS

78. A number is a *prime number* when it is not divisible except by itself and one (**53**): 1, 2, 3, 5, 7, 11, 13, 17 . . . are prime numbers.

79. All numbers, 21, which are not prime numbers are the product of several prime factors larger than unity: $21 = 3 \times 7$.

80. Several numbers are said to be *prime to each other* when they have no other common divisor than unity (**53**): such are the numbers 4 and 9; also 6, 10, and 15. The numbers 6, 8, and 12 being all divisible by 2, are not prime to each other.

81. All prime numbers which do not divide a whole number are prime with that number: such are 7 and 15.

82. The *greatest common divisor* of several numbers is the largest number which will divide each of the numbers.

Remark. The greatest common divisor of several numbers prime to each other is one.

83. The *least common multiple* of several numbers is the smallest number which is a multiple of each of the numbers (**38**).

84. The *separation of a number into its factors, factoring,* is to find several numbers, the product of which will equal the number. Thus, having $24 = 2 \times 3 \times 4$, the number 24 is separated into three factors 2, 3, and 4.

85. The product of several factors each equal to a given number is a *power* of that number. Thus, having $27 = 3 \times 3 \times 3$, and $81 = 3 \times 3 \times 3 \times 3$, 27 and 81 are powers of 3.

86. The *degree* of the power of a number is the number of factors of that power. Thus 3 and 4 are the degrees of the powers 27 and 81 of the number 3.

Remark. All powers of 10 are equal to one followed by as many ciphers as there are units in the degree of the power. Thus the third power of 10 is 1000; $10 \times 10 \times 10 = 1000$ (**44**).

87. The second power, $7 \times 7 = 49$, of a number, 7, is the *square* of the number, 7; the third power, $4 \times 4 \times 4 = 64$, of a number, 4, is the *cube* of the number, 4.

88. The *exponent* of a number raised to a certain power is the degree of this power written to the right and a little above the number. Thus, to express, in an abbreviated manner, that the number 5 is raised to the fourth power, write 5^4 instead of $5 \times 5 \times 5 \times 5$.

REMARK. The first power of a number is the number itself, which may be considered as having the exponent one, although properly speaking it is no power and has no exponent.

89. *To obtain a quotient and a remainder by dividing a number by a power of* 10, separate on the right of the number as many figures as there are units in the degree of the power; the part to the left and the part to the right considered as expressing simple units, are respectively the desired quotient and remainder. Thus having to divide 97,845 by $10^3 = 1000$, separate three figures, which will give 97.845; the quotient is then 97 and the remainder 845.

COROLLARY. If a *number be divisible by a power of* 10, it must end in at least as many ciphers as there are units in the degree of the power (**66**).

90. *To obtain the remainder in the division of a number by* 2 *or* 5, it suffices to find the remainder in the division of the first figure at the right by 2 or 5. Thus the number 45,737 divided by 2 gives 1 for a remainder, and divided by 5 gives 2, because the first figure 7 divided by 2 or 5 gives respectively 1 or 2 for a remainder; the figure 0 is considered as divisible by 2 and by 5 (**55**).

91. In general, *to obtain the remainder in the division of a number by any power of* 2 *or* 5, it suffices to find the remainder in the division of the number, obtained by pointing off as many figures on the right of the number as there are units in the degree of the power, by the power. Thus, to obtain the remainder in the division of 45,737 by $2^3 = 8$, or by $5^3 = 125$, find the remainder in the division of 737 by 8 or by 125, which gives respectively 1 and 112 (**50** and **66**).

In order that a number be divisible by any power of 2 or 5, the number, obtained by pointing off at the right of the number in question as many figures as there are units in the degree of the power, must be 0 or divisible by the power. Thus, for example, a number is divisible by 125 if the three figures at the right form the numbers 000, 125, 250, 375, 500 . . .

92. To obtain the *remainder in the division of a number by* 9, add the figures considering them as simple units; operate on this sum as upon the first number, and so on until a result is obtained which does not exceed 9. When this result is less than 9, it is the required remainder; and if it is 9, the remainder is 0. Thus to obtain the remainder in the division of 75,487 by 9, for instance, add $7 + 5 + 4 + 8 + 7 = 31$; then add $3 + 1 = 4$, and 4 is the required remainder. It is immaterial how the sum is made, commencing at the right or left.

The operation is shortened by taking 9 from each successive sum which is greater than or equal to 9. Thus, one says: 7 and 8, 15 (less 9), 6 and 4, 10 (less 9), 1 and 5, 6 and 7, 13 (less 9), 4.

The operation may be shortened still more by neglecting the figures 9 and any group of which the sum is 9. Thus in the preceding example neglecting 4 and 5: 7 and 8, 15, 6 and 7, 13; 4.

Finally, a step still more expeditive consists in neglecting the figures 9 and those of which the sum is 9 and continuing the addition until all the figures have been used, reducing the successive sums which are multiples of 9 to 0, and those which are not, to numbers in the tens. Thus according as a sum is 27, 29, or 20 it may be reduced to 0, 2, or 2. Given the following number to find the remainder when dividing by 9:

$$8{,}562{,}647{,}683{,}568{,}697,$$

one says: 7, 13, 21, 27; 5, 8, 16, 22, 29; 2, 6, 12, 14, 20; 2, 7, 15, 6.

If for one reason or another the above short methods are not used and the successive sum becomes too large, it may be reduced by adding its figures and proceeding as before. If, for instance, one has 75, one says: 5 and 7, 12; 2 and 1, 3, and continues the addition with the number 3.

93. If a number is divisible by 9, the sum of the figures which express the simple units must be divisible by 9, that is, be a multiple of 9 (**38** and **53**).

94. *To obtain the remainder in the division of a number by* 3, firstly, find its remainder in its division by 9 (**92**); then the remainder in the division of this first remainder by 3. Thus the number 45,847 giving 4 for a remainder in its division by 9, and 4 divided by 3 giving 1 for a remainder, 1 is the required remainder in the division of the number in question by 3.

95. If a number is divisible by 3, the sum of the figures which express the simple units is divisible by 3, that is, must be a multiple of 3 (**38** and **53**).

96. *To obtain the remainder in the division of a number by* 11, commencing at the right point off the figures in periods of two figures each; and add these numbers, considering them as expressing simple units; operate on this sum as before and so on until a result is obtained which does not exceed 99; the remainder in the division of this last sum by 11 is the required remainder. Thus, it being given to find the remainder in the division of 7,345,798 by 11, separate the number into periods of two figures each, which gives 7, 34, 57, 98; adding, we get

$$98 + 57 + 34 + 7 = 196, \quad \text{then} \quad 96 + 1 = 97;$$

the remainder 9 in the division of 97 by 11 is the required remainder.

It is evident that this sum of periods of two figures each may be obtained by adding them directly, in saying 98 and 57, 155 and 34, 189 and 7, 196, if one is accustomed to calculating, or one can add the right-hand figures considered as units, $8 + 7 + 4 + 7 = 26$, and then the others taken as tens, $2 + 9 + 5 + 3 = 19$, the 2 being carried from the first sum ; writing these according to their orders, that is, 19 before the 6, we get the same result 196; upon which the operation may be continued.

If a number is divisible by 11, the sum of the periods of two figures each must be a multiple of 11, that is, divisible by 11.

Another rule for finding the remainder in the division of a number 7,395,748 by 11: commencing at the right with the first figure, add every other figure, $8 + 7 + 9 + 7 = 31$, then do the same thing, commencing with the second figure, $4 + 5 + 3 = 12$; subtract the second result from the first, $31 - 12 = 19$, and divide the difference by 11, which gives 8, the required remainder. Operating on this remainder 19 as on the original number, the required remainder is $9 - 1 = 8$. If a number 7391 gives a sum $3 + 1 = 4$, which is less than $7 + 9 = 16$, the subtraction is made possible by increasing the first by a number which is a multiple of 11. Thus $[4 + 22] - 16 = 10$, 10 being the remainder. Operating as in Ex. 2d (31), for the number 7,395,748, one would say without writing a single figure: 8, 15, 24, 31; less

4, 27, less 5, 22, less 3, 19. Having obtained the difference 19, one says, 9 less 1, 8, and 8 is the required remainder. With this manner of operating, when applied to the number 7391, where 11 is added to make the subtraction possible, one says: 1, 4; (4 + 11 or 15) less 9, 6; 17 less 7, 10.

97. *The proof of the addition of several whole numbers by the rule of* 9. Find the remainders 8, 3, 1, 4, in the division of the numbers to be added by 9; add these remainders, and if the remainder 7 in the division of this sum 16 by 9 is equal to the remainder 7 in the division of the sum 2437 of the whole numbers by 9, the result 2437 is correct (**26**).

NUMBERS	REMAINDERS
827	8
453	3
325	1
832	4
2437	16
16	7
7	

REMARK. This proof may be done more rapidly by adding the remainder of the first number directly to the figures of the second; the remainder obtained for the first two directly to the third and so on. Thus, using the abbreviations as in (**92**), one says (leaving out 7 and 2 in the first and 5 and 4 in the second): 8, 11, 16, 18; 3, 5, 8, 16; 7, which ought to be equal to the remainder in the division of 2437 by 9.

98. *The proof of the subtraction of two whole numbers by the rule of* 9. Consider the larger number, 845, as being the sum of the smaller, 258, and the remainder, 587, then proceed as in addition (**97**).

845	8
258	6
587	2
	8

Thus the sum 6 + 2 = 8 of the remainders in the division of the smaller number and the difference by 9 being equal to the remainder 8 in the division of the larger number 845 by 9, the operation is correct (**30**).

The remark under (**97**) applies here as well, but the ordinary proof of subtraction being so simple, the proof by 9 is seldom used.

99. *The proof of the multiplication of two whole numbers by*

the rule of 9. Find the remainders 6 and 2 in the division of the numbers 357 and 65 by 9 (**92**); multiply these two remainders together, and the remainder 3, in the division of the product 12 by 9, is equal to the remainder in the division of the product 23,205 by 9, if the calculations are correct (**48**).

357	6	
65	2	
1785	12	3
2142		
23205		3

REMARK. This proof is often used. Like all proofs by 9, it does not show errors equal to a multiple of 9. It is a probability but not a mathematical certainty.

100. *The proof of the division of two whole numbers by the rule of* 9. Consider the dividend as being the product of the divisor 85, and the quotient 59 plus the remainder 48, the proof is a combination of the proof for addition and that for multiplication (**97** and **99**). Thus, find the remainders 4 and 5 in the division of the divisor and quotient by 9; multiply them together, and the remainder 2, in the division of this product 20 by 9, increased by the remainder 3, in the division of the remainder 48 by 9, should equal the remainder 5 in the division of the dividend by 9 (**65**). Instead of finding the remainders 2 and 3 in the division of the product 20 and the remainder 48 by 9 and adding them $2 + 3 = 5$, the same result may be obtained by finding the remainder in the division of the sum $48 + 20$ by 9. One says (**97**): 2, 10, 14; 5.

5063	85	4	
813	59	5	
48	3	20	2
			3
			5

101. *The proof of the four operations is the same by the rule of* 11 *as by that of* 9 (**97** to **100**), but is rarely used. However, if the correctness of the results is of very great importance, both methods of proof may be used.

102. *To find the greatest common divisor of two whole numbers,* 876 *and* 360 (**82**), divide the greater number by the smaller, writing the quotient obtained, 2, and those following over the corresponding divisors; then divide the smaller number by the remainder obtained 156; and this first remainder by the next 48; and so on until a remainder of 0 is obtained. The last divisor 12 is the greatest common divisor (**125**).

	2	2	3	4
876	360	156	48	12
156	48	12	0	

Generally the greatest common divisor of two numbers is found simply to determine the quotient of these numbers by their greatest common divisor (146). In performing the operation of finding the greatest common divisor, these quotients are easily obtained, as are also those of the remainders or successive divisors 156, 48, and 12. Thus, on a horizontal line under 12 write 1; under the divisor 48, on the same horizontal line, write the last quotient obtained 4; under the divisor 156, the number $4 \times 3 + 1 = 13$, obtained by adding the preceding number 1 to the product of the number 4, just written, and the quotient 3 written above in the same column; under the divisor 360, the number $13 \times 2 + 4 = 30$, obtained by adding the preceding number 4 to the product of the last number obtained, 13, and the quotient 2 in the same column, and under the number 876, the numbers $30 \times 2 + 13 = 73$, obtained in the same manner. The number 1, 4, 13, 30, and 73 are respectively the quotients in the divisions of the divisors 12, 48, 156, and the given numbers 360 and 876 by the greatest common divisor 12.

	2	2	3	4
876	360	156	48	12
156	48	12	0	
73	30	13	4	1

REMARK. The greatest common divisor of two numbers, 36 and 144, of which one divides the other, is the smaller, 36, of the numbers.

103. All divisors, 3, common to two numbers, 384 and 36, divide their greatest common divisor, 12, also the successive remainders, 24, 12, obtained in the process of finding the greatest common divisor.

104. *To find the greatest common divisor of any number of numbers*, find the greatest common divisor of two of the numbers (102), then the greatest common divisor of that greatest common divisor and another of the numbers, and so on until all of the numbers have been used; the last greatest common divisor is the one desired (125).

105. The greatest common divisor of several numbers is multiplied or divided by a number when those numbers are multiplied or divided by the same number.

It follows that the quotients of several numbers divided by their greatest common divisor are prime to each other.

106. Any number, 4, which divides a product, 7×16, of two

factors, and which is prime to one of the factors, 7, divides the other factor, 16.

107. Any prime number, 5, which divides a product, $12 \times 13 \times 25$, divides at least one of the factors of the product; and all prime numbers which divide a power, 15^3, of a number, 15, divide the number.

108. Any number, 4, prime to each factor of a product, 7×15 15×23, is prime to the product. Any number, 4, prime with another, 15, is prime to any power of that number.

109. When two numbers, 4 and 15, are prime to each other, all powers of one are prime to any power of the other.

110. Any number, 720, divisible by two numbers, 4 and 9, prime to each other (80), is divisible by their product, 36.

111. Any number, 7200, divisible by several numbers, 4, 9, 25, prime to each other in pairs, is divisible by their product.

112. The least common multiple of several whole numbers, 4, 9, 25, prime to each other in pairs, is equal to their product, $4 \times 9 \times 25 = 900$ (83).

113. Any common multiple, 192, of two numbers, 24 and 16, is a multiple of the product, $8 \times 3 \times 2$, whose factors are the greatest common divisor, 8, of these numbers and the quotients, 3 and 2, of their division by this greatest common divisor; and, conversely, any multiple of this product is a common multiple of the two numbers, 24 and 16.

114. The least common multiple of two numbers, 24 and 16, is equal to the product, $8 \times 3 \times 2 = 48$, whose factors are the greatest common divisor, 8, of these numbers and the quotients, 3 and 2, of their division by this greatest common divisor. In the same manner the least common multiple may be determined (112 and 126).

115. Any common multiple of two, 24 and 16, is a multiple of their least common multiple, 48.

116. The least common multiple, 48, of two numbers, 24 and 16, is equal to the product of either one of the numbers and the quotient of the division of the other number by their greatest common divisor, 8 (114).

117. The product of the greatest common divisor, 8, of two numbers, 24 and 16, and their least common multiple, 48, is equal to the product, 24×16, of the two numbers.

118. When two numbers, 24 and 16, are multiplied or divided

by the same number, their least common multiple, 48, is multiplied or divided by that number.

119. *To find the least common multiple of several whole numbers*, 6, 8, 9, 10, find the least common multiple, 24, of the first two, 6 and 8 (**114**), then the least common multiple, 72, of that least common multiple, 24, and the third number, 9, and so on; the last least common multiple, 360, is the one required (**126**).

120. When the least common multiple, 72, of several numbers 8, 12, 18, is divided by each one of the numbers, the quotients, 9, 6, 4, are prime to each other; and, conversely, when a number, 72, is such that in dividing it by several others, 8, 12, 18, quotients, 9, 6, 4, are obtained which are prime to each other, this number is the least common multiple of all the others.

121. Any whole number, 43, is prime when, being between the squares, 25 and 49, of two consecutive prime numbers, 5 and 7, it is neither divisible by the smaller of these prime numbers, nor by any number which precedes it, except one.

122. In general, *to determine a prime number*, divide by 2, 3, 5, 7, etc., until a quotient is obtained which is equal to or less than the last prime number used as divisor (**121**).

123. The series of prime numbers is unlimited. In the following tables on the next pages are given:

1st. Prime numbers from 1 to 10,000.

2d. Numbers less than 10,000 which do not contain the prime factors 2, 3, 5, 7, and 11, and their prime factors.

Table of Prime Numbers between 1 and 10,000

1	367	839	1367	1907	2467	3061	3643	4243	4889	5501	6121	6761	7433	8069	8713	9349
2	73	53	73	13	73	67	59	53	4903	03	31	63	51	81	19	71
3	79	57	81	31	77	79	71	59	09	07	33	79	57	87	31	77
5	83	59	99	33	2503	83	73	61	19	19	43	81	59	89	37	91
7	89	63	1409	49	21	89	77	71	31	21	51	91	77	93	41	97
11	97	77	23	51	31	3109	91	73	33	27	63	93	81	8101	47	9403
13	401	81	27	73	39	19	97	83	37	31	73	6803	87	11	53	13
17	09	83	29	79	43	21	3701	89	43	57	97	23	89	17	61	19
19	19	87	33	87	49	37	09	97	51	63	99	27	99	23	79	21
23	21	907	39	93	51	63	19	4327	57	69	6203	29	7507	47	83	31
29	31	11	47	97	57	67	27	37	67	73	11	33	17	61	8803	33
31	33	19	51	99	79	69	33	39	69	81	17	41	23	67	07	37
37	39	29	53	2003	91	81	39	49	73	91	21	57	29	71	19	39
41	43	37	59	11	93	87	61	57	87	5623	29	63	37	79	21	61
43	49	41	71	17	2609	91	67	63	93	39	47	69	41	91	31	63
47	57	47	81	27	17	3203	69	73	99	41	57	71	47	8209	37	67
53	61	53	83	29	21	09	79	91	5003	47	63	83	49	19	39	73
59	63	67	87	39	33	17	93	97	09	51	69	99	59	21	49	79
61	67	71	89	53	47	21	97	4409	11	53	71	6907	61	31	61	91
67	79	77	93	63	57	29	3803	21	21	57	77	11	73	33	63	97
71	87	83	99	69	59	51	21	23	23	59	87	17	77	37	67	9511
73	91	91	1511	81	63	53	23	41	39	69	99	47	83	43	87	21
79	99	97	23	83	71	57	33	47	51	83	6301	49	89	63	93	33
83	503	1009	31	87	77	59	47	51	59	89	11	59	91	69	8923	39
89	09	13	43	89	83	71	51	57	77	93	17	61	7603	73	29	47
97	21	19	49	99	87	99	53	63	81	5701	23	67	07	87	33	51
101	23	21	53	2111	89	3301	63	81	87	11	29	71	21	91	41	87
03	41	31	59	13	93	07	77	83	99	17	37	77	39	93	51	9601
07	47	33	67	29	99	13	81	93	5101	37	43	83	43	97	63	13
09	57	39	71	31	2707	19	89	4507	07	41	53	91	49	8311	69	19
13	63	49	79	37	11	23	3907	13	13	43	59	97	69	17	71	23
27	69	51	83	41	13	29	11	17	19	49	61	7001	73	29	99	29
31	71	61	97	43	19	31	17	19	47	79	67	13	81	53	9001	31
37	77	63	1601	53	29	43	19	23	53	83	73	19	87	63	07	43
39	87	69	07	61	31	47	23	47	67	91	79	27	91	69	11	49
49	93	87	09	79	41	59	29	49	71	5801	89	39	99	77	13	61
51	99	91	13	2203	49	61	31	61	79	07	97	43	7703	87	29	77
57	601	93	19	07	53	71	43	67	89	13	6421	57	17	89	41	79
63	07	97	21	13	67	73	47	83	97	21	27	69	23	8419	43	89
67	13	1103	27	21	77	89	67	91	5209	27	49	79	27	23	49	97
73	17	09	37	37	89	91	89	97	27	39	51	7103	41	29	59	9719
79	19	17	57	39	91	3407	4001	4603	31	43	69	09	53	31	67	21
81	31	23	63	43	97	13	03	21	33	49	73	21	57	43	91	33
91	41	29	67	51	2801	33	07	37	37	51	81	27	59	47	9103	39
93	43	51	69	67	03	49	13	39	61	57	91	29	89	61	09	43
97	47	53	93	69	19	57	19	43	73	61	6521	51	93	67	27	49
99	53	63	97	73	33	61	21	49	79	67	29	59	7817	8501	33	67
211	59	71	99	81	37	63	27	51	81	69	47	77	23	13	37	69
23	61	81	1709	87	43	67	49	57	97	79	51	87	29	21	51	81
27	73	87	21	93	51	69	51	63	5303	81	53	93	41	27	57	87
29	77	93	23	97	57	91	57	73	09	97	63	7207	53	37	61	91
33	83	1201	33	2309	61	99	73	79	23	5903	69	11	67	39	73	9803
39	91	13	41	11	79	3511	79	91	33	23	71	13	73	43	81	11
41	701	17	47	33	87	17	91	4703	47	27	77	19	77	63	87	17
51	09	23	53	39	97	27	93	21	51	39	81	29	79	73	99	29
57	19	29	59	41	2903	29	99	23	81	53	99	37	83	81	9203	33
63	27	31	77	47	09	33	4111	29	87	81	6607	43	7901	97	09	39
69	33	37	83	51	17	39	27	33	93	87	19	47	07	99	21	51
71	39	49	87	57	27	41	29	51	99	6007	37	53	19	8609	27	57
77	43	59	89	71	39	47	33	59	5407	11	53	83	27	23	39	59
81	51	77	1801	77	53	57	39	83	13	29	59	97	33	27	41	71
83	57	79	11	81	57	59	53	87	17	37	61	7307	37	29	57	83
93	61	83	23	83	63	71	57	89	19	43	73	09	49	41	77	87
307	69	89	31	89	69	81	59	93	31	47	79	21	51	47	81	9901
11	73	91	47	93	71	83	77	99	37	53	89	31	63	63	83	07
13	87	97	61	99	99	93	4201	4801	41	67	91	33	93	69	93	23
17	97	1301	67	2411	3001	3607	11	13	43	73	6701	49	8009	77	9311	29
31	809	03	71	17	11	13	17	17	49	79	03	51	11	81	19	31
37	11	07	73	23	19	17	19	31	71	89	09	69	17	89	23	41
47	21	19	77	37	23	23	29	61	77	91	19	93	39	93	37	49
49	23	21	79	41	37	31	31	71	79	6101	33	7411	53	99	41	67
53	27	27	89	47	41	37	41	77	83	13	37	17	59	8707	43	73
59	29	61	1901	59	49											

Table of Numbers between 1 and 10,000 which do not Contain the Prime Factors 2, 3, 5, 7, and 11 and Their Prime Factors.

No.	Factors.	No.	Factors.	No.	Factors.	No.	Factors.
169	13 × 13	1333	31 × 43	2171	13 × 167	2951	13 × 227
221	13 × 17	39	13 × 103	73	41 × 53	77	13 × 229
47	13 × 19	43	17 × 79	83	37 × 59	83	19 × 157
89	17 × 17	49	19 × 71	97	13 × 13 × 13	87	29 × 103
99	13 × 23	57	23 × 59	2201	31 × 71	93	41 × 73
323	17 × 19	63	29 × 47	09	47 × 47	3007	31 × 97
61	19 × 19	69	37 × 37	27	17 × 131	13	23 × 131
77	13 × 29	87	19 × 73	31	23 × 97	29	13 × 233
91	17 × 23	91	13 × 107	49	13 × 173	43	17 × 179
403	13 × 31	1403	23 × 61	57	37 × 61	53	43 × 71
37	19 × 23	11	17 × 83	63	31 × 73	71	37 × 83
81	13 × 37	17	13 × 109	79	43 × 53	77	17 × 181
93	17 × 29	57	31 × 47	91	29 × 79	97	19 × 163
527	17 × 31	69	13 × 113	2323	23 × 101	3103	29 × 107
29	23 × 23	1501	19 × 79	27	13 × 179	07	13 × 239
33	13 × 41	13	17 × 89	29	17 × 137	27	53 × 59
51	19 × 29	17	37 × 41	53	13 × 181	31	31 × 101
59	13 × 43	37	29 × 53	63	17 × 139	33	13 × 241
89	19 × 31	41	23 × 67	69	23 × 103	39	43 × 73
611	13 × 47	77	19 × 83	2407	29 × 83	49	47 × 67
29	17 × 37	91	37 × 43	13	19 × 127	51	23 × 137
67	23 × 29	1633	23 × 71	19	41 × 59	61	29 × 109
89	13 × 53	43	31 × 53	49	31 × 79	73	19 × 167
97	17 × 41	49	17 × 97	61	23 × 107	93	31 × 103
703	19 × 37	51	13 × 127	79	37 × 67	97	23 × 139
13	23 × 31	79	23 × 73	83	13 × 191	3211	13 × 13 × 19
31	17 × 43	81	41 × 41	89	19 × 131	33	53 × 61
67	13 × 59	91	19 × 89	91	47 × 53	39	41 × 79
79	19 × 41	1703	13 × 131	2501	41 × 61	47	17 × 191
93	13 × 61	11	29 × 59	07	23 × 109	63	13 × 251
99	17 × 47	17	17 × 101	09	13 × 193	77	29 × 113
817	19 × 43	39	37 × 47	33	17 × 149	81	17 × 193
41	29 × 29	51	17 × 103	37	43 × 59	87	19 × 173
51	23 × 37	63	41 × 43	61	13 × 197	93	37 × 89
71	13 × 67	69	29 × 61	67	17 × 151	3317	31 × 107
93	19 × 47	81	13 × 137	73	31 × 83	37	47 × 71
99	29 × 31	1807	13 × 139	81	29 × 89	41	13 × 257
901	17 × 53	17	23 × 79	87	13 × 199	49	17 × 197
23	13 × 71	19	17 × 107	99	23 × 113	79	31 × 109
43	23 × 41	29	31 × 59	2603	19 × 137	83	17 × 199
49	13 × 73	43	19 × 97	23	43 × 61	97	43 × 79
61	31 × 31	49	43 × 43	27	37 × 71	3401	19 × 179
89	23 × 43	53	17 × 109	41	19 × 139	03	41 × 83
1003	17 × 59	91	31 × 61	69	17 × 157	19	13 × 263
07	19 × 53	1909	23 × 83	2701	37 × 73	27	23 × 149
27	13 × 79	19	19 × 101	43	13 × 211	31	47 × 73
37	17 × 61	21	17 × 113	47	41 × 67	39	19 × 181
73	29 × 37	27	41 × 47	59	31 × 89	73	23 × 151
79	13 × 83	37	13 × 149	71	17 × 163	81	59 × 59
81	23 × 47	43	29 × 67	73	47 × 59	97	13 × 269
1121	19 × 59	57	19 × 103	2809	53 × 53	3503	31 × 113
39	17 × 67	61	37 × 53	13	29 × 97	23	13 × 271
47	31 × 37	63	13 × 151	31	19 × 149	51	53 × 67
57	13 × 89	2021	43 × 47	39	17 × 167	69	43 × 83
59	19 × 61	33	19 × 107	67	47 × 61	87	17 × 211
89	29 × 41	41	13 × 157	69	19 × 151	89	37 × 97
1207	17 × 71	47	23 × 89	73	13 × 13 × 17	99	59 × 61
19	23 × 53	59	29 × 71	81	43 × 67	3601	13 × 277
41	17 × 73	71	19 × 109	99	13 × 223	11	23 × 157
47	29 × 43	77	31 × 67	2911	41 × 71	29	19 × 191
61	13 × 97	2117	29 × 73	21	23 × 127	49	41 × 89
71	31 × 41	19	13 × 163	23	37 × 79	53	13 × 281
73	19 × 67	47	19 × 113	29	29 × 101	67	19 × 193
1313	13 × 101	59	17 × 127	41	17 × 173	79	13 × 283

Table of Numbers between 1 and 10,000 which do not Contain the Prime Factors — Continued.

No.	Factors.	No.	Factors.	No.	Factors.	No.	Factors.
3683	29 × 127	4453	61 × 73	5207	41 × 127	5947	19 × 313
3713	47 × 79	69	41 × 109	13	13 × 401	59	59 × 101
21	61 × 61	71	17 × 263	19	17 × 307	63	67 × 89
37	37 × 101	89	67 × 67	21	23 × 227	69	47 × 127
43	19 × 197	4511	13 × 347	39	13 × 13 × 31	77	43 × 139
49	23 × 163	31	23 × 197	49	29 × 181	83	31 × 193
57	13 × 17 × 17	37	13 × 349	51	59 × 89	89	53 × 113
63	53 × 71	41	19 × 239	63	19 × 277	93	13 × 461
81	19 × 199	53	29 × 157	67	23 × 229	6001	17 × 353
91	17 × 223	59	47 × 97	87	17 × 311	19	13 × 463
99	29 × 131	73	17 × 269	93	67 × 79	23	19 × 317
3809	13 × 293	77	23 × 199	5311	47 × 113	31	37 × 163
11	37 × 103	79	19 × 241	17	13 × 409	49	23 × 263
27	43 × 89	89	13 × 353	21	17 × 313	59	73 × 83
41	23 × 167	4601	43 × 107	29	73 × 73	71	13 × 467
59	17 × 227	07	17 × 271	39	19 × 281	77	59 × 103
69	53 × 73	19	31 × 149	53	53 × 101	6103	17 × 359
87	13 × 13 × 23	33	41 × 113	59	23 × 233	07	31 × 197
93	17 × 229	61	59 × 79	63	31 × 173	09	41 × 149
3901	47 × 83	67	13 × 359	71	41 × 131	19	29 × 211
37	31 × 127	81	31 × 151	77	19 × 283	37	17 × 19 × 19
53	59 × 67	87	43 × 109	89	17 × 317	57	47 × 131
59	37 × 107	93	13 × 19 × 19	5429	61 × 89	61	61 × 101
61	17 × 233	99	37 × 127	47	13 × 419	69	31 × 199
73	29 × 137	4709	17 × 277	59	53 × 103	79	37 × 167
77	41 × 97	17	53 × 89	61	43 × 127	87	23 × 269
79	23 × 173	27	29 × 163	73	13 × 421	91	41 × 151
91	13 × 307	47	47 × 101	91	17 × 17 × 19	6227	13 × 479
4009	19 × 211	57	67 × 71	97	23 × 239	33	23 × 271
31	29 × 139	69	19 × 251	5513	37 × 149	39	17 × 367
33	37 × 109	71	13 × 367	39	29 × 191	41	79 × 79
43	13 × 311	77	17 × 281	43	23 × 241	53	13 × 13 × 37
61	31 × 131	4811	17 × 283	49	31 × 179	83	61 × 103
63	17 × 239	19	61 × 79	61	67 × 83	89	19 × 331
69	13 × 313	41	47 × 103	67	19 × 293	6313	59 × 107
87	61 × 67	43	29 × 167	87	37 × 151	19	71 × 89
97	17 × 241	47	37 × 131	97	29 × 193	31	13 × 487
4117	23 × 179	49	13 × 373	5603	13 × 431	41	17 × 373
21	13 × 317	53	23 × 211	09	71 × 79	71	23 × 277
41	41 × 101	59	43 × 113	11	31 × 181	83	13 × 491
63	23 × 181	67	31 × 157	17	41 × 137	6401	37 × 173
71	43 × 97	83	19 × 257	27	17 × 331	03	19 × 337
81	37 × 113	91	67 × 73	29	13 × 433	07	43 × 149
83	47 × 89	97	59 × 83	33	43 × 131	09	13 × 17 × 29
87	53 × 79	4901	13 × 13 × 29	71	53 × 107	31	59 × 109
89	59 × 71	13	17 × 17 × 17	81	13 × 19 × 23	37	41 × 157
99	13 × 17 × 19	27	13 × 379	99	41 × 139	39	47 × 137
4223	41 × 103	79	13 × 383	5707	13 × 439	43	17 × 379
37	19 × 223	81	17 × 293	13	29 × 197	63	23 × 281
47	31 × 137	97	19 × 263	23	59 × 97	67	29 × 223
67	17 × 251	5017	29 × 173	29	17 × 337	87	13 × 499
4303	13 × 331	29	47 × 107	59	13 × 443	93	43 × 151
07	59 × 73	41	71 × 71	67	73 × 79	97	73 × 89
09	31 × 139	53	31 × 163	71	29 × 199	99	67 × 97
13	19 × 227	57	13 × 389	73	23 × 251	6509	23 × 283
21	29 × 149	63	61 × 83	77	53 × 109	11	17 × 383
31	61 × 71	69	37 × 137	5809	37 × 157	27	61 × 107
43	43 × 101	83	13 × 17 × 23	33	19 × 307	33	47 × 139
51	19 × 229	5111	19 × 269	37	13 × 449	39	13 × 503
69	17 × 257	23	47 × 109	91	43 × 137	41	31 × 211
79	29 × 151	29	23 × 223	93	71 × 83	57	79 × 83
81	13 × 337	41	53 × 97	99	17 × 347	83	29 × 227
87	41 × 107	43	37 × 139	5909	19 × 311	93	19 × 347
93	23 × 191	49	19 × 271	11	23 × 257	6613	17 × 389
99	53 × 83	61	13 × 397	17	61 × 97	17	13 × 509
4427	19 × 233	77	31 × 167	21	31 × 191	23	77 × 179
29	43 × 103	83	71 × 73	33	17 × 349	31	19 × 349
	23 × 193	91	29 × 179	41	13 × 457	41	29 × 229

Table of Numbers between 1 and 10,000 which do not Contain the Prime Factors — Continued.

No.	Factors.	No.	Factors.	No.	Factors.	No.	Factors.
6647	17 × 17 × 23	7363	37 × 199	8033	29 × 277	8759	19 × 461
49	61 × 109	67	53 × 139	47	13 × 619	73	31 × 283
67	59 × 113	73	73 × 101	51	83 × 97	77	67 × 131
83	41 × 163	79	47 × 157	77	41 × 197	91	59 × 149
97	37 × 181	87	83 × 89	83	59 × 137	97	19 × 463
6707	19 × 353	91	19 × 389	8119	23 × 353	8801	13 × 677
31	53 × 127	97	13 × 569	31	47 × 173	09	23 × 383
39	23 × 293	7409	31 × 239	37	79 × 103	43	37 × 239
49	17 × 397	21	41 × 181	43	17 × 479	51	53 × 167
51	43 × 157	23	13 × 571	49	29 × 281	57	17 × 521
57	29 × 233	29	17 × 19 × 23	53	31 × 263	73	19 × 467
67	67 × 101	39	43 × 173	59	41 × 199	79	13 × 683
73	13 × 521	53	29 × 257	77	13 × 17 × 37	81	83 × 107
99	13 × 523	63	17 × 439	89	19 × 431	91	17 × 523
6817	17 × 401	71	31 × 241	8201	59 × 139	8903	29 × 307
21	19 × 359	93	59 × 127	03	13 × 631	09	59 × 151
47	41 × 167	7501	13 × 577	07	29 × 283	17	37 × 241
51	13 × 17 × 31	19	73 × 103	13	43 × 191	27	79 × 113
59	19 × 19 × 19	31	17 × 443	27	19 × 433	47	23 × 389
77	13 × 23 × 23	43	19 × 397	49	73 × 113	57	13 × 13 × 53
87	71 × 97	71	67 × 113	51	37 × 223	59	17 × 17 × 31
89	83 × 83	97	71 × 107	57	23 × 359	77	47 × 191
93	61 × 113	7613	23 × 331	79	17 × 487	83	13 × 691
6901	67 × 103	19	19 × 401	99	43 × 193	89	89 × 101
13	31 × 223	27	29 × 263	8303	19 × 19 × 23	93	17 × 23 × 23
29	13 × 13 × 41	31	13 × 587	21	53 × 157	9017	71 × 127
31	29 × 239	33	17 × 449	33	13 × 641	19	29 × 311
43	53 × 131	57	13 × 19 × 31	39	31 × 269	47	83 × 109
53	17 × 409	61	47 × 163	41	19 × 439	61	13 × 17 × 41
73	19 × 367	63	79 × 97	47	17 × 491	71	47 × 193
89	29 × 241	97	43 × 179	57	61 × 137	73	43 × 211
7003	47 × 149	7709	13 × 593	59	13 × 643	77	29 × 313
09	43 × 163	29	59 × 131	81	17 × 17 × 29	83	31 × 293
31	79 × 89	39	71 × 109	83	83 × 101	89	61 × 149
33	13 × 541	47	61 × 127	99	37 × 227	9101	19 × 479
37	31 × 227	51	23 × 337	8401	31 × 271	13	13 × 701
61	23 × 307	69	17 × 457	11	13 × 647	31	23 × 397
67	37 × 191	71	19 × 409	13	47 × 179	39	13 × 19 × 37
81	73 × 97	81	31 × 251	17	19 × 443	43	41 × 223
87	19 × 373	83	43 × 181	41	23 × 367	67	89 × 103
93	41 × 173	87	13 × 599	53	79 × 107	69	53 × 173
97	47 × 151	7801	29 × 269	71	43 × 197	79	67 × 137
99	31 × 229	07	37 × 211	73	37 × 229	93	29 × 317
7111	13 × 547	11	73 × 107	79	61 × 139	97	17 × 541
23	17 × 419	13	13 × 601	83	17 × 499	9211	61 × 151
41	37 × 193	31	41 × 191	89	13 × 653	17	13 × 709
53	23 × 311	37	17 × 461	97	29 × 293	23	23 × 401
57	17 × 421	49	47 × 167	8507	47 × 181	53	19 × 487
63	13 × 19 × 29	59	29 × 271	09	67 × 127	59	47 × 197
69	67 × 107	71	17 × 463	31	19 × 449	63	59 × 157
71	71 × 101	91	13 × 607	49	83 × 103	69	13 × 23 × 31
81	43 × 167	97	53 × 149	51	17 × 503	71	73 × 127
99	23 × 313	7913	41 × 193	57	43 × 199	87	37 × 251
7201	19 × 379	21	89 × 89	67	13 × 659	99	17 × 547
23	31 × 233	39	17 × 467	79	23 × 373	9301	71 × 131
41	13 × 557	43	13 × 13 × 47	87	31 × 277	07	41 × 227
61	53 × 137	57	73 × 109	93	13 × 661	13	67 × 139
67	13 × 13 × 43	61	19 × 419	8611	79 × 109	29	19 × 491
77	19 × 383	67	31 × 257	21	37 × 233	47	13 × 719
79	29 × 251	69	13 × 613	33	89 × 97	53	47 × 199
89	37 × 197	79	79 × 101	39	53 × 163	67	17 × 19 × 29
91	23 × 317	81	23 × 347	51	41 × 211	79	83 × 113
7303	67 × 109	91	61 × 131	53	17 × 509	89	41 × 229
13	71 × 103	99	19 × 421	71	13 × 23 × 29	9407	23 × 409
19	13 × 563	8003	53 × 151	83	19 × 457	09	97 × 97
27	17 × 431	21	13 × 617	8711	31 × 281	51	13 × 727
39	41 × 179	23	71 × 113	17	23 × 379	69	17 × 557
61	17 × 433	27	23 × 349	49	13 × 673	81	19 × 499

Table of Numbers between 1 and 10,000 which do not Contain the Prime Factors — Continued.

No.	Factors.	No.	Factors.	No.	Factors.	No.	Factors.
9487	53 × 179	9599	29 × 331	9731	37 × 263	9893	13 × 761
9503	13 × 17 × 43	9607	13 × 739	61	43 × 227	99	19 × 521
09	37 × 257	17	59 × 163	63	13 × 751	9913	23 × 431
17	31 × 307	37	23 × 419	73	29 × 337	17	47 × 211
23	89 × 107	41	31 × 311	97	97 × 101	37	19 × 523
29	13 × 733	59	13 × 743	99	41 × 239	43	61 × 163
53	41 × 233	71	19 × 509	9809	17 × 577	53	37 × 263
57	19 × 503	73	17 × 569	27	31 × 317	59	23 × 433
63	73 × 131	83	23 × 421	41	13 × 757	71	13 × 13 × 59
71	17 × 563	9701	89 × 109	47	43 × 229	79	17 × 587
77	61 × 157	03	31 × 313	53	59 × 167	83	67 × 149
89	43 × 223	07	17 × 571	69	71 × 139	91	97 × 103
93	53 × 181	27	71 × 137	81	41 × 241	97	13 × 769

124. *The general rule for separating a number into its prime factors greater than one.* Divide successively, as many times as possible, by each of the numbers 2, 3, 5, 7 . . . which may be used as divisors, until a prime number is obtained in the quotient; this last quotient and all the numbers which have been used as divisors are the prime factors of the number. For example, to separate the number 540 into its prime factors, the calculation is arranged as shown, which gives the factors 2, 2, 3, 3, 3, 5; or $540 = 2 \times 2 \times 3 \times 3 \times 3 \times 5 = 2^2 \times 3^3 \times 5$.

540	2
270	2
135	3
45	3
15	3
5	5
1	

The table on page 32 permits of an easy separation into its factors of a number, 2,031,810 for instance, which contains only prime factors 2, 3, 5, 7, and 11, and other prime factors of which the product is not greater than 10,000. It is seen immediately that the number contains the factors 2 and 5 (90), then the factor 3 (95), and the factor 11. The last quotient, 6157, may be found in the table, which indicates that it does not contain any of the factors 2, 3, 5, 7, and 11, and gives its prime factors 47 and 131, which could not have been obtained without proving that the number did not contain any prime number less than 47. The prime factors are:

2,031,810	2 × 5
203,181	3
67,727	11
6,157	47 × 131

$$2{,}031{,}810 = 2 \times 3 \times 5 \times 11 \times 47 \times 131.$$

Remark 1. When a number, 8100, is the product of known numbers, 81 and 100, the process of separating it into its prime

factors may be shortened by finding the prime factors of 81 and of 100.

$$81 = 3^4, \quad 100 = 2^2 \times 5^2, \quad 8100 = 2^2 \times 3^4 \times 5^2.$$

REMARK 2. This last example shows that when a number, $8100 = 90^2$, is an exact power, the exponents of its prime factors are divisible by the degree of the power.

125. The greatest common divisor of several numbers, 240, 180, 72, is equal to the product of the prime factors common to these numbers, each of these factors being raised to the power corresponding to the smallest exponent which it bears as a factor of the numbers. Thus, having given:

$$240 = 2^4 \times 3 \times 5, \quad 180 = 2^2 \times 3^2 \times 5, \quad 72 = 2^3 \times 3^2,$$

the greatest common divisor of these numbers is

$$2^2 \times 3 = 12.$$

This gives another method for determining the greatest common divisor of several numbers (102 and 104).

126. The least common multiple of several numbers is equal to the product of their prime factors, each of the factors being raised to the power corresponding to the largest exponent which it bears as a factor of the numbers. Thus the least common multiple of the numbers in the above example, 240, 180, and 72, is

$$2^4 \times 3^2 \times 5.$$

This being another method of finding the least common multiple of several numbers (114 and 119).

127. *To find all the divisors of a number*, 360, separate the number into its prime factors (124), writing them in a vertical column; multiply the first factor 2 by the second 2, the first two factors and their product 4 by the third, omitting the multiplications which would give the products already obtained; multiply in the same manner the first three factors and the products obtained by the fourth factor, and so on until the last factor has been used as multiplier; all the unequal prime factors of the number, and the products that have been obtained, are the required divisors. The operation is carried on as follows; the

number 1 being always a divisor, is written at the top of the table:

	1
360	2
180	2, 4
90	2, 8
45	3, 6, 12, 24
15	3, 9, 18, 36, 72
5	5, 10, 20, 40, 15, 30, 60, 120, 45, 90, 180, 360.

1	3	5
2	9	10
4	6	20
8	12	40
	24	15
	18	45
	36	30
	72	60
		120
		90
		180
		360

The prime factors of a number being known, given for example $360 = 2^3 \times 3^2 \times 5$, it is simpler, in obtaining all its divisors, to write 1 and the successive powers 2, 4, 8, of 2 contained in the number in the first column; in the second the products of the numbers in the first with the powers 3 and 9 of 3 contained in 360, and in the third column the products of the numbers in the first two columns with the first power 5 of 5 contained in 360.

The numbers forming this table, when completed, are all the divisors of 360.

128. *The number of divisors of a number* is equal to the product of the sums obtained by increasing the exponent of each prime factor by 1 (**124**). Thus, given $360 = 2^3 \times 3^2 \times 5$, the number of divisors counting 1 and 360 is

$$(3 + 1)\ (2 + 1)\ (1 + 1) = 24.$$

129. *To find all the common divisors of several numbers,* find the greatest common divisor of the numbers, then all the divisors of this greatest common divisor (**125** and **127**).

BOOK III

FRACTIONS AND DECIMALS

FRACTIONS

130. A *fraction* or a *fractional number* is one or several parts of a unit which has been divided into equal parts. Thus, a unit having been divided into 9 equal parts, the number formed with 5 of these parts is a fraction.

131. The *denominator* of a fraction is the number which indicates into how many parts the unit has been divided.

The *numerator* is the number which indicates how many of these equal parts are contained in the fraction. Thus, in the preceding example, 9 is the denominator and 5 the numerator. The numerator and denominator are the two *terms* of the fraction.

Conceive that a fraction may contain all the parts of one or several units, and even all the parts of one or several units plus the parts of another unit these units; being the same and being all divided into the same number of equal parts.

When a fraction does not contain all the parts of one, that is, when its numerator is less than its denominator, it is less than unity. If it contains all the parts of one, its terms are equal, and it is equal to unity. Finally, if the numerator is greater than the denominator, the fraction is larger than unity.

According as a fraction is smaller or larger than unity, it is called a *proper* or an *improper fraction* (**130**).

132. To pronounce a fraction, pronounce the numerator, then the denominator, adding the termination *th*. Thus the fraction in (**130**) is pronounced five ninths. There are exceptions for the denominators 2, 3, and 4; thus we say one half, one third, one quarter, or fourth.

133. In writing a fraction, write the numerator above the denominator and separate them by a line. Thus five ninths is written $\frac{5}{9}$.

134. A fraction represents the quotient of the division of its

numerator by its denominator (51). Thus $\frac{5}{9}$ is equal to 5 divided by 9.

Any whole number, 7, may be considered as a fraction, $\frac{7}{1}$, with the number 7 for a numerator and unity 1 for a denominator.

135. *To reduce an improper fraction to a whole number and a proper fraction, or to a mixed number,* divide the numerator by the denominator, and add to the quotient a fraction, having the remainder for a numerator and the denominator of the improper fraction for a denominator. Thus:

$$\frac{63}{9}=7, \text{ and } \frac{37}{5}=7+\frac{2}{5}.$$

136. *To reduce a whole number to an equivalent fraction having a given denominator* 9; for the numerator of the fraction take the product 63 of its denominator 9 with the whole number 7. Thus:

$$7=\frac{7\times 9}{9}=\frac{63}{9}.$$

137. *In adding the terms of several equal fractions,* the resulting fraction is equal to any one of those fractions:

$$\frac{3}{7}=\frac{3}{7}=\frac{3}{7}=\frac{3}{7}=\frac{12}{28}, \quad \frac{4}{6}=\frac{10}{15}=\frac{14}{21}=\frac{4+10+14}{6+15+21}=\frac{28}{42}.$$

In subtracting the terms of two equal fractions which have not the same terms, a resulting fraction is obtained which is equal to both of the given fractions:

$$\frac{28}{42}=\frac{10}{15}=\frac{28-10}{42-15}=\frac{18}{27}.$$

138. *When the terms of any two unequal fractions are added,* generally the value of the resulting fraction lies between that of the two fractions added:

$$\frac{4}{7}<\frac{4+9}{7+5}<\frac{9}{5}, \quad \frac{4}{7}<\frac{4+8+9}{7+5+5}<\frac{9}{5}.$$

139. *When the same quantity is added to both terms of a fraction,* the fraction is increased or diminished according as the fraction

is proper or improper (131). In each case unity is the *limit* which it approaches as the terms become larger, but which can never be attained because the terms can never become equal:

$$\frac{5}{9} < \frac{5+3}{9+3}, \quad \text{and} \quad \frac{11}{4} > \frac{11+2}{4+2}.$$

On the contrary, *if the same quantity is subtracted from both terms of a fraction,* the fraction is diminished or increased according as the fraction is proper or improper. In each case the fraction departs farther and farther from unity:

$$\frac{8}{12} > \frac{8-3}{12-3}, \quad \text{and} \quad \frac{13}{6} < \frac{13-2}{6-2}.$$

When the fraction is equal to unity its value is not altered by adding to, or subtracting the same quantity from each term.

140. *To multiply a fraction by a whole number,* multiply the numerator, or, if it is possible without a remainder, divide the denominator by the number. Thus:

$$\frac{3}{7} \times 4 = \frac{3 \times 4}{7} = \frac{12}{7}, \quad \text{and} \quad \frac{3}{8} \times 4 = \frac{3}{8 \div 4} = \frac{3}{2}.$$

141. *To divide a fraction by a whole number,* multiply the denominator, or, if it is possible without a remainder, divide the numerator by the number. Thus:

$$\frac{3}{7} : 4 = \frac{3}{7 \times 4} = \frac{3}{28}, \quad \text{and} \quad \frac{8}{7} : 4 = \frac{8 \div 4}{7} = \frac{2}{7}.$$

142. *It does not alter the value of a fraction to multiply or divide both its terms by the same number* (73):

$$\frac{3}{4} = \frac{3 \times 2}{4 \times 2} = \frac{6}{8}, \quad \text{and} \quad \frac{8}{12} = \frac{8 \div 4}{12 \div 4} = \frac{2}{3}.$$

IRREDUCIBLE FRACTIONS

143. *To simplify or reduce a fraction to a simpler form,* is to diminish the value of its terms without changing value as a fraction.

144. A fraction is *irreducible,* or *reduced to its simplest form,*

when it cannot be made simpler. Such are the fractions $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{11}$ (146).

145. The terms of an irreducible fraction, $\frac{7}{8}$, are prime to each other (80).

146. *To reduce a fraction,* $\frac{30}{45}$, *to a simpler form,* divide the two terms by a common divisor (142):

$$\frac{30}{45} = \frac{30 \div 3}{45 \div 3} = \frac{10}{15}.$$

To reduce a fraction, $\frac{30}{45}$, *to its simplest form,* divide its terms by their greatest common divisor, 15 (102):

$$\frac{30}{45} = \frac{30 \div 15}{45 \div 15} = \frac{2}{3};$$

or cancel all the prime factors common to the two terms (125):

$$\frac{30}{45} = \frac{2 \times 3 \times 5}{3 \times 3 \times 5} = \frac{2}{3}.$$

Applying what was said in (102), not only the greatest common divisor, 12, of the terms of the fraction, $\frac{360}{876}$, is obtained, but also the quotient, 30 and 73, of the two terms divided by 12, and it may be written

$$\frac{360}{876} = \frac{30}{73}.$$

In practice, to reduce a fraction, $\frac{168}{252}$, to a simpler form, its

$$\frac{168}{252} = \frac{84}{126} = \frac{42}{63} = \frac{14}{21} = \frac{2}{3}$$

terms being even, divide by 2; for the same reason divide the terms of the resulting fraction, $\frac{84}{126}$, by 2; it is now seen that the terms of the resulting fraction, $\frac{42}{63}$, are divisible by 3 (95), and those of the fraction $\frac{14}{21}$ by 7. Thus a fraction may often be reduced to its simpler form by dividing out its common factors.

147. *The least common multiple,* 36, *of the denominators of several irreducible fractions,* $\frac{5}{6}, \frac{4}{9}, \frac{7}{12}$, is the least common denominator to which the fractions may be reduced (**151**).

148. *The greatest common divisor of several irreducible fractions,* $\frac{6}{5}, \frac{9}{4}, \frac{12}{7}$, is the fraction, $\frac{3}{140}$, whose numerator 3 is the greatest common divisor of the numerators (**104**), and whose denominator is the least common multiple 140 of their denominators (**119**).

149. *The least common multiple of several irreducible fractions,* $\frac{5}{6}, \frac{4}{9}, \frac{7}{12}$, is the irreducible fraction $\frac{140}{3}$, whose numerator is the least common multiple 140 of the numerators, and whose denominator is the greatest common divisor 3 of the denominators.

REDUCTION OF FRACTIONS TO THE SAME DENOMINATOR

150. *To reduce fractions to the same denominator* is to find fractions equal to the given fractions, with denominators equal to each other (**131**).

151. To reduce two fractions to the same denominator, multiply the terms of each fraction by the denominator of the other. And, in general, to *reduce several fractions to the same denominator,* multiply each numerator by the product of the denominators of the others, and as common denominator use the product of all the denominators:

$$\frac{2}{3} = \frac{2 \times 6}{3 \times 6} = \frac{12}{18} \qquad \frac{1}{2} = \frac{3 \times 5 \times 6}{2 \times 3 \times 5 \times 6} = \frac{90}{180}$$

$$\frac{5}{6} = \frac{5 \times 3}{6 \times 3} = \frac{15}{18} \qquad \frac{2}{3} = \frac{2 \times 2 \times 5 \times 6}{180} = \frac{120}{180}$$

$$\frac{4}{5} = \frac{4 \times 2 \times 3 \times 6}{180} = \frac{144}{180}$$

$$\frac{5}{6} = \frac{5 \times 2 \times 3 \times 5}{180} = \frac{150}{180}$$

When it is seen that a number is divisible by all of the denominators of the given fractions, that is, is common multiple of the denominators (**126**), it is taken as common denominator, and the numerator of each fraction is multiplied by the quotient obtained in dividing this common denominator by the denomi-

nator of the fraction. Thus, in the preceding examples, it is seen immediately that 6 and 30 may be taken as common denominators, and then we have:

$$\frac{2}{3} = \frac{2 \times 2}{6} = \frac{4}{6} \qquad \frac{5}{6} = \frac{5}{6} = \frac{5}{6}$$

$$\frac{1}{2} = \frac{1 \times 15}{30} = \frac{15}{30}$$
$$\frac{2}{3} = \frac{2 \times 10}{30} = \frac{20}{30}$$
$$\frac{4}{5} = \frac{4 \times 6}{30} = \frac{24}{30}$$
$$\frac{5}{6} = \frac{5 \times 5}{30} = \frac{25}{30}$$

It is always possible to find the least common multiple of the denominators (126), and use it as common denominator as was done above.

The number, 2×3^2, by which the numerator of the fraction $\frac{7}{20}$ must be multiplied, for example, is obtained simply by canceling in the common denominator $2^3 \times 3^2 \times 5$, the factors of the denominator $2^2 \times 5$ of the fraction $\frac{7}{2^2 \times 5}$.

$$\frac{7}{20} = \frac{7}{2^2 \times 5} = \frac{7 \times 2 \times 3^2}{2^3 \times 3^2 \times 5} = \frac{126}{360}$$
$$\frac{11}{24} = \frac{11}{2^3 \times 3} = \frac{11 \times 3 \times 5}{2^3 \times 3^2 \times 5} = \frac{165}{360}$$
$$\frac{23}{36} = \frac{23}{2^2 \times 3^2} = \frac{23 \times 2 \times 5}{2^3 \times 3^2 \times 5} = \frac{230}{360}$$
$$\frac{17}{45} = \frac{17}{3^2 \times 5} = \frac{17 \times 2^3}{2^3 \times 3^2 \times 5} = \frac{136}{360}$$

In this example the general rule would have given 777,600 for the common denominator.

When the denominators of the given fractions are prime to each other (80), their least common multiple is equal to their product, and then to reduce the fractions to the same denominator, follow the general rule without any possible simplification (147).

ADDITION OF FRACTIONS

152. *To add fractions,* reduce them, if necessary, to the same common denominator (151); and add the numerators which result; then the result of the operation is a fraction whose numerator is the sum of the reduced numerators and whose denominator is the common denominator. Example:

$$\begin{array}{r} \frac{5}{12} \\ +\frac{7}{12} \\ +\frac{17}{12} \\ \hline \frac{5+7+17}{12}=\frac{29}{12} \text{ sum.} \end{array} \qquad \begin{array}{r} \frac{2}{3}=\frac{20}{30} \\ +\frac{7}{5}=\frac{42}{30} \\ +\frac{5}{6}=\frac{25}{30} \\ \hline \frac{87}{30} \text{ sum.} \end{array}$$

153. *To add a whole number and a fraction,* reduce the whole number to an equivalent fraction, having for a denominator the denominator of the fraction (136), and proceed as in the preceding case. This amounts to adding to the numerator of the given fraction the product of the denominator and the whole number:

$$\frac{4}{5}+7=\frac{4+5\times 7}{5}=\frac{39}{5}.$$

154. *To add any number of fractions and whole numbers together,* add the fractions and whole numbers separately, and then operate with the sums as in the preceding case:

$$\frac{1}{4}+5+3+\frac{2}{3}=(5+3)+\left(\frac{1}{4}+\frac{2}{3}\right)=8+\frac{11}{12}=\frac{107}{12}.$$

SUBTRACTION OF FRACTIONS

155. *To obtain the difference between two fractions,* reduce them, if necessary, to the same denominator (151); subtract the numerators of the reduced fractions, and the required result will have this difference for a numerator and the common denominator for a denominator:

$$\frac{7}{9}-\frac{2}{9}=\frac{7-2}{9}=\frac{5}{9}, \qquad \frac{3}{4}-\frac{1}{7}=\frac{21}{28}-\frac{4}{28}=\frac{17}{28}.$$

156. *To subtract a fraction from a whole number, or conversely,* reduce the whole number to an equivalent fraction, having for a denominator that of the fraction (136), and proceed as in the preceding case. Thus:

$$8-\frac{4}{7}=\frac{56}{7}-\frac{4}{7}=\frac{56-4}{7}=\frac{52}{7}, \qquad \frac{15}{4}-3=\frac{15}{4}-\frac{12}{4}=\frac{3}{4}.$$

157. *To subtract a whole number plus a fraction from a whole number plus a fraction,* $4 + \frac{1}{3}$ from $7 + \frac{3}{5}$ for example, reduce each of the quantities to an equivalent fraction (153), then take the difference of the fractions obtained (155 and 156). However, it is simpler to subtract, first the fractions

$$\frac{3}{5} - \frac{1}{3} = \frac{9}{15} - \frac{5}{15} = \frac{4}{15},$$

then the whole numbers, $7 - 4 = 3$; which gives $3 + \frac{4}{15}$.

When the fraction from which the subtracting is to be done is the lesser, it is increased by a unit, which means that the numerator is to be increased by a number equal to the denominator, and to compensate this the whole number to be subtracted is reduced by one unit. As a special case, the fraction may be zero. Examples:

$$\begin{array}{lcl} 7 + \frac{3}{5} & \text{or} & 7 + \frac{9}{15} \\ 4 + \frac{1}{3} & & 4 + \frac{5}{15} \\ \hline \text{Difference } 3 + \frac{4}{15} & & \end{array} \qquad \begin{array}{lclcl} 7 + \frac{1}{3} & \text{or} & 7 + \frac{5}{15} & \text{or} & 6 + \frac{20}{15} \\ 4 + \frac{3}{4} & & 4 + \frac{9}{15} & & 4 + \frac{9}{15} \\ \hline & & \text{Difference } 2 + \frac{11}{15} & & \end{array} \qquad \begin{array}{lcl} 8 & \text{or} & 7 + \frac{5}{5} \\ 3 + \frac{2}{5} & & 3 + \frac{2}{5} \\ \hline \text{Difference } 4 + \frac{3}{5} & & \end{array}$$

To subtract several whole numbers plus fractions from several whole numbers plus fractions, reduce all the plus quantities to one whole number and fraction (154), the same with the negative quantities, and proceed as in the preceding case.

MULTIPLICATION OF FRACTIONS

158. *To multiply a quantity by a fraction,* multiply it by the numerator of the fraction and divide the product by the denominator.

Remark. In multiplying a quantity by a fraction, the product is equal to, greater or less than the multiplicand, according as the fraction multiplier is equal to, greater, or less than unity.

159. *To multiply a whole number by a fraction,* is the same as a fraction by a whole number (140). Thus:

$$9 \times \frac{3}{4} = \frac{9 \times 3}{4} = \frac{27}{4}, \qquad 3 \times \frac{7}{9} = \frac{7}{9 \div 3} = \frac{7}{3}.$$

160. *To multiply one fraction by another,* multiply the numerators together for the numerator, and the denominators for the denominator:

$$\frac{3}{4} \times \frac{7}{5} = \frac{3 \times 7}{4 \times 5} = \frac{21}{20}.$$

161. *The product of any number of whole numbers and fractions* is a fraction whose numerator is the product of the whole numbers and the numerators of the given fractions, and whose denominator is equal to the product of their denominators:

$$5 \times \frac{3}{4} \times 2 \times \frac{2}{7} = \frac{5 \times 3 \times 2 \times 2}{4 \times 7} = \frac{60}{28}.$$

In practice, before going through the calculations, write out the multiplication and cancel the common factors of the two terms (146). This shortens the operation, and gives a product reduced to its lowest terms providing all common factors are canceled. In the preceding example, canceling 2×2 in the numerator and 4 in the denominator, we have $\frac{5 \times 3}{7} = \frac{15}{7}$ for a result. In the example

$$\frac{4}{9} \times \frac{5}{7} \times \frac{42}{35} \times \frac{11}{8} = \frac{\cancel{4} \times \cancel{5} \times \overset{\cancel{6}}{\cancel{42}} \times 11}{\underset{3}{\cancel{9}} \times \cancel{7} \times \underset{7}{\cancel{35}} \times \underset{\cancel{2}}{\cancel{8}}} = \frac{11}{21},$$

cancel 4 in the numerator and replace 8 by 2 in the denominator (confusion is avoided by drawing a line through the canceled factors); cancel 5 in the numerator and replace 35 by 7 in the denominator; then a 7 in the denominator, replacing the 42 by 6 in the numerator; finally 6 in the numerator, by canceling 2 and replacing 9 by 3 in the denominator. The result is

$$\frac{11}{3 \times 7} = \frac{11}{21}.$$

162. *To find a certain fraction of a fraction of any quantity,* multiply the quantity by the product of the fractions (161). Thus:

$\frac{2}{3}$ of 5 are $5 \times \frac{2}{3} = \frac{10}{3}$.

$\frac{1}{4}$ of $\frac{2}{3}$ of 5 are $5 \times \frac{1}{4} \times \frac{2}{3} = \frac{10}{12}$.

$\frac{3}{7}$ of $\frac{1}{4}$ of $\frac{2}{3}$ of 5 are $5 \times \frac{2}{3} \times \frac{1}{4} \times \frac{3}{7} = \frac{30}{84}$.

REMARK. To multiply a fraction which has unity 1 for a numerator by a quantity is to divide the quantity by the denominator of the fraction. Thus:

$$\frac{1}{6} \text{ of } 15 = \frac{15}{6} \ (64).$$

163. Articles (33, 34, 41, 42, 43) are equally true with whole numbers and fractions.

DIVISION OF FRACTIONS

164. *To divide a quantity by a fraction*, multiply by the divisor fraction inverted (159 and 160).

$$7 \div \frac{3}{4} = 7 \times \frac{4}{3} = \frac{28}{3}, \quad \frac{4}{7} \div \frac{2}{5} = \frac{4}{7} \times \frac{5}{2} = \frac{20}{14}.$$

REMARK. The quotient is equal to, less, or greater than the dividend according as the divisor fraction is equal to, greater, or less than unity.

165. The articles (56, 59, 60, 61, 62, 63, 67, 68, 69, 70, 71, 72, 73), and some which are immediate consequences of them, being founded upon principles applicable to fractions as well as whole numbers, apply to both sorts of numbers.

166. *To divide whole numbers plus fractions by whole numbers plus fractions*, reduce the dividend to one fraction (154 and 157), and the divisor to another, and divide, proceeding as in the preceding case (164). Thus:

$$\left(3 + \frac{2}{5}\right) \div \left(2 + \frac{1}{4}\right) = \frac{17}{5} \div \frac{9}{4} = \frac{17}{5} \times \frac{4}{9} = \frac{68}{45}.$$

DECIMAL NUMBERS

167. *A decimal fraction* is a fraction whose denominator is a power of 10 (85 and 86). Such are the fractions $\frac{3}{10}$ and $\frac{278}{100}$.

168. A decimal number is a number composed of a whole number, which may be zero, and one or several decimal fractions, whose numerators are less than the base, 10, and whose denominators are powers of that base. Such are:

$$\left(37 + \frac{5}{10} + \frac{8}{1000}\right), \quad \text{and} \quad \left(\frac{3}{10} + \frac{5}{100} + \frac{7}{1000}\right).$$

169. *Numeration of decimals.* To simplify the writing of a decimal number, the several figures composing the number are written on a horizontal line and separated into two parts by a period; the part at the left expresses whole units; the first figure at the right of the period expresses tenths, or decimals of the first order; the second, hundredths, or decimals of the second order, and so on; thus in a decimal, as in a whole number, any figure placed at the left of another figure expresses units ten times as great as those at its right (7). According to this method, the number $\left(37 + \frac{5}{10} + \frac{8}{1000}\right)$ is written 37.508, and $\left(\frac{3}{10} + \frac{5}{100} + \frac{7}{1000}\right)$ is written 0.357.

To pronounce a decimal number written in figures, pronounce successively the part at the left and right of the period, adding to each the units expressed by the first figure to the right of each part. Thus the number 37.508 is pronounced thirty-seven units five hundred eight thousandths, and 0.357 is pronounced no units, three hundred fifty-seven thousandths. When the decimal part contains more than 5 or 6 figures, in pronouncing, it is convenient to divide it into periods of 3 figures each, commencing at the decimal point; then, commencing at the left, pronounce successively each period of figures, giving each the name of the units expressed by the figure at the right.

Thus, the number

$$37.32504645769$$

is pronounced: 37 units, 325 thousandths, 46 millionths, 457 billionths, 69 hundred billionths, or 690 trillionths, adding a cipher in the last period.

170. Each figure placed at the right of the *decimal point,* or period, is a decimal, or decimal figure of the given number. Its form indicates its absolute value, and its position its relative value (8).

171. It does not alter the value of a decimal to suppress or add ciphers at the right:

$$32.45 = 32.4500, \quad \text{and} \quad 3.12500 = 3.125.$$

172. *To reduce a decimal to the form of a decimal fraction* (167), take the given number for numerator, omitting the decimal

point, and for denominator 1 followed by as many ciphers as there are decimals in the given number:

$$27.347 = \frac{27347}{1000}.$$

173. Conversely, *to reduce a decimal fraction to the form of a decimal number*, write the numerator and separate on the right as many decimal figures as there are ciphers in the denominator. In the case where there are less figures in the numerator than ciphers in the denominator, write ciphers at the left of the figures:

$$\frac{2348}{1000} = 2.348, \quad \text{and} \quad \frac{37}{1000} = 0.037.$$

174. The value of a given quantity is near the value of another quantity by less than a third quantity, when the difference of the first two is less than the third quantity. Thus 24.37 is less than a hundredth, .01, smaller than 24.376, because 24.376 − 24.37 = 0.006 is less than 0.01.

175. *The nearest value of a decimal, at least of a decimal of a certain order*, is the result which is obtained by suppressing in the given number all the decimals written at the right of the figure which expresses the units of the given order. Thus the value of the number 7.46537 to the thousandths place is 7.465.

176. *In getting the nearest possible value of a decimal, retaining a certain number of decimal figures*, there are three cases: *First*, if the first figure which follows the last which is to be retained is less than 5, suppress the 5 with the figures which follow; *second*, if it is larger than 5, or if it is 5 followed by other significative figures, suppress it with those which follow and increase the last figure by 1; *third*, finally, if it is 5 and not followed by other figures, suppress it, and add either one or nothing to the last figure. In any case the error can not be greater than a half a unit of the order of the last figure. The value of 4.8365 to the first decimal place is 4.8; to the second decimal place, 4.84; to the third place, 4.836 or 4.837.

177. To multiply or divide a decimal by one, followed by several ciphers, move the decimal point to the right or left as many places as there are ciphers after the one:

$$3.127 \times 100 = 312.7; \quad 25.83 \div 1000 = 0.02583.$$

REMARK. The same rule applies where the dividend is a whole number, $453 \div 100 = 4.53$.

THE FOUR FUNDAMENTAL OPERATIONS ON DECIMAL NUMBERS

178. *To add decimals,* proceed in the same manner as in the addition of whole numbers (**25**), placing the point in the result on the same vertical line with the points in the numbers. (This rule applies equally well where some of the numbers are whole numbers.)

$$\begin{array}{r} 37.425 \\ 8.72 \\ 436 \\ 0.54 \\ 68.034 \\ \hline 550.719 \end{array}$$

179. *To find the difference of two decimals,* or of a whole number and a decimal, operate as with whole numbers (**29**), placing the decimal point in the result on the same vertical line with the points in the numbers. (When there are more decimals in one of the numbers than in the other, write or imagine to be written at the right of the number ciphers sufficient to make the number of decimal figures the same in each number.)

$$\begin{array}{r} 68.740 \\ 53.837 \\ \hline 14.903 \end{array} \qquad \begin{array}{r} 837 \\ 73.534 \\ \hline 763.466 \end{array}$$

180. *To multiply several decimal numbers or decimals and whole numbers together,* disregard the decimal points and operate as with whole numbers (**47**), pointing off at the right of the result as many decimal figures as there are decimals in all the factors:

$$\begin{array}{r} 3.27 \\ 4.005 \\ \hline 1635 \\ 130800 \\ \hline 13.09635 \end{array} \qquad \begin{array}{r} 0.2 \\ 0.3 \\ \hline 0.06 \end{array} \qquad 8.75 \times 4 \times 6.3 = 220.500 = 220.5.$$

REMARK. Since all decimals may be reduced to the form of decimal fractions (**172**), all rules and principles which apply to fractions apply also to decimals (**163**). Thus, for example, the value of a product of several decimals is not changed by changing the order of its factors.

181. *To divide a decimal by a whole number,* write the figures

the same as in the operation on whole numbers (64). Then divide the whole number part of the dividend by the divisor, which gives the whole part of the quotient; reduce the remainder to tenths, adding the tenths in the dividend by placing the tenths figure at the right of the remainder; divide this number by the divisor, which gives the first decimal (tenths) of the quotient; reduce this remainder to hundredths and proceed as before until a remainder zero is obtained or a figure expressing units of an indicated order. If the remainder is less than one-half the divisor, it is neglected; if it is greater, the last figure of the quotient is increased by 1; and if it is equal to half the divisor, the last figure may be increased by one or left as it is (176).

This rule still holds where the dividend is a whole number and it is desired to have decimals in the quotient:

$$\begin{array}{l|l} 35.427 & 12 \\ \cline{2-2} 11\,4 & 2.95225 \\ \;\;62 & \\ 27 & \\ 30 & \\ 60 & \\ 0 & \end{array} \qquad\qquad \begin{array}{r|l} 135 & 12 \\ \cline{2-2} 15 & 11.25 \\ 30 & \\ 60 & \\ 0 & \end{array}$$

If in the first example a quotient to the thousandths place had been desired, the operation would have been completed when 2.952 was obtained in the quotient. The last remainder 3 being smaller than half the divisor 12, 2.952 is the nearest true value to the thousandths place.

182. *To divide a whole number or a decimal by a decimal*, take the given divisor for a divisor, removing the decimal point; and the given dividend multiplied by 1 followed by as many ciphers as there are decimals in the divisor (177) for a dividend, and proceed as in the division of whole numbers (181). Thus to divide 3.3756 by 0.45, operate in the following manner:

$$\begin{array}{r|l} 337.56 & 45 \\ \cline{2-2} 22\,5 & 7.501 \\ 060 & \\ 15 & \end{array}$$

REMARK 1. Article (165) applies to decimals.

REMARK 2. The proof of the operations with decimals is the same as with whole numbers (26, 30, 48, 65). In the proofs by the rule of 9 and 11 neglect the decimal point (97, 98, 99, 100, 101).

183. Two numbers are reciprocals of each other when their product is equal to unity 1. Thus the reciprocal of the number 7 is $\frac{1}{7}$.

THE REDUCTION OF FRACTIONS TO DECIMALS

184. A decimal number is *periodic,* when one or several decimal figures reappear in the same order indefinitely: such is the number 2.37474 . . . The number 74, formed by the figures 7 and 4, reappears in the same order indefinitely, and is the *period* of the decimal.

185. A decimal number is *simple periodic* or *mixed periodic,* according as it commences or not with the tenths figure. Thus the number 3.4545 . . . is simple periodic, and 2.37474 . . . is mixed periodic.

186. *A constant quantity is the limit of a variable quantity,* when the difference of the two quantities may become infinitely small without reaching zero. The unit 1 is the limit of the decimal 0.9999 . . . Because by taking an infinite number of 9's the difference between the resulting number and 1 will be infinitely small, but never can equal zero (**38**, **139**).

REMARK. A variable quantity can have but one limit.

187. *To reduce a fraction to decimals,* is to put the fraction in the form of a decimal.

188. *To reduce a fraction to decimals,* divide its numerator by its denominator, operating as in the division of a decimal by a whole number (**182**):

$$\frac{27}{8} = 3.375.$$

189. When the denominator of an irreducible fraction (**144**) contains only the factor 2 and 5, the reduction of the fraction to decimals will give an exact quotient, in which the number of decimal figures is equal to or greater than the exponents of the factors 2 and 5 in the denominator.

$$\frac{127}{40} = \frac{127}{2^3 \times 5} = 3{,}175.$$

190. Any irreducible fraction of which the denominator contains one or several prime factors other than 2 and 5, cannot be

reduced exactly to decimals, and the division of its numerator by its denominator gives a periodic quotient (**184**):

$$\frac{127}{30} = \frac{127}{2 \times 3 \times 5} = 4.23333\ldots$$

191. Any fraction, $\frac{127}{30}$, is the limit (**186**) of the periodic quotient 4.2333, ..., obtained in reducing the fraction to decimals (**187**).

192. When the denominator of an irreducible fraction, $\frac{8}{3}$, does not contain the factors, 2 nor 5, the reductions of the fraction to decimals gives a simple periodic quotient (**185**):

$$\frac{8}{3} = 2.666\ldots$$

193. When the denominator of an irreducible fraction contains one or several of the factors 2 and 5, together with other prime factors, the reduction of the fraction to decimals gives a mixed periodic quotient in which the number of non-periodic decimal figures is equal to or greater than the exponents of the factors 2 and 5 in the denominator. Thus the irreducible fraction $\frac{95}{84} = \frac{95}{2^2 \times 3 \times 7}$ gives two non-periodic decimals.

194. *The number of figures contained in the period* can not exceed the product of the prime factors of the denominator other than 2 and 5, less 1. Thus in the preceding example it cannot exceed $3 \times 7 - 1 = 20$.

195. *The generant of any simple periodic decimal* 0.2727 *less than unity* and whose period is not 9, is that fraction $\frac{27}{99}$ which has the period for a numerator and as many 9's as there are figures in the period for a denominator. Thus:

$$\frac{27}{99} = 0.2727\ldots \text{ (197, Remark).}$$

196. Any simple periodic decimal 4.2727 . . . greater than unity and whose period is not 9, results from the reduction of a fraction to decimals. The same holds true for any mixed periodic decimal 4.342727 . . . whose period is not 9.

To obtain the generant fraction of a simple periodic decimal

4.2727 *greater than unity,* take the difference between the whole part followed by the period and the whole part for the numerator, and as many 9's as there are figures in the period for the denominator. Thus:

$$\frac{427-4}{99}=\frac{423}{99}.$$

To obtain the generant fraction of a mixed decimal 15.273434 . . .. for the numerator take the whole number followed by the non-periodic figures and the first period less the whole number followed by the non-periodic part, and for a denominator as many 9's as there are figures in the period followed by as many ciphers as there are figures in the non-periodic part of the decimal. Thus,

$$\frac{152{,}734-1527}{9900}=\frac{151{,}207}{9900}.$$

Remark. When the period is the figure 9, the decimal has no generant; the limit is obtained by suppressing the periods and increasing the last figure to the right by one. Thus:

$$0.999\ldots=\frac{9}{9}=1;\qquad 4.999\ldots=\frac{49-4}{9}=5;$$

$$4.34999\ldots=\frac{4349-434}{900}=4.35.$$

OPERATIONS ON COMBINED FRACTIONS AND DECIMALS, COMPLEX DECIMALS

197. *To add complex decimals,* reduce each decimal to the form of a fraction (172), and proceed as in the addition of fractions (152).

Remark. When given decimals have a limited number of figures, and the fractions are exactly reducible to decimals (188), operate as in the addition of decimals.

The same methods hold true for the subtraction, multiplication and division of complex decimals.

NUMERICAL APPROXIMATIONS. SHORT METHODS OF OPERATING

198. When a quantity is replaced by an approximate value, the difference between the exact value and the approximate value is called the *absolute error,* and the quotient obtained by dividing the absolute error by the exact value is called the *rela-*

tive error. Thus, the distance between two points being 40 meters, if we suppose it to be 42 or 38 the absolute error is two meters, $42^m - 40^m = 2^m$, $40^m - 38^m = 2^m$, and the relative error $\frac{2}{40} = \frac{1}{20}$. The relative error is the error in each unit of the exact number.

199. When a whole number 314,159 is replaced by 314,100, or a decimal 3.14159 by 3.141, or 0.0314159 by 0.03141, that is, when figures at the right are replaced by ciphers if the number is whole or a decimal, the absolute error is respectively 59, 0.00059, 0.0000059, numbers formed by the suppressed figures, and the relative error is

$$\frac{59}{314{,}159} = \frac{0.00059}{3.14159} = \frac{0.0000059}{0.0314159}.$$

From the foregoing examples it is seen that for numbers, which differ simply in position of the decimal point, the relative error depends only upon the suppressed figures and not upon the position of the point; but the absolute error depends both upon the figures suppressed and the position of the point.

The absolute error is respectively less than 100, 0.001, 0.000001, *that is, than a unit of the order of the last figure retained,* and *the relative error is less than* $\frac{100}{314{,}159} = \frac{0.001}{3.14159} = \frac{0.000001}{0.0314159}$, and evidently less than $\frac{100}{300{,}000} = \frac{1}{3000}$ and less than $\frac{1}{1000} = 0.001$, that is, than a decimal unit of an order, which is one less than the number of figures retained, not counting the ciphers at the left of the first significative figure. It follows that in order to obtain an approximate value of a whole or decimal number, which is less than the number, and has a relative error less than 0.1, 0.01, 0.001, 0.00001 . . . , retain at the left 2, 3, 4, 5 . . . figures commencing with the first significative figure. Thus the approximate value of the numbers 314,159, 31415.9, 3.14159, 0.0314159 with a relative error less than 0.001 is respectively, 314,100, 31,410, 3.141, and 0.03141.

REMARK 1. When the first significative figure at the left of the number is greater than 1, the relative error as found by the preceding rule is less than half a decimal unit of an order, which is one less than the number of figures retained. In replacing

the number 0.0314159 by 0.03141, the relative error being less than $\frac{1}{3000}$, is evidently less than $\frac{1}{2000}$ or than a half a thousandth.

REMARK 2. When the first significative figure at the left of the part retained is 1, and the first figure at the left of the part suppressed is less than 5 or is 5 not followed by significative figures, the relative error is less than one-half a decimal unit of an order, which is one less than the number of units retained. In replacing the number 1.14137 by 1.141, the absolute error, 0.00037, is less than one-half of a thousandth, and as the given number exceeds 1000 thousandths the relative error is less than a half a thousandth divided by 1000 thousandths or by 1, that is, than a half a thousandth.

REMARK 3. From the two preceding remarks, it follows that in the majority of cases, the relative error of a whole or decimal number, at the right of which one or several figures have been suppressed, is less than half of a decimal unit of an order, which is one less than the number of figures retained commencing with the first significative figure at the left.

REMARK 4. In retaining a certain number of figures, it is evident that the relative error will be as much smaller as the absolute error is less; therefore, approximate values should be taken which give the smallest absolute error (177).

200. *Addition. The absolute error of the sum of several numbers,* whose values are approximate, is equal to the sum of the absolute errors of the numbers.

When the numbers have approximate values, some greater and some smaller than the number, add the plus and minus errors separately, and the difference of the two sums will be the absolute error of the sum, bearing the sign of the greater sum.

The relative error of the sum of several numbers is equal to the absolute error divided by the sum.

To find the sum of less than 11 numbers, with an absolute error of less than a unit of a certain order, add the numbers including the figures of the next lower order, neglecting all others at the right. Thus, to find the sum of the following numbers with an absolute error less than 0.1,

$$5.347 + 8.7537 + 0.0425 = 14.1432,$$

take simply

$$5.34 + 8.75 + 0.04 = 14.13.$$

The absolute error of each number is less than 0.01, and there being less than 11 numbers, the absolute error of the sum will be less than $0.01 \times 10 = 0.1$.

If there are more than 10 numbers and less than 101, take one more still in making the addition. Given, the numbers 75.347, 8.7537, 0.6435, to find their sum with a relative error less than 0.01.

$$75.347 + 8.7537 + 0.6435 = 84.7432.$$

First add:

$$70 + 8 + 0.6 = 78.6,$$

the first figures at the left of each number; divide this sum by 100, formed by one followed by as many ciphers as indicated by the order desired (0.01), which gives 0.786; divide this sum by the number 3 of numbers to be added, and the first figure at the left of the quotient 0.262 expressing tenths it shows that it is sufficient to take each of the given numbers with one decimal only. If the first figure to the left had expressed hundreds, the given number would have to be taken with two decimals, and so on. Thus in the given example:

$$75.3 + 8.7 + 0.6 = 84.6.$$

Since the relative error of the sum of the numbers is less than 0.01 when the absolute error is less than the hundredth part of the sum, as the sum of the given numbers is greater than $70 + 8 + 0.6 = 78.6$, and, therefore, the hundredth part is greater than $78.6 \times 0.01 = 0.786$, in taking each of the given numbers with an absolute error less than $\frac{0.786}{3} = 0.262$, and certainly less than 0.1 by taking a decimal figure, the absolute error is certainly less than 0.786 and evidently less than a hundredth of the sum. Therefore, the sum thus obtained satisfies the conditions given.

201. *Subtraction.* The greater number being the sum of the smaller and the difference, according as the absolute errors of the two numbers have or have not the same sign, the absolute error is equal to the difference or the sum of the absolute errors of the two numbers:

$$\begin{array}{ccccc} 8.67 & 8.6 & 0.07 & 8.7 & 0.03 \\ \underline{3.24} & \underline{3.2} & \underline{0.04} & \underline{3.2} & \underline{0.04} \\ 5.43 & 5.4 & 0.03 & 5.5 & 0.07 \end{array}$$

It follows from what was said concerning addition (**200**), that to find the difference of two numbers with a relative error less than 0.01, for example,

$$75.3478 - 26.5363 = 48.8115,$$

take the difference,

$$70 - 20 = 50,$$

of the numbers formed by the first figures at the left of the numbers; multiply this difference by the given error 0.01, which gives 0.5; take half 0.25 of the product, and since the first figure 2 at the left of this half expresses tens, one decimal is all that need be retained in the operation; which gives for a result,

$$75.3 - 26.5 = 48.8.$$

202. *Multiplication.* 1st. *The absolute error of the product of two factors, one of whose values has been approximated to a certain degree,* is equal to the absolute error of the approximated factor multiplied by the other factor. The *relative error of the product* is equal to the relative error of the approximated factor (**200**).

Calculate, correct to 0.01, *the product,*

$$3.1415926\ldots \times 271.8.$$

The absolute error of the product being equal to the absolute error of the multiplicand multiplied by the multiplier 271.8, it suffices to take the multiplicand with an absolute error less than $\frac{0.01}{271.8}$ and even better if less than $\frac{0.01}{1000} = 0.00001$; which gives 3.14159.

This amounts to taking the approximated number with a number $2 + 3 = 5$ of decimal figures equal to the number of decimal figures 2 desired in the product plus the number 3 of whole number figures of the other factor.

To find the same product with a relative error less than 0.01, take the approximated factor with a relative error less than 0.01, that is, with 3 decimal figures (**199**), which gives 3.141×271.8.

2d. *When the two factors of a product are replaced by approximate values, one of which is less than the exact value, the absolute error of the product* is less than the sum of the products of each of the factors and the absolute error of the other factor, by the product of the absolute errors of the factors.

The relative error of a product is less than the sum of the relative errors of the two factors.

Calculate, with an absolute error less than 0.01, *the product,*

$$314.15926\ldots \times 27.18281828\ldots$$

The problem is satisfied when the absolute error of the product is less than

$$\frac{0.005}{28} + \frac{0.005}{315}.$$

Therefore, taking the first factor with four decimal figures and the second with five, we have an absolute error less than

$$\frac{0.005}{30} + \frac{0.005}{400} = 0.00016\ \ldots + 0.000012\ \ldots$$

Instead of dividing the absolute error 0.01 into two parts, it may be divided in any manner as long as the sum of the two parts is equal to 0.01.

To find the preceding product with a relative error less than 0.01, it suffices if the relative error of each factor is less than 0.005, and still more if less than 0.001, which would be the case in taking four figures at the left of each of the factors, and we have

$$314.1 \times 27.18.$$

The relative error of the product of several approximated factors, whose approximate values are less than the exact values, is less than the sum of the relative errors of all the factors; *the relative error of a power of an approximated number, whose approximate value is less than the exact,* is less than the relative error of the number multiplied by the degree of the power.

Calculate, with a relative error less than 0.01, *the product,*

$$314.15926\ldots \times 27.18281828\ldots \times 2.34246735\ldots$$

It suffices if the sum of the relative errors of the factors is less than 0.01; consequently, taking each of the factors with a relative error less than $\frac{0.01}{3}$ or less than 0.001,

$$314.1 \times 27.18 \times 2.342,$$

one is sure of satisfying the conditions of the problem (**199**).

For a product of approximate values,

$$314.15 \times 27.18 \times 2.34,$$

the relative errors of the factors being respectively less than 0.0001, 0.001, and 0.01, the sum of which is 0.0111, the relative error of the product is less than 0.1, and probably even less than 0.01.

If the product

$$314.15926\ldots \times 27.18281828\ldots \times 2.34246735\ldots$$

is desired with an absolute error less than 0.1, it suffices if the relative error is less than 0.1 divided by a number $320 \times 30 \times 3 = 28{,}800$ greater than the product; this gives a relative error for each factor of less than $\frac{0.1}{3 \times 28{,}800} = \frac{0.1}{86{,}400}$, and when each factor is taken with seven figures to the left, the relative error is less than $\frac{0.1}{100{,}000} = 0.000001$,

$$314.1592 \times 27.18281 \times 2.342467.$$

Remark. *The relative error of the product of several approximated factors, which approximations are greater than the exact values,* is greater than the sum of the relative errors of all the factors; the *relative error of a power of an approximated number, which approximation is greater than the exact value,* is greater than the relative error of the number multiplied by the degree of the power.

203. *Oughtred's short method of multiplication.* To calculate a product of two whole numbers or decimals, $3.1415926\ldots \times 32.18642$ (see below), with an absolute error (198) less than a whole or decimal unit, 0.1 for example, write in an inverse order, the figures of the multiplier under the multiplicand in such a manner that the figure 2 of the simple units in the multiplier corresponds to the figure 1 in the multiplicand which expresses units (0.001) one hundred times smaller than those of the order desired, 0.1; then commencing at the right multiply successively the multiplicand by each figure of the multiplier, neglecting the figures of the multiplicand which are at the right of the figure which serves as multiplier (for the figure 3, for example, neglect 926 . . .); this leads to the fact that no figures in the multiplier at the left of the last figure 3 in the multiplicand are used as multipliers. Write the partial products under the multiplier, placing the first right-hand figures in the same vertical column; in adding,

consider them to express units of an order one hundred times smaller than that desired, 0.1; in this example two figures, 07, are suppressed at the right of the result, and the last figure on the left is increased by one unit. Thus the product is 101.2.

```
        3.1 4 159 26 . . .
 . . . 46 8 1 23
 ------------------------
        9 4 2 45
          6 2 82
            3 14
            2 48
              18
      ----------
      10 1.1 07
      10 1.2
```

REMARK. The preceding rule is given for a general case. The case where the sum $3 + 2 + 1 + 8 + 6 = 20$ of the figures employed in the multiplier, plus the first figure, 4, which was neglected, gives 24, which is greater than 10 and less than 101.

In the case where this sum is less than 10, and in that one where it is between 100 and 1001, operate in the same manner as above, but writing the units figure of the multiplier respectively under the figure of the multiplicand which expresses units ten or one thousand times smaller than those of the order desired in the result.

204. *Division. When the dividend is replaced by an approximate value, which is greater or less than the exact value,* the absolute error of the quotient is equal to the absolute error of the dividend divided by the divisor, and its relative error is equal to that of the dividend. Thus replacing

$$\frac{3.14159}{38} \text{ by } \frac{3.14}{38},$$

the absolute and the relative error of the quotient are respectively,

$$\frac{0.00159}{38} \text{ and } \frac{0.00159}{3.14159}.$$

When the divisor is replaced by an approximate value, which is larger or smaller than the exact value, the absolute error of the quotient is equal to the quotient multiplied by the absolute error of the divisor divided by the approximate value, and the relative

error is equal to the absolute error of the divisor divided by its approximate value. Thus replacing

$$\frac{38}{3.14159} \text{ by } \frac{38}{3.14},$$

the absolute and relative error are respectively,

$$\frac{38}{3.14159} \times \frac{0.00159}{3.14} \text{ and } \frac{0.00159}{3.14}.$$

From the form of the relative error, it follows that according as the approximation is less or greater than the exact value, the relative error of the quotient is greater or less than that of the divisor; and from the form of the absolute error, it follows that when the whole part of the divisor is greater than the quotient multiplied by a number a, if the divisor is replaced by its whole part, the absolute error of the quotient is less than $\frac{1}{a}$.

Thus, in replacing $\frac{8}{6.7}$ by $\frac{8}{6}$, as $6 > \frac{8}{6.7} \times 5$, the absolute error will be less than $\frac{1}{5}$.

The dividend being equal to the product of the divisor and the quotient, *the relative error of the quotient may be considered as being equal to the difference between the relative errors of the dividend and divisor* (2d, 202), *and consequently*, at least, *less than one of them.* Therefore, to obtain a quotient with an error less than 0.1, 0.01, 0.001 . . ., the relative errors of the two numbers must be taken less than these same quantities, that is, respectively the 2, 3, 4 . . ., first figures at the left of the dividend and the divisor. Thus, to find the quotient of 3.1415926 . . . divided by 32.1864 . . ., with a relative error less than 0.001, divide 3.141 by 32.18.

205. *Short method of division.* To find the quotient of a whole or decimal number divided by a whole or decimal number, with an absolute error (198) less than a given whole or decimal unit, 0.001 for instance (see example below), commence by determining the number of figures 1 in the whole part of the quotient (64), and then, the total number $n = 1 + 3 = 4$ of figures in the required quotient. If the whole part were 0, n would equal 3; if the figure in tenths place were 0, n would equal 2; and if the

figure in hundredths place were 0, n would equal 1 (the highest order of units in the quotient is easily determined by inspection, and thus the value of n). Then, removing the decimal points, take, at the left of the divisor, just enough figures so that the number 32 which results is at least equal to $n = 4$; at the right of 32 write the $n = 4$ following figures of the divisor, and the resulting number 321,864 is the first partial divisor. To form the first partial dividend, separate at the left of the dividend just enough figures so that the decimal number 3,141,592.65 . . . which results is at least equal to the decimal number 321,864.18 . . ., formed by placing in the given divisor a point at the right of the first partial divisor, and the part 3,141,592 separated at the left of the dividend is the first partial dividend. The quotient 9 in the division of the first partial dividend by the first partial divisor, is the first left-hand figure in the required quotient. Take the remainder 244,816 obtained for a second partial dividend, and neglecting the first figure 4 at the right of the first partial divisor, the number 32,186 thus formed is the second partial divisor; dividing the second partial dividend by the second partial divisor, the second figure 7 of the required quotient is obtained. Taking the new remainder, 19,514, for the third partial dividend, and the number 3218, obtained by suppressing the first right-hand figure in the second partial divisor, for the third partial divisor, and continuing thus until the $n = 4$ figures of the quotient have been obtained, the required quotient is correct to the given place (0.001), when the decimal point is so placed that the first figure on the right expresses units of the given order.

3.141 592 65 . . .	0.321 864 18 . . .	3 141 592	$321^{1}8^{1}6^{1}4$
		244 816	9.760
		19 514	
		206	

It can happen that a partial dividend contains a corresponding partial divisor 10 times; then take 10 for a partial quotient, that is, write 0 in the quotient and increase the figure immediately preceding by one unit; continuing the process ciphers are obtained for all the following figures. The quotient obtained in this case is always larger than the exact value by less than a unit of the given order. An example of this case is: 26.389292 . . . divided by 3.1415926 correct to the third place (0.001).

$$\begin{array}{r|l} 2\,638\,929 & \underline{314^{1}1^{1}5^{1}9} \\ 125\,657 & 83 \\ 31\,412 & \underline{\ \ 10\qquad} \\ 2 & 8.400 \end{array}$$

206. The relative error of the power of an approximate number, which approximation is greater than the exact value, being greater than the product $e \times n$ of the relative error e of the number and the degree n of the power (**202**, Remark), it follows that *the relative error of the root of an approximate number, which approximation is greater than the exact value,* is less than the relative error e' of the number divided by the index n of the root.

Example. Extract $\sqrt[3]{65.36874}\ldots$

If four figures at the left are taken, increasing the last by one unit, we have 65.37, which gives a relative error,

$$e' < \frac{1}{6000},$$

and for the root,

$$e < \frac{e'}{3} < \frac{1}{6000 \times 3} < \frac{1}{10{,}000} \text{ or } 0.0001.$$

Thus, in taking, in this example, the given number with two exact decimal figures, four exact figures are obtained in the root, that is, the figure in the whole part and three decimals.

DEFINITIONS RELATIVE TO MEASURES

207. The *ratio* of two quantities of the same kind is a number such that in multiplying the second of the two by this number, the first is obtained. Thus, for example, when a length contains another just 5 times, the ratio of the first to the second is 5, and the ratio of the second to the first is $\frac{1}{5}$.

Remark. The ratio of one number to another is equal to the quotient obtained by dividing the first by the second, or a fraction with the first number for numerator and the second for denominator (**134**).

208. *To compare one quantity with another,* find the ratio of the first to the second.

209. All quantities with which others of the same kind are compared so as to form an idea of their extent, are called *units of measure.* The number *one* is the numerical unit (**1** and **5**).

210. *To measure a quantity* is to compare it with the unit of its kind.

211. The ratio of a quantity to the unit of its kind is the measure of the quantity.

212. A quantity is the *common measure* of several quantities when it is contained one or several times in each one of them without a remainder.

213. Two quantities are *commensurable* or *incommensurable,* according as they have or have not a common measure. The ratio of these quantities is also called *commensurable* or *incommensurable.*

214. *The arithmetical mean of several* like quantities is the quotient obtained in dividing the sum of the quantities by their number. Thus the arithmetical mean of the numbers 3, 7 and 5 is $\frac{3 + 7 + 5}{3} = 5$.

THE METRIC SYSTEM

215. The base of the *metric system* is the *meter,* which is a ten-millionth part of a quadrant of a meridian circle (**209**).

216. The metric system contains five principal units, which are: the unit of *length;* the unit of *surface;* the unit of *volume;* the unit of *weight;* and the unit of *money.*

1st. The unit of length is the *meter* (**215**).

2d. The unit of surface is the *square meter,* or a square which has a meter for a side. Land is measured in *ares;* an are is a square whose side is 10 meters; it is equivalent to 100 square meters.

3d. The unit of volume is the *cubic meter,* or a cube whose side is a meter.

In measuring wood the cubic meter is called a *stere.*

In measuring grains and liquids the *liter* is used, which is a hollow cylinder, the capacity of which is a cubic decimeter, or a cube whose side is the tenth part of a meter; it is equivalent to a thousandth part of a cubic meter.

4th. The unit of weight is the *gramme,* or the weight in a vacuum of a cubic centimeter of distilled water at its maximum density, which corresponds to 4° C. above 0°.

The *cubic centimeter* is a cube whose side is a hundredth part of a meter; it is equivalent to the thousandth part of a cubic decimeter or the millionth part of a cubic meter.

5th. The monetary unit in the United States is the dollar.

217. *In the metric system, to express multiples of a unit,* the name of the unit is preceded by the words deca, hecto, kilo, myria, which signify respectively, 10, 100, 1000, 10,000. Thus, to express 1000 grammes, one says kilogramme, and to express 10,000 meters, one says myriameter. These prefixes do not apply to the monetary units.

To express the under-multiples of a unit (38), the name of the unit is preceded by the words *deci, centi, milli,* which signify respectively $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$. Thus, one-hundredth of a gramme is a centigramme, the thousandth of a meter is a millimeter.

The hundredth part of a dollar is called a *cent;* there are 10 cents in a dime, and 10 dimes in a dollar.

218. In the metric system, the multiple units of the principal unit being, as in the decimal numbers, each ten times greater than the other, and the under-multiple being each ten times smaller than the other, it follows that:

1st. *A concrete decimal number* (12) *is pronounced* as an abstract decimal (169), but replacing the name of the abstract unit by that of the concrete unit which it represents. Thus the number 325.87 considered as expressing meters is pronounced 325 meters, 87 centimeters.

2d. *A concrete decimal number is written* as an abstract decimal, but the initial letter of the word which expresses the concrete unit is placed at the right of the units figure. Thus the number given above is written 325 m., 87 cm.

219. *The units of measure which are principally used are:*

1st. *For lengths:*

Myriameter, kilometer, decameter, meter, decimeter, centimeter, millimeter, whose values are respectively in meters:

10,000 m. 1000 m. 10 m. 1 m. 0.1 m. 0.01 m. 0.001 m.

In the industries the meter is most ordinarily used; in surveying, the decameter; geographical distances are generally given in myriameter or kilometer, and sometimes in leagues. The league is equal to 4 kilometers or 4000 meters. There is also the league 25 to the degree, whose value is 4444.44 m.; the *marine league* 20 to the degree, which is 5555.56 m.; and the *sea mile* 60 to a degree, which is 1851.85 m.

The speed of vessels is given in knots of 15 meters, per half minute.

2d. *For surfaces:*

Square meter, square decimeter, square centimeter, square millimeter, whose values are respectively in square meters:

1 sq. m.	0.01 sq. m.
0.0001 sq. m.	0.000001 sq. m.

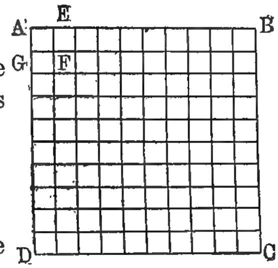


Fig. 1

It is seen, as shown in Fig. 1, that the square decimeter is simply the 0.01 $\left(\text{or } \frac{1}{100}\right)$ of the square meter; that the square centimeter is 0.01 $\left(\text{or } \frac{1}{100}\right)$ of the square decimeter, and so on.

In the same manner the value of the units for land measure,

	hectare,	are,	centare,
in ares are:	100 a.	1 a.	0.01 a.

3d. *For volumes:*

Cubic meter, cubic decimeter, cubic centimeter, cubic millimeter, which in cubic meters are:

1 cb. m. 0.001 cb. m. 0.000001 cb. m. 0.000000001 cb. m.

It is seen that the cubic decimeter is simply the 0.001 or $\frac{1}{1000}$ of the cubic meter; the cubic centimeter the 0.001 of the cubic decimeter, etc.

The hectoliter,	decaliter,	liter,	deciliter,	centiliter,
in liters are:				
100 l.	10 l.	1 l.	0.1 l.	0.01 l.

and the decastere,	stere,	decistere,
in cubic meters or steres are:		
10 st.	1 st.	0.1 st.

4th. *For weights* we have:

myriag., kilog., hectog., decag., gramme, decig., centig., millig., which in grammes are:

10,000 g. 1000 g. 100 g. 10 g. 1 g. 0.1 g. 0.01 g. 0.001 g.

In the industries, the *metric quintal* is sometimes used, which is 100 kilogrammes. In commerce and engineering the *metric ton*, 1000 kilogrammes, is frequently used.

The weight of precious stones is given in *carats*. The carat is divided into $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$, and varies so little in the different countries, that it may be considered as universal. One carat is equal to 205.5000 mg. or 4 grains.

The approximate value of rough diamonds in dollars is obtained by multiplying the price of one carat by the square of their weight in carats. Thus a rough diamond of three carats is worth $40 \times 3 \times 3 =$ \$360.00, one carat being worth \$40.00.

Formerly the value of cut diamonds was also calculated from the price of a one-carat stone, but, owing to an abnormal demand for small stones and a supply of very large ones, the large diamonds are most often cut up into smaller sizes. This process entails loss, so that a one-carat diamond more often costs more by weight than either a one and one-half or a two-carat diamond.

5th. *For money:*

Eagle,	dollar,	dime,	cent,	mill,

which in dollars is:

\$10	\$1	\$0.1	\$0.01	\$0.001

The coins of the United States are:

Gold: double eagle, eagle, half-eagle, quarter-eagle, three-dollar and one-dollar piece.

Silver: dollar, half-dollar, quarter-dollar, and ten-cent piece.

Nickel: five-cent piece.

Bronze: one-cent piece.

220. *Real or effective measures* are those which exist in the form of instruments or objects authorized by law.

Effective measures, marked with the official stamp, are established with certain forms and dimensions which are best suited to facilitate their use.

I. The *effective measures of length*, which are most commonly used, are:

1st. The *chain*, which is ordinarily one decameter (10 m.) and sometimes a double decameter (20 m.) long.

2d. The *tape,* which is rolled on an axle and protected by a housing made of leather or paper, is divided into meters which are subdivided into decimeters and centimeters, and the first decimeter is even divided into millimeters. Dressmakers and others use tapes 1 m., 1.5 m., 2 m. long. Civil engineers, etc., use tapes 5 m., 10 m., and 20 m. long.

3d. The *double meter* is a rule of wood or metal, sometimes jointed so as to be carried in the pocket, and generally divided into decimeters and centimeters.

4th. The *meter,* a straight rule, sometimes jointed in 2, 5, or 10 parts. It is divided into centimeters and ordinarily into millimeters on the first decimeter. It is made of wood, whalebone, bone, ivory, and metal.

5th. The *half-meter,* a straight rule, of one piece or jointed in the middle.

6th. The *double decimeter* and the *decimeter,* made of boxwood, bone, or ivory. They are divided into millimeters and sometimes into half-millimeters.

7th. The *scale* is made of steel and generally ½ or 1 decimeter long, and divided into millimeters and half-millimeters.

II. There are no effective measures of surfaces; their measure is obtained by the use of geometry (Part III).

III. The *effective measures of volumes.*

In measuring the solids it is necessary to have recourse to geometry (Part III); but for the liquids and the grains there are effective measures.

For the liquids there are 13 effective measures, of which five are called *large measures* and eight *small measures.*

The 5 large measures are cylindrical vessels, the depth of which is equal to the interior diameter. According to their use they are made of copper, galvanized iron, and tin plate.

Table of the Five Large Liquid Measures

Name.	Capacity in Liters.	Depth and Diameter in mm.
Hectoliter	100	503.1
Half-hectoliter	50	399.3
Double-decaliter	20	294.2
Decaliter	10	233.5
Half-decaliter	5	185.3

The 8 small measures for liquids other than milk and oil are made of an alloy containing 95 parts tin and 5 parts lead; the tin alone would be too breakable, and lead alone would be poisonous. They are hollow cylinders whose depth is twice their interior diameter. For milk and oil these 8 measures are made of tin plate, and their depth is equal to their interior diameter.

Table of Eight Small Liquid Measures

Name.	Capacity in Liters.	Depth, mm.	Diameters, mm.	Milk and Oil, Depth and Diameter, mm.
Double-liter	2	216.8	108.4	136.6
Liter	1	172.1	86.0	108.4
Half-liter	0.5	136.6	68.3	86.0
Double-deciliter	0.2	100.6	50.3	63.4
Deciliter	0.1	79.9	39.9	50.3
Half-deciliter	0.05	63.4	31.7	39.9
Double-centiliter	0.02	46.7	23.4	29.4
Centiliter	0.01	37.1	18.5	23.4

For the grains, etc., there are 11 effective measures, which according to their use are constructed of wood, copper, or iron. They are ordinarily made of oak staves secured by metal fastenings. All are cylindrical in form and have an internal diameter equal to the depth.

Table of Dry Measure

Name.	Capacity, Liters.	Diameters and Depths, mm.	Name.	Capacity, Liters.	Diameters and Depths, mm.
Hectoliter	100	503.1	Liter	1	108.4
Half-hectoliter	50	399.3	Half-liter	0.5	86.0
Double-decaliter	20	294.2	Double-deciliter	0.2	63.4
Decaliter	10	233.5	Deciliter	0.1	50.3
Half-decaliter	5	185.3	Half-deciliter	0.05	39.9
Double-liter	2	136.6	. . . .	. . .	. . .

Prices of grains are usually based upon the hectoliter or metric quintal. In measuring grains, seeds, and small fruits, the measure is *level full* or *stricken*. The mean weight of a hectoliter of wheat is 75 kg.; of barley, 64 kg.; of oats, 47 kg.

Coal is measured in half-hectoliters, hectoliters, and tons.

Fire-wood is measured in half-decasteres, double steres, and steres, which are respectively, 5, 2, and 1 cubic meters.

Each of these measures is constructed of wood, in the following manner. Upon a rectangular base two upright ends are fastened and braced. The distance between the uprights is respectively, 1, 2, or 3 meters for the stere, double stere, and half-decastere; the height varies with the length of the pieces of wood.

4th. *Effective measures of weight.* The 24 official weights which are used in commerce and industry are divided according to the following table into 5 large weights, 9 medium weights, and 10 small weights.

Large Weights.	Medium Weights.		Small Weights.	
kilog.		kilog.		gramme.
50	1 kilogr.	1 = 1	1 gramme	1 = 1
20	5 hectogr.	0.5 = $\frac{1}{2}$	5 decigr.	0.5 = $\frac{1}{2}$
10	2 hectogr.	0.2 = $\frac{1}{5}$	2 decigr.	0.2 = $\frac{1}{5}$
5	1 hectogr.	0.1 = $\frac{1}{10}$	1 decigr.	0.1 = $\frac{1}{10}$
2	5 decagr.	0.05 = $\frac{1}{20}$	5 centigr.	0.05 = $\frac{1}{20}$
. .	2 decagr.	0.02 = $\frac{1}{50}$	2 centigr.	0.02 = $\frac{1}{50}$
. .	1 decagr.	0.01 = $\frac{1}{100}$	1 centigr.	0.01 = $\frac{1}{100}$
. .	5 grammes	0.005 = $\frac{1}{200}$	5 milligr.	0.005 = $\frac{1}{200}$
. .	2 grammes	0.002 = $\frac{1}{500}$	2 milligr.	0.002 = $\frac{1}{500}$
. .	. . . . .	. . . . . .	1 milligr.	0.001 = $\frac{1}{1000}$

Ten of these weights, from 50 kg. to 5 decagrammes or a half-hectogramme, are made of cast iron. The 50 and 20 kg. weights have the form of a frustum of a rectangular pyramid with rounded edges, the 8 others have the form of a frustum of a hexagonal pyramid. All of these weights are supplied with a ring on top which lies below the surface when not in use, and thus does not interfere with the piling of the weights one upon the other.

Fourteen weights, from 20 kg. to 1 gramme, are made of brass. They are cylindrical in form and have a button on top to take hold of. The height of the cylinder is equal to its diameter, and the height of the button is half of that. The diameter of the double gramme and gramme is often greater than the height. Weights are also made in the form of conical goblets which fit one over the other, and are inclosed in a box of the same form. The box itself represents a legal weight.

The nine weights under the half-gramme are made of little, thin square or octagonal pieces of brass, aluminum, silver, or platinum. One corner is slightly raised so as to facilitate handling with pincers. They are mostly employed in chemical analysis and experimental physics.

221. *Units of time.* The different units of time are not of the decimal order, and do not belong to the metric system.

The *solar day* is the time included between two consecutive crossings of a certain meridian by the sun.

The *solar year* is the time required by the earth to make one complete revolution around the sun, and is equal to a number of solar days which lies between 365 and 366. The *solar year* is constant, but the solar days are not, for the two following reasons: *first,* the non-uniform velocity of the earth in its orbit, by which the apparent diurnal movement of the sun is more rapid in winter than in summer; *second,* the obliquity of the ecliptic, which makes the apparent diurnal movement of the sun in right ascension, that is, in the plane of the terrestrial equator, slower at the equinoxes than at the solstices.

The *principal unit of time is the mean day,* or the mean value of the 365 solar days. The mean day is divided into 24 equal parts called *hours,* the hour into 60 equal parts called *minutes,* the minute into 60 seconds, the second into fifths, tenths, or hundredths.

In writing units which express time, write the abbreviations for the different units after each number. The minutes and seconds are sometimes denoted by $'$ or $''$. Thus 3 da. 8 hr. 35 min. 45 sec. or 3 da. 8 hrs. $35' \, 45''$ represents 3 days 8 hours 35 minutes 45 seconds.

The *sidereal day* is the interval of time between two consecutive transits of a certain meridian by a star. Its duration is constant, and equal to 23 hrs. $56' \, 4''$ mean time.

Remark. The solar year contains approximately 365.24225 mean days.

The *civil year* is the legal year; the solar year is increased or decreased enough so that it contains exactly 365 or 366 days. One hundred consecutive years form a *century.* The civil year is divided into twelve parts called months, the names of which are January, February, March, April, May, June, July, August, September, October, November, December. The number of days in each month is easily remembered by memorizing the following:

"Thirty days has September,
April, June, and November;
All the rest have thirty-one,
Except February, which has but twenty-eight in fine,
Until leap year gives it twenty-nine."

The solar year is 0.24225 mean day longer than the civil year, and if the civil always had 365 days, at the end of 4 years it would be 0.969 day ahead of the solar year; it is to compensate for this that one day is added every fourth year, such a year being called leap year. From this correction it follows that every four years the civil year is placed $1 - .969 = 0.031$ days behind the solar year, and at the end of a century is $0.031 \times 25 = 0.775$ day behind; for this reason the last year of each century is not leap year. From this it again follows that at the end of each century the civil day is $1 - 0.775 = 0.225$ day ahead of the solar year, and every fourth century is $0.225 \times 4 = 0.9 = 1 - 0.1$ ahead; thus it is that we have a leap year every fourth century. After this third correction the civil year is 0.1 day behind the solar year every 400 years, which is 1 day at the end of 4000 years; thus by suppressing a leap year every 4000 years, the civil year terminates at the same instant as the solar year if we accept 365.24225 as the exact value, which in reality is only an approximation.

These four successive corrections may be represented by putting the ratio of the solar year to the mean day in the form

$$365 + \frac{1}{4} - \frac{1}{100} + \frac{1}{400} - \frac{1}{4000}.$$

The *Julian calendar* was established by Julius Cæsar forty-six years before Christ, and was in use in the Roman world until 1582, at which time the pope, Gregory, instituted the *Gregorian calendar*, which is in use to-day in nearly every country.

To-day the Julian dates are 12 days behind the Gregorian dates; and when writing to countries which still employ the Julian calendar (Russia and Greece), it is customary to write $\frac{1}{13}$ Jan., $\frac{9}{21}$ Feb., which gives the dates according to both calendars.

222. The circumference of a circle is divided into 360 equal parts called *degrees;* the degree into 60 equal parts called *minutes;* the minute into 60 equal parts called *seconds.* The *quadrant* of a circumference is 90 degrees.

Degrees, minutes, and seconds are units used to measure angles and arcs (see Geometry).

In writing degrees, minutes, and seconds, the signs $^\circ$, $'$, and $''$,

respectively, are placed above and a little to the right of the number; thus $3° 17' 28''$ is read 3 degrees 17 minutes 28 seconds.

Often the circumference of a circle is divided into 400 equal parts called *grades,* and each grade into 100 equal parts, which parts are again divided by 100. The quadrant equals 100 grades.

These measures conform with the law of decimals. Thus 74.3705 g. reads 74 grades 37 hundredths of a grade 5 hundredths of a hundredth of a grade.

223. A *complex quantity* is a quantity composed of several parts, compared with different units of its kind. Such are the quantities 7 da. 16 hr. 34 m. and $42° 21' 15''$.

PROBLEMS RELATING TO MEASURES

224. In general, concrete decimals may be operated upon in the same manner as abstract decimals (**178** to **182**).

225. *Application to the payment of workmen.* A workman earns \$4.75 per day; in a month of 26 working days he will earn

$$\$4.75 \times 26 = \$123.50.$$

The following table gives the sum earned by a workman, working 10 hours a day for a certain number of days at a certain wage.

To find what is due a man for a certain number of hours, 7, for example, at \$4.75 per day, take as many days as there are hours and divide by ten, which in this case (referring to the table) gives \$3.33.

Therefore in 26 days and 7 hours the workman will earn

$$\$123.50 + 3.33 = \$126.83.$$

Wage Table

Days.	$0.50	$0.60	$0.70	$0.75	$0.90	$1.00	$1.25	$1.50	$1.75	$2.00
1	0.50	0.60	0.70	0.75	0.90	1.00	1.25	1.50	1.75	2.00
2	1.00	1.20	1.40	1.50	1.80	2.00	2.50	3.00	3.50	4.00
3	1.50	1.80	2.10	2.25	2.70	3.00	3.75	4.50	5.25	6.00
4	2.00	2.40	2.80	3.00	3.60	4.00	5.00	6.00	7.00	8.00
5	2.50	3.00	3.50	3.75	4.50	5.00	6.25	7.50	8.75	10.00
6	3.00	3.60	4.20	4.50	5.40	6.00	7.50	9.00	10.50	12.00
7	3.50	4.20	4.90	5.25	6.30	7.00	8.75	10.50	12.25	14.00
8	4.00	4.80	5.60	6.00	7.20	8.00	10.00	12.00	14.00	16.00
9	4.50	5.40	6.30	6.75	8.10	9.00	11.25	13.50	15.75	18.00
10	5.00	6.00	7.00	7.50	9.00	10.00	12.50	15.00	17.50	20.00
11	5.50	6.60	7.70	8.25	9.90	11.00	13.75	16.50	19.25	22.00
12	6.00	7.20	8.40	9.00	10.80	12.00	15.00	18.00	21.00	24.00
13	6.50	7.80	9.10	9.75	11.70	13.00	16.25	19.50	22.75	26.00
14	7.00	8.40	9.80	10.50	12.60	14.00	17.50	21.00	24.50	28.00
15	7.50	9.00	10.50	11.25	13.50	15.00	18.75	22.50	26.25	30.00
16	8.00	9.60	11.20	12.00	14.40	16.00	20.00	24.00	28.00	32.00
17	8.50	10.20	11.90	12.75	15.30	17.00	21.25	25.50	29.75	34.00
18	9.00	10.80	12.60	13.50	16.20	18.00	22.50	27.00	31.50	36.00
19	9.50	11.40	13.30	14.25	17.10	19.00	23.75	28.50	33.25	38.00
20	10.00	12.00	14.00	15.00	18.00	20.00	25.00	30.00	35.00	40.00
21	10.50	12.60	14.70	15.75	18.90	21.00	26.25	31.50	36.75	42.00
22	11.00	13.20	15.40	16.50	19.80	22.00	27.50	33.00	38.50	44.00
23	11.50	13.80	16.10	17.25	20.70	23.00	28.75	34.50	40.25	46.00
24	12.00	14.40	16.80	18.00	21.60	24.00	30.00	36.00	42.00	48.00
25	12.50	15.00	17.50	18.75	22.50	25.00	31.25	37.50	43.75	50.00
26	13.00	15.60	18.20	19.50	23.40	26.00	32.50	39.00	45.50	52.00
27	13.50	16.20	18.90	20.25	24.30	27.00	33.75	40.50	47.25	54.00
28	14.00	16.80	19.60	21.00	25.20	28.00	35.00	42.00	49.00	56.00
29	14.50	17.40	20.30	21.75	26.10	29.00	36.25	43.50	50.75	58.00
30	15.00	18.00	21.00	22.50	27.00	30.00	37.50	45.00	52.50	60.00

Days.	$2.25	$2.50	$2.75	$3.00	$3.25	$3.50	$3.75	$4.00	$4.25	$4.50
1	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50
2	4.50	5.00	5.50	6.00	6.50	7.00	7.50	8.00	8.50	9.00
3	6.75	7.50	8.25	9.00	9.75	10.50	11.25	12.00	12.75	13.50
4	9.00	10.00	11.00	12.00	13.00	14.00	15.00	16.00	17.00	18.00
5	11.25	12.50	13.75	15.00	16.25	17.50	18.75	20.00	21.25	22.50
6	13.50	15.00	16.50	18.00	19.50	21.00	22.50	24.00	25.50	27.00
7	15.75	17.50	19.25	21.00	22.75	24.50	26.25	28.00	29.75	31.50
8	18.00	20.00	22.00	24.00	26.00	28.00	30.00	32.00	34.00	36.00
9	20.25	22.50	24.75	27.00	29.25	31.50	33.75	36.00	38.25	40.50
10	22.50	25.00	27.50	30.00	32.50	35.00	37.50	40.00	42.50	45.00
11	24.75	27.50	30.25	33.00	35.75	38.50	41.25	44.00	46.75	49.50
12	27.00	30.00	33.00	36.00	39.00	42.00	45.00	48.00	51.00	54.00
13	29.25	32.50	35.75	39.00	42.25	45.50	48.75	52.00	55.25	58.50
14	31.50	35.00	38.50	42.00	45.50	49.00	52.50	56.00	59.50	63.00
15	33.75	37.50	41.25	45.00	48.75	52.50	56.25	60.00	63.75	67.50
16	36.00	40.00	44.00	48.00	52.00	56.00	60.00	64.00	68.00	72.00
17	38.25	42.50	46.75	51.00	55.25	59.50	63.75	68.00	72.25	76.50
18	40.50	45.00	49.50	54.00	58.50	63.00	67.50	72.00	76.50	81.00
19	42.75	47.50	52.25	57.00	61.75	66.50	71.25	76.00	80.75	85.50
20	45.00	50.00	55.00	60.00	65.00	70.00	75.00	80.00	85.00	90.00
21	47.25	52.50	57.75	63.00	68.25	73.50	78.75	84.00	89.25	94.50
22	49.50	55.00	60.50	66.00	71.50	77.00	82.50	88.00	93.50	99.00
23	51.75	57.50	63.25	69.00	74.75	80.50	86.25	92.00	97.75	103.50
24	54.00	60.00	66.00	72.00	78.00	84.00	90.00	96.00	102.00	108.00
25	56.25	62.50	68.75	75.00	81.25	87.50	93.75	100.00	106.25	112.50
26	58.50	65.00	71.50	78.00	84.50	91.00	97.50	104.00	110.50	117.00
27	60.75	67.50	74.25	81.00	87.75	94.50	101.25	108.00	114.75	121.50
28	63.00	70.00	77.00	84.00	91.00	98.00	105.00	112.00	119.00	126.00
29	65.25	72.50	79.75	87.00	94.25	101.50	108.75	116.00	123.25	130.50
30	67.50	75.00	82.50	90.00	97.50	105.00	112.50	120.00	127.50	135.00

Wage Table — (*Continued*)

Days.	$4.75	$5.00	$5.25	$5.50	$5.75	$6.00	$6.25	$6.50	$6.75	$7.00
1	4.75	5.00	5.50	5.50	5.75	6.00	6.25	6.50	6.75	7.00
2	9.50	10.00	10.50	11.00	11.50	12.00	12.50	13.00	13.50	14.00
3	14.25	15.00	15.75	16.50	17.25	18.00	18.75	19.50	20.25	21.00
4	19.00	20.00	21.00	22.00	23.00	24.00	25.00	26.00	27.00	28.00
5	23.75	25.00	26.25	27.50	28.75	30.00	31.25	32.50	33.75	35.00
6	28.50	30.00	31.50	33.00	34.50	36.00	37.50	39.00	40.50	42.00
7	33.25	35.00	36.75	38.50	40.25	42.00	47.75	45.50	47.25	49.00
8	38.00	40.00	42.00	44.00	46.00	48.00	50.00	52.00	54.00	56.00
9	42.75	45.00	47.25	49.50	51.75	54.00	56.25	58.50	60.75	63.00
10	47.50	50.00	52.50	55.00	57.50	60.00	62.50	65.00	67.50	70.00
11	52.25	55.00	57.75	60.50	63.25	66.00	68.75	71.50	74.25	77.00
12	57.00	60.00	63.00	66.00	69 00	72.00	75.00	78.00	81.00	84.00
13	61.75	65.00	68.25	71.50	74.75	78.00	81.25	84.50	87.75	91.00
14	66.50	70.00	73.50	77.00	80.50	84.00	87.50	91.00	94.50	98.00
15	71.25	75.00	78.75	82.50	86.25	90.00	93.75	97.50	101.25	105.00
16	76.00	80.00	84.00	88.00	92.00	96.00	100.00	104.00	108.00	112.00
17	80.75	85.00	89.25	93.50	97.75	102.00	106.25	110.50	114.75	119.00
18	85.50	90.00	94.50	99.00	103.50	108.00	112.50	117.00	121.50	126.00
19	90.25	95.00	99.75	104.50	109.25	114.00	118.75	123.50	128.25	133.00
20	95.00	100.00	105.00	110.00	115.00	120.00	125.00	130.00	135.00	140.00
21	99.75	105.00	110.25	115.50	120.75	126.00	131.25	136.50	141.75	147.00
22	104.50	110.00	115.50	121.00	126.50	132.00	137.50	143.00	148.50	154.00
23	109.25	115.00	120.75	126.50	132.25	138.00	143.75	149.50	155.25	161.00
24	114.00	120.00	126.00	132.00	138.00	144.00	150.00	156.00	162.00	168.00
25	118.75	125.00	131.25	137.50	143.75	150.00	156.25	162.50	168.75	175.00
26	123.50	130.00	136.50	143.00	149.50	156.00	162.50	169.00	175.50	182.00
27	128.25	135.00	141.75	148.50	155.25	162.00	168.75	175.50	182.25	189.00
28	133.00	140.00	147.00	154.00	161.00	168.00	175.00	182.00	189.00	196.00
29	137.75	145.00	152.25	159.50	166.75	174.00	181.25	188.50	195.75	203.00
30	142.50	150.00	157.50	165.00	172.50	180.00	187.50	195.00	202.50	210.00

Days.	$7.25	$7.50	$7.75	$8.00	$8.25	$8.50	$8.75	$9.00	$9.50	$10.00
1	7.25	7.50	7.75	8.00	8.25	8.50	8.75	9.00	9.50	10.00
2	14.50	15.00	15.50	16.00	16.50	17.00	17.50	18.00	19.00	20.00
3	21.75	22.50	23.25	24.00	24.75	25.50	26.25	27.00	28.50	30.00
4	29.00	30.00	31.00	32.00	33.00	34.00	35.00	36.00	38.00	40.00
5	36.25	37.50	38.75	40.00	41.25	42.50	43.75	45.00	47.50	50.00
6	43.50	45.00	46.50	48.00	49.50	51.00	52.50	54.00	57.00	60.00
7	50.75	52.50	54.25	56.00	57.75	59.50	61.25	63.00	66.50	70.00
8	58.00	60.00	62.00	64.00	66.00	68.00	70.00	72.00	76.00	80.00
9	65.25	67.50	69.75	72.00	74.25	76.50	78.75	81.00	85.50	90.00
10	72.50	75.00	77.50	80.00	82.50	85.00	87.50	90.00	95.00	100.00
11	79.75	82.50	85.25	88.00	90.75	93.50	96.25	99.00	104.50	110.00
12	87.00	90.00	93.00	96.00	99.00	102.00	105.00	108.00	114.00	120.00
13	94.25	97.50	100.75	104.00	107.25	110.50	113.75	117.00	123.50	130.00
14	101.50	105.00	108.50	112.00	115.50	119.00	122.50	126.00	133.00	140.00
15	108.75	112.50	116.25	120.00	123.75	127.50	131.25	135.00	142.50	150.00
16	116.00	120.00	124.00	128.00	132.00	136.00	140.00	144.00	152.00	160.00
17	123.25	127.50	131.75	136.00	140.25	144.50	148.75	153.00	161.50	170.00
18	130.50	135.00	139.50	144.00	148.50	153.00	157.50	162.00	171.00	180.00
19	137.75	142.50	147.25	152.00	156.75	161.50	166.25	171.00	180.50	190.00
20	145.00	150.00	155.00	160.00	165.00	170.00	175.00	180.00	190.00	200.00
21	152.25	157.50	162.75	168.00	173.25	178.50	183.75	189.00	199.50	210.00
22	159.50	165.00	170.50	176.00	181.50	187.00	192.50	198.00	209.00	220.00
23	166.75	172.50	178.25	184.00	189.75	195.50	201.25	207.00	218.50	230.00
24	174.00	180.00	186.00	192.00	198.00	204.00	210.00	216.00	228.00	240.00
25	181.25	187.50	193.75	200.00	206.25	212.50	218.75	225.00	237.50	250.00
26	188.50	195.00	201.50	208.00	214.50	221.00	227.50	234.00	247.00	260.00
27	195.75	202.50	209.25	216.00	222.75	229.50	236.25	243.00	256.50	270.00
28	203.00	210.00	217.00	224.00	231.00	238.00	245.00	252.00	266.00	280.00
29	210.25	217.50	224.75	232.00	239.25	246.50	253.75	261.00	275.50	290.00
30	217.50	225.00	232.50	240.00	247.50	255.00	262.50	270.00	285.00	300.00

226. *To compare a quantity expressed by a concrete decimal with one of the units of its kind,* remove the decimal point to the right of the figure which represents the units. Thus, to express the quantity 365.867 m. in centimeters, advance the decimal point two places towards the right, giving 36586.7 cm., that is, a number one hundred times greater and which expresses units a hundred times smaller than the given number.

227. *To reduce a complex quantity* 5 *years,* 7 *months, and* 8 *days to one of its units.* Let it be required to reduce the given quantity to years. The year has 12 months, 5 yrs. + 7 mo. $= 5 \times 12 + 7 = 67$ mo., and as a month has 30 days, 67 mo. + 8 da. $= 67 \times 30 + 8 = 2018$ da. But 1 yr. $= 12 \times 30 = 360$ da., therefore,

$$5 \text{ yrs.} + 7 \text{ mo.} + 8 \text{ da.} = \frac{2018}{360} \text{yrs.} = 5.60555 \text{ yrs.} \ldots (181).$$

Since the month contains 30 days,

$$5 \text{ yrs.} + 7 \text{ mo.} + 8 \text{ da.} = \frac{2018}{30} \text{ mo.} = 67.2666 \text{ mo.} \ldots$$

228. *The inverse of the preceding problem.* Reduce 2018 days to years, months, days, etc. Divide 2018 by 360:

$$\begin{array}{r|l}
2018 \text{ da.} & 360 \\
218 & \overline{\phantom{5 \text{ yrs. } 7 \text{ mo. } 8 \text{ da.}}} \\
12 & 5 \text{ yrs. } 7 \text{ mo. } 8 \text{ da.} \\
\hline
436 & \\
218 & \\
\hline
2616 & \\
96 & \\
30 & \\
\hline
2880 & \\
\hline
00 &
\end{array}$$

The division of 2018 by 360 gives 5 for the quotient and 218 for the remainder, thus:

$$\frac{2018}{360} \text{yrs.} = 5 \text{ yrs.} + \frac{218}{360} \text{yrs.} = 5 \text{ yrs.} + \frac{218 \times 12}{360} \text{mo.}$$

$$= 5 \text{ yrs.} + \frac{2616}{360} \text{mo.} = 5 \text{ yrs.} + 7 \text{ mo.} \frac{96}{360} \text{mo.}$$

$$= 5 \text{ yrs.} + 7 \text{ mo.} + \frac{96 \times 30}{360} \text{da.} = 5 \text{ yrs.} + 7 \text{ mo.} + 8 \text{ da.}$$

229. *The same problem,* the number of years 5.60555 . . . yrs. being expressed in decimals.

Putting the decimal in the form of a decimal fraction $\frac{560,555}{100,000}$,

proceed as before. In this case the division by 1 followed by ciphers renders the operation more simple, as we have but one series of multiplications (177). Thus:

$$5.60555 \text{ yrs.} = 5 \text{ yrs.} + 0.60555 \text{ yrs.} = 5 \text{ yrs.} + 0.60555 \times 12 \text{ mo.} = 5 \text{ yrs.} + 7.2666 \text{ mo.} = 5 \text{ yrs.} + 7 \text{ mo.} + 0.2666 \times 30 \text{ da.} = 5 \text{ yrs. } 7 \text{ mo. } 8 \text{ da.}$$

230. *The four operations on the complex numbers* are performed by following the same methods as with whole or decimal numbers, remembering that the different units are no longer equal to 10 of the units of next lower order, when reducing the partial results to units of the next higher order (addition and multiplication), or next lower order (division), and when a number has to be increased in order to make a subtraction possible. It may be noted also that the numbers of each order of units may have more than one figure.

ADDITION

7 hrs.	5 min.	54.8 sec.
2	10	40.4
5	18	47.6
14 hrs.	35 min.	22.8 sec.

SUBTRACTION

9 hrs.	25 min.	14.8 sec.
3	31	30.4
5 hrs.	53 min.	44.4 sec.

MULTIPLICATION

8 da.	3 hrs.	19 min.	16.3 sec.
			7
37 da.	10 hrs.	14 min.	54.1 sec.

DIVISION (231)

```
7 hrs. 18 min. 13.5 sec. | 4
  3                      |------------------------
 60                      | 1 hr. 49 min. 33.375 sec.
180
 18
198
 38
  2
 60
120
 13.5
133.5
 13
  1 5
    30
     20
      0
```

If the multiplier is a fraction, multiply the complex quantity by the numerator and divide the product by the denominator.

If the divisor is a fraction, multiply the complex dividend by the fraction inverted. When a problem involves *the multiplication or division of one complex number by another,* reduce one of them to a common unit (227), and proceed as when dividing a complex number by a fraction.

An example in division. A movement takes 5 hrs. 10 m. 3 s. to turn 2° 18′ 15″; how long will it take for it to turn 1°, its velocity being constant?

From the question it is seen that 2° 18′ 15″ should be reduced to degrees, which gives $\frac{8295}{3600} = \frac{553}{240}$ in dividing by the common factors 3 and 5. The time is then

$$(5 \text{ hrs. } 10 \text{ min. } 3 \text{ sec.}) \times \frac{240}{553} = \frac{1240 \text{ hrs. } 12 \text{ min.}}{553}$$
$$= 2 \text{ hrs. } 14 \text{ min. } 33.63 \text{ sec.}$$

BRITISH SYSTEM OF WEIGHTS AND MEASURES

231. Although the British system of measures is in general use in this country, the values of the individual units, in some cases, differ from those used in Great Britain.

Therefore, in the tables that follow the values, assigned to the units apply to those used in the United States unless otherwise stated.

MEASURES OF LENGTH

232. Linear measure has but one dimension, and is used for comparing lines and distances.

Table of Common Linear Measure

12	inches (in. ″)	= 1 foot (ft. ′).
3	feet	= 1 yard (yd.) = 36 in.
5½	yards	= 1 rod (rd.) = 16½ ft. = 198″.
320	rods	= 1 mile (mi.) = 1760 yds. = 5280 ft. = 63,360 in.

233. *Table of Surveyor's Linear Measure.*

7.92	inches (in.)	= 1 link (l.).
25	links	= 1 rod (rd.) = 198 in.
4	rods	= 1 chain (ch.) = 100 l. = 792 in.
80	chains	= 1 mile (mi.) = 320 rds. 8000 l. = 63,360 in.

234. *Miscellaneous Units*

$\frac{1}{12}$ inch	= 1 line.
$\frac{1}{3}$ inch	= 1 barleycorn or size (boot and shoe measure).
3 inches	= 1 palm.
4 inches	= 1 hand (for measuring the height of horses).
9 inches	= 1 span.
18 inches	= 1 cubit.
28 inches	= 1 pace (military pace).
3 feet	= 1 pace (ordinary).
6 feet	= 1 fathom (for measuring depths at sea).
120 fathoms	= 1 cable length.
1.15 statute mile	= 1 nautical or geographical mile.
1 nautical mile	= 1 knot (for measuring speed of vessels).
3 knots	= 1 league (for measuring distances at sea).
60 nautical miles	= 1 degree = 69.16 statute miles.
$\frac{1}{8}$ statute mile	= 1 furlong.
360 degrees	= 1 circumference of the earth.

MEASURES OF SURFACE

235. *Surface* has two linear dimensions, length and breadth.

Table of Common Square Measure

144 square inches (sq. in., □″)	= 1 square foot (sq. ft.).
9 square feet	= 1 square yard (sq. yd.) = 1296 sq. in.
$30\frac{1}{4}$ square yards	= 1 square rod (sq. rd.) = $272\frac{1}{4}$ sq. ft. = 39,204 sq. in.
160 square rods	= 1 acre (A.) = 4840 sq. yds. = 43,560 sq.ft.
640 acres	= 1 square mile (sq. mi.) = 102,400 sq. rds. = 3,097,600 sq. yds. = 27,878,400 sq. ft.

236. *Table of Surveyor's Square Measure*

625 square links (sq. l.)	= 1 square rod (sq. rd.).
160 square rods	= 1 acre (A.) = 100,000 sq. l.
640 acres	= 1 section (sec.) = 102,400 sq. rds. = 64,000,000 sq. l.
36 sections	= 1 township (Tp.) = 23,040 A. = 368,640 sq. rds. = 2,304,000,000 sq. l.

MEASURES OF VOLUME

237. *Volume* has three linear dimensions, length, breadth, and thickness.

Table of Common Cubic Measure

1728 cubic inches (cu. in.)	= 1 cubic foot (cu. ft.).
27 cubic feet	= 1 cubic yard (cu. yd.) = 46,656 cu. in.

238. *Table of Wood Measure*

16 cubic feet	= 1 cord foot (cd. ft.).
8 cord feet	= 1 cord (cd.) = 128 cu. ft.

These measures are also used in measuring small, irregular stones. A cord is a pile 8 ft. long, 4 ft. wide, and 4 ft. high. Wood cut in lengths of 4 feet is called cord wood.

239 *Stone Measure*

$24\frac{3}{4}$ cubic feet = 1 perch.

A perch of stone in masonry is $16\frac{1}{2}$ feet long, $1\frac{1}{2}$ feet wide, and 1 foot high.

MEASURES OF CAPACITY

240. Measures of capacity are divided into *liquid* and *dry measures.*

Liquid measures are used for measuring liquids. There are two kinds of liquid measure, namely, *common liquid measure*, used for measuring water, milk, etc., and *apothecaries' liquid measure*, used for measuring liquid medicines.

241. *Table of Common Liquid Measure*

4 gills (gi.)	= 1 pint (pt.).
2 pints	= 1 quart (qt.).
4 quarts	= 1 gallon (gal.) = 8 pts. = 231 cu. in.
$31\frac{1}{2}$ gallons	= 1 barrel (bbl.) = 126 qts. = 252 pts.
2 barrels	= 1 hogshead (hhd.) = 63 gal. = 252 qts. = 504 pts.

REMARK. Casks holding from 28 gal. to 43 gal. are called barrels, and those holding from 54 gal. to 63 gal. are called hogsheads, but whenever barrels or hogsheads are used as *measures*, a barrel means $31\frac{1}{2}$ gal. and a hogshead 63 gal.

242. *Table of Apothecaries' Liquid Measure*

60 minims (M.)	= 1 fluid dram (ƒ ʒ).
8 fluid drams	= 1 fluid ounce (ƒ ℥).
16 fluid ounces	= 1 pint (0).
8 pints	= 1 gallon (cong.) = 231 cu. in.

243. *Dry measure* is used for measuring grains, seeds, fruit, vegetables, etc.

Table of Dry Measure

2 pints (pt.)	= 1 quart (qt.).
8 quarts	= 1 peck (pk.) = 16 pts.
4 pecks	= 1 bushel (bu.) = 32 qts. = 64 pts.

REMARK. In measuring grains, seeds, and small fruits, the measure must be *even* full; but in measuring apples, potatoes, and other large articles, it must be *heaping* full.

244. *Comparative Table*

U. S. liquid measure, 1 gal.	= 231 cu. in.
U. S. liquid measure, 1 qt.	= $57\frac{3}{4}$ cu. in.
U. S. dry measure, $\frac{1}{2}$ pk.	= $268\frac{4}{5}$ cu. in.
U. S. dry measure, 1 qt.	= $67\frac{1}{5}$ cu. in.
U. S. apothecaries' liquid measure, 1 gal.	= 231 cu. in.
Great Britain liquid measure, 1 qt.	= 69.3185 cu. in.
Great Britain liquid measure, 1 gal.	= 277.274 cu. in.
Great Britain dry measure, 1 qt.	= 69.3185 cu. in.
Great Britain dry measure, 1 bu.	= 2218.192 cu. in.

MEASURES OF WEIGHT

245. There are three systems of units used for measuring weights, namely, *avoirdupois*, *apothecaries'*, and *troy*.

246. *Avoirdupois weight* is used in weighing all ordinary articles.

Table

16 ounces (oz.)	= 1 pound (lb.).
100 pounds	= 1 hundredweight (cwt.).
20 hundredweight	= 1 ton (T.) = 2000 lbs.

247. *Apothecaries' weight* is used in weighing dry medicines and drugs.

Table

20 grains (gr.)	= 1 scruple (sc. or ℈).
3 scruples	= 1 dram (dr. or ʒ).
8 drams	= 1 ounce (oz. or ℥).
12 ounces	= 1 pound (lb. or ℔).

248. *Troy weight* is used in weighing precious stones and metals, such as gold, silver, etc.

Table

24 grains (gr.)	= 1 pennyweight (pwt.).
20 pennyweights	= 1 ounce (oz.).
12 ounces	= 1 pound (lb.).

249. *Comparative Table*

	Avoirdupois	Apothecaries'	Troy
1 pound	= 7000 gr.	5760 gr.	5760 gr.
1 ounce	= 437.5 gr.	480 gr.	480 gr.

CONVERSION TABLES

Metric-English and English-Metric

Linear Measure

250. *Common linear measure*

1 inch	=	25.40 mm.	1 meter	=	39.37 in.
1 foot	=	0.30 m.	1 meter	=	3.28 ft.
1 yard	=	0.91 m.	1 meter	=	1.09 yds.
1 mile	=	1.61 km.	1 km.	=	0.62 mi.

251. *Surveyors' linear measure*

1 link	=	20.12 cm.	1 meter	=	4.97 l.
1 rod	=	5.03 m.	1 meter	=	0.19 rds.
1 chain	=	20.12 m.	1 km.	=	0.05 ch.
1 nautical mile	=	1.85 km.	1 km.	=	0.54 n. mi.

Square Measure

252. *Common square measure*

1 sq. inch	=	6.45 cm.^{2}	1 sq. cm.	=	0.16 sq. in.
1 sq. foot	=	9.29 dm.^{2}	1 sq. m.	=	10.76 sq. ft.
1 sq. yard	=	0.84 m.^{2}	1 sq. m.	=	1.20 sq. yd.
1 sq. mile	=	2.59 km.^{2}	1 sq. km.	=	0.39 sq. mi.

253. *Surveyors' square measure*

1 sq. link	=	404.81 cm.^{2}	1 m.^{2}	=	22.23 sq. l.
1 sq. rod	=	25.30 m.^{2}	1 km.^{2}	=	247.11 acres
1 acre	=	0.41 hectares	1 hectare	=	2.47 acres

Measures of Volume

254.

1 cubic inch	=	16.39 cm.^{3}	1 cm.^{3}	=	0.06 cu. in.
1 cubic foot	=	0.03 m.^{3}	1 dm.^{3}	=	0.04 cu. ft.
1 cubic yard	=	0.77 m.^{3}	1 m.^{3}	=	1.31 cu. yds.

Measures of Capacity

255. *Dry measure*

1 pint	=	0.55 l.	1 liter = 1 dm.^{3}	=	0.91 qts.
1 quart	=	1.10 l	1 liter	=	61.02 cu. in.
1 peck	=	8.81 l.	1 decoliter	=	1.13 pks.
1 bushel	=	35.24 l.			

256. *Liquid measure.* (*Common*)

1 pint	=	0.47 l.	1 liter	=	2.11 pts.
1 quart	=	0.95 l.	1 liter	=	1.06 qts.
1 gallon (U. S.)	=	3.79 l.	1 hectoliter	=	26.42 gals (U.S.)
1 gallon (Br.)	=	4.55 l.	1 hectoliter	=	22.00 gal. (Br.)

Liquid measure. (Apothecaries')

1 dram	= 3.66 cm.^{3}	1 cm.^{3}	= 0.27 fl. ʒ
1 ounce	= 29.37 cm.^{3}	1 liter = 1 dm.^{3}	= 34.48 fl. ℥
1 pint	= 0.47 l.	1 liter	= 2.12 O.
1 gallon	= 3.79 l.	1 deciliter	= 2.64 gal.

Weights

257. *Avoirdupois weights*

1 ounce	= 28.35 g.	1 hectogramme	= 3.53 oz.
1 pound	= 0.45 kg.	1 kilogramme	= 2.21 lbs.
1 hundredweight	= 50.80 kg.	1 ton	= 2204.62 lbs.
1 ton (short)	= 0.91 t.	1 ton	= 1.10 tons.

258. *Troy weights*

1 grain	= 64.79 mg.	1 gramme	= 15.43 gr.
1 pennyweight	= 1.55 g.	1 gramme	= 0.65 pwt.
1 ounce	= 31.10 g.	1 hectogramme	= 3.22 oz.
1 pound	= 0.37 kg.	1 kilogramme	= 2.68 lbs.

259. *Apothecaries' weights*

1 grain	= 64.79 mg.	1 milligramme°	= 0.015 gr.
1 scruple	= 1.30 g.	1 gramme	= 15.432 gr.
1 dram	= 3.89 g.	1 gramme	= 0.77 sc.
1 ounce	= 3.10 g.	1 gramme	= 0.25 dr.
1 pound	= 0.37 kg.	1 kilogramme	= 2.68 lbs.

BOOK IV

POWERS AND ROOTS

DEFINITIONS

260. That which was said in articles **85** to **88**, concerning powers of whole numbers, applies to any number, fraction, decimal, or complex. Thus,

$$3.25^2,\ \left(\frac{3}{14}\right)^3,\ \left(4 \times \frac{5}{7}\right)^4,\ \left(4 + \frac{5}{7}\right)^5,$$

are respectively the square of 3.25, the cube of $\frac{3}{14}$, the fourth power of $4 \times \frac{5}{7}$, and the fifth power of $4 + \frac{5}{7}$.

261. Any number which has a given number for a power is the root of that number.

262. If, of two numbers, the first is a power, of a certain degree, of the second, the second is the root, of the same *degree*, of the first. Thus, 3 giving 3, 9, 27, 81 . . . for 1st, 2d, 3d, 4th powers, these respective numbers have 3 for 1st, 2d, 3d, 4th roots.

263. The roots of the second and third degree are designated as *square root* and *cube root*.

264. To indicate the root of a number, write the number under the sign $\sqrt{\ }$, called a *radical*, at the upper left-hand corner of which the *index* of the root is written. Thus,

$$\sqrt[2]{9},\quad \sqrt[3]{27 \times 3},\quad \sqrt[4]{4 + 16},\quad \sqrt[5]{\frac{35}{74}},$$

express respectively the square, cube, fourth, and fifth roots of the quantities 9, 27×3, $4 + 16$, and $\frac{35}{74}$.

Remark. The first root of a number being equal to the number, the radical sign and index are discarded. For the square root it is customary to write simply the radical sign without the index. Thus, instead of writing $\sqrt[2]{9}$, write simply $\sqrt{9}$.

SQUARES AND SQUARE ROOTS

265. *To square a number, and, in general, to raise a number to any power,* multiply the number by itself and the successive products until as many multiplications have been performed as are indicated by the index of the power. Thus, to square multiply $\frac{3}{4}$ by itself (**160**):

$$\frac{3}{4} \times \frac{3}{4} = \frac{3 \times 3}{4 \times 4} = \frac{3^2}{4^2} = \frac{9}{16}.$$

266. *Directions for using a table of squares and cubes, of the consecutive whole numbers from* 1 *to* 1000, *for squaring or cubing whole, decimal, or fractional numbers.*

Assume that the table gives directly the square and cube of numbers not greater than 1000, which covers all cases in general practice.

In an abstract or concrete decimal, if, neglecting the decimal point, the whole number which results is not greater than 1000, by the use of the table find the square or cube of this whole number, and separate on the right two or three times as many decimal figures as there are decimals in the given number.

1. EXAMPLE. Determine the area of a square the side of which is 7.96 m. Taking the centimeter as unity, we have the length of the side equal to 796 cm., and from the table the area is 633,616 sq. cm. = 63.3616 m^2.

2. Find the volume of a cube whose side is 0.796 m. Taking the millimeter as unity, the side of the cube is 796 mm., and the table gives the volume as 504,358,336 mm^3. = 0.504358336 m^3.

If the given number, on removing the decimal point, is larger than 1000, reduce it to units such that the whole part will be as large as possible without exceeding 1000, and the square or cube of this whole part as given by the table may be taken as an approximation, which in ordinary cases is quite sufficiently accurate.

Thus, in the first example the side of the square being 7.963 m., take the centimeter for unity, which gives 796.3 cm. Neglecting the 3 millimeters, proceed as in the above example, which gives 633,616 cm^2. = 63.3616 m^2., or the square of 7.96 m., and may be taken as an approximation to the square of 7.963 m.

If the side were 7.968 m., instead of taking 7.96 m. take 7.97 m.,

so as to have the nearest approximation. For a fraction find the square or cube of each of the terms (265).

267. Table of cubes and squares of whole numbers between 1 and 10.

Roots,	0	1	2	3	4	5	6	7	8	9	10
Squares,	0	1	4	9	16	25	36	49	64	81	100
Cubes,	0	1	8	27	64	125	216	343	512	729	1000

268. The square of a whole number, of a single figure, has two figures; that of one having two figures has three or four; that of three has five or six, etc. From this it follows that in order *to obtain the number of figures in the square root of a given number,* separate the number into periods of two figures each, commencing at the simple units. The number of periods gives the number of figures in the square root.

269. The square of a quantity composed of two parts is made up of the following:

1. The square of the first; 2. Twice the product of the first and the second; 3. The square of the second. Thus:

$$(3 + 5)^2 = 3^2 + 2 \times 3 \times 5 + 5^2 = 9 + 30 + 25 = 64.$$

As a special case, *the square of a number composed of tens and units* is made up as follows:

1. Of the square of the tens; 2. Of twice the product of the tens and units; 3. Of the square of the units.

$$54^2 = 50^2 + 2 \times 50 \times 4 + 4^2 = 2500 + 400 + 16 = 2916;$$
$$273^2 = 270^2 + 2 \times 270 \times 3 + 3^2 = 72{,}900 + 1620 + 9 = 74{,}529.$$

270. *The difference of the squares of two consecutive whole numbers is equal to twice the smaller of the numbers, plus one.*

$$(26 + 1)^2 - 26^2 = 26 \times 2 + 1 = 53.$$

271. *To extract the square root of a whole number,* 74,529 for example, commencing at the right separate the number into periods of two figures each (the number of periods is the number of figures in the root) (268), and draw a vertical line at the right, to separate it from the root. Take the square root, 2, of the greatest square, 4, contained in the first period at the left, 7; this root, 2, which can have but one figure (268), is the first figure at the left of the root. Subtract the square of the first figure found from the first period at the left; at the right of the remainder,

3, write the next following period, 45; separate the first figure, 5, at the right of the resulting number; divide the part at the left, 34, considered as expressing simple units, by twice the number obtained in the root, which gives 8 for a quotient; this quotient is either the next figure of the root, or it is too large.

To prove it, write it at the right of double that part of the root already obtained; multiply the number 48 which results by 8, and the product 384, being greater than the number 345, shows that 8 is too large. Operating on the figure 7 as on the figure 8, the product 329, obtained by multiplying the number made up of double the part of the root already found and 7 by 7, being less than 345, 7 is the next figure in the root. Subtract the product 329 from 345; at the right of the remainder, 16, write the next period, 29; separate the figure 9 from the others, and divide the part at the left, 162, considered as expressing simple units, by double, $27 \times 2 = 54$, the part of the root already obtained, which gives as quotient the next figure in the root or one too large. This is proved as was the preceding figure, and so on until all the periods of the number have been operated upon.

7.45.29	273		
4	48	47	543
34.5	8	7	3
32.9	384	329	1629
1 62.9			
1 62 9			
0			

7.4 5.29	273
3 4.5	48
1	47
162.9	543
0	

Generally the products of the figures and the root are not written, but they are subtracted as fast as they are obtained; this was done in the second operation shown above.

272. *Limit of the remainder of a square root.* In the operation of extracting the square root, if the remainder which corresponds to the part of the root already obtained is not less than twice that part of the root plus one, that part of the root is too small by at least one unit; and when the remainder is less than twice that part of the root plus one, that part of the root cannot be increased by one.

Thus the last remainder should always be less than twice the root, plus one. When it is less than the root, the root is correct to half a unit and is less than the exact value. In the opposite

case, the root is increased by one and is then correct to a half unit, but is greater than the exact value (175, 206).

273. If, as in the last example, a remainder of zero is obtained, the given number is a *perfect square.*

If, on the contrary, the last remainder is not zero, the given number is not a perfect square. The root obtained is the root of the number, but less than the exact root by less than one unit (272), that is, of the whole part (175). It is the exact root of the largest perfect square contained in the given number, and the remainder is the difference between this number and the largest perfect square. The exact root of the given number cannot be expressed exactly by any number, whole, fractional, or decimal; it is incommensurable (213), and consequently cannot be expressed by a periodic decimal (195, 196, 206). It can be expressed only by approximation.

274. *A whole number is not a perfect square:*

1st. When it does not contain all the prime factors of a power of an even degree (124, 273).

2d. When, being an even number, it is not divisible by $2^2 = 4$.

3d. When the zeros which terminate it are not in even numbers.

4th. When it is terminated by one of the four figures 2, 3, 7, 8.

5th. When, terminating with 5, it has not the figure 2 in tens' place.

CUBES AND CUBE ROOTS

275. Since the cube of a number of a single figure does not contain more than three figures; of one of two figures contains four, five, or six, etc., it follows that in order *to obtain the number of figures in the cube root of a whole number,* the number is divided into periods of three figures each, the number of periods giving the number of figures in the root (268).

In general, *to obtain the number of figures in the mth root of a whole number,* divide the number into periods of m figures, and the number of periods will be the number of figures in the root.

276. *The cube of a quantity composed of two parts is made up of the following:*

First, the cube of the first part; *second,* the triple product of the square of the first and the second; *third,* the triple product of the first and square of the second; *fourth,* the cube of the second. Thus:

$(4+5)^3 = 4^3 + 3 \times 4^2 \times 5 + 3 \times 4 \times 5^2 + 5^3 = 64 + 240 + 300 + 125 = 729.$

As a special case, the cube of a number composed of tens and units is made up of four parts:

First, the cube of the tens; *second,* the triple product of the square of the tens and the units; *third,* the triple product of the tens and the square of the units; *fourth,* the cube of the units. Thus:

$$\begin{aligned} 145^3 &= 140^3 + 3 \times 140^2 \times 5 + 3 \times 140 \times 5^2 + 5^2 \\ &= 2{,}744{,}000 + 294{,}000 + 10{,}500 + 125 = 3{,}048{,}625. \end{aligned}$$

277. The difference of the cubes of two consecutive whole numbers is equal to the triple square of the smaller, plus the triple of the smaller, plus one:

$$(26 + 1)^3 - 26^3 = 26^2 \times 3 + 26 \times 3 + 1.$$

278. *To extract the cube root of a whole number,* 3,048,625 for example, commencing at the right, separate the number into periods of three figures each (the number of periods indicates the number of figures in the root) (**275**).

Take the cube root, 1, of the greatest cube 1, contained in the first period, 3, at the left; the root, which can have but one figure (**275**), is the first figure at the left of the root.

3.048.6 25	145		
1	$3 \times 1^3 = 3$	$3 \times 1^2 = 3$	$3 \times 14^2 = 588$
20.48	$3 \times 10^2 \times 6 = 1800$	$3 \times 10^2 \times 4 = 1200$	$3 \times 140^2 \times 5 = 294000$
17 44	$3 \times 10 \times 6^2 = 1080$	$3 \times 10 \times 4^2 = 480$	$3 \times 140 \times 5^2 = 10500$
3 04 6.25	$6^3 = 216$	$4^3 = 64$	$5^3 = 125$
3 04 6 25	3096	1744	304625
0			

Subtract the cube of the first figure, 1, from the first period at the left; at the right of the remainder, 2, write the next period, 048; separate two figures at the right of the resulting number; divide the part at the left, 20, considered as expressing simple units, by three times the square of that part of the root already obtained, which gives for a quotient a figure 6, which is either the next figure of the root or too large. To determine which, finish the operation of constructing the cube, that is, since the cube of the tens has been subtracted, three times the square of the tens times the units $3.10^2.6 = 1800$, three times the tens times the square of the units $3.10.6^2 = 1080$, and the cube of

the units $6^3 = 216$; adding, the sum 3096 being greater than 2048 shows that 6 is too large.

By the same process it is found that 5 is also too large. Trying 4 the sum of the three parts, 1744, being less than 2048, 4 is taken as the next figure in the root. Subtract 1744 from 2048; at the right of the remainder, 304, write the figures 625 of the next period; separate two figures, 25, on the right of the resulting number; divide the part at the left, 3046, considered as expressing simple units, by three times the square of that part of the root already obtained, $3 \times 14^2 = 588$, which gives 5 for a quotient, this being either too large or the next figure in the root.

The truth may be established in the same manner as above, considering 140 as one part and 5 as the other, and constructing the three parts: $[3 \times 140^2 \times 5 = 294{,}000] + [3 \times 140 \times 5^2 = 10{,}500] + [5^3 = 125] = 304{,}625$; since this sum is not greater than 304,625, 5 is the next figure of the root. Continue thus until all the periods of the root have been used.

279. *Limit of the remainder of a cube root.* The largest remainder which can be obtained in the process of extracting the cube root of a number, cannot be as great as three times the square of that part of the root already obtained, plus three times that part of the root, plus one. If the remainder is equal to or greater than this sum, the last figure in the root is too small, and should be increased.

280. At any point in the operation of extracting the cube root of a number, the remainder, followed by all the periods which have not been operated upon, is equal to the number of which the root is desired less the cube of that part of the root already obtained.

Analogous to the square root (**273**), if the cube root falls between two consecutive whole numbers, it cannot be expressed by any number, whole, fractional, or decimal; it is incommensurable. This root can only be expressed by approximation.

281. An even number cannot be a perfect cube unless it is divisible by $2^3 = 8$. A number terminating with ciphers cannot be a perfect cube unless the number of ciphers be a multiple of 3 (**274**).

282. *Proof by the rule of* 9. A power of a number being the result of the multiplication of this number taken several times as factor, the proof by 9 of the raising of a number to a certain

power, is made in the same manner as the proof by 9 in multiplication (99).

To prove by 9 the extraction of a root, the given number being equal to a certain power of the root, plus the remainder, proceed in the same manner as in the proof by 9 of a division (100). Thus, to prove by 9 the example in (278), find the remainder 1 of the root 145 by 9, take the cube 1 of this remainder, and the remainder 1 of this cube by 9, added to the remainder 0 by 9 of the remainder obtained in the extraction of the root, gives the sum 1, of which the remainder, 1 by 9, should be equal to the remainder by 9 of the given number 3,048,625.

The proofs by 11 of powers and roots are calculated in the same manner as the proofs by 9 (101).

SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS OF FRACTIONS AND DECIMALS

283. *The square of a fraction* being the product of the fraction and itself, it is obtained by squaring each of the terms (160, 266):

$$\left(\frac{4}{7}\right)^2 = \frac{4^2}{7^2} = \frac{16}{49}.$$

284. *The cube of a fraction* being the product obtained by using the fraction three times as a factor, it is obtained by cubing each of the terms (266):

$$\left(\frac{4}{5}\right)^3 = \frac{4^3}{5^3} = \frac{64}{125}$$

285. From the manner in which the squares and cubes of fractions are formed, it follows that in order to extract the square or cube root of a fraction, it suffices to extract the square or cube root of its terms (262). Thus:

$$\sqrt{\frac{16}{49}} = \frac{\sqrt{16}}{\sqrt{49}} = \frac{4}{7} \text{ and } \sqrt[3]{\frac{64}{125}} = \frac{\sqrt[3]{64}}{\sqrt[3]{125}} = \frac{4}{5}.$$

286. *The extraction of the square or cube root of a fraction may be reduced to the extraction of the root of but one number.*

To do this, multiply the two terms of the given fraction by the

denominator, for the square root, or by the square of the denominator for the cube root. Thus,

$$\sqrt{\frac{4}{7}} = \sqrt{\frac{4 \times 7}{7 \times 7}} = \frac{\sqrt{28}}{\sqrt{7^2}} = \frac{\sqrt{28}}{7},$$

and

$$\sqrt[3]{\frac{4}{5}} = \sqrt[3]{\frac{4 \times 5^2}{5 \times 5^2}} = \frac{\sqrt[3]{4 \times 5 \times 5}}{\sqrt[3]{5^3}} = \frac{\sqrt[3]{100}}{5}.$$

It is seen that in this method of operating, the denominator of the root is the same as that of the given fraction (275).

This method holds for all fractions; but if the denominator of the given fraction is not a prime number, it may be better to reduce it to a perfect square or cube, by multiplying the two terms by any convenient factors:

$$\sqrt{\frac{19}{504}} = \sqrt{\frac{19}{2^3 \times 3^2 \times 7}} = \sqrt{\frac{19 \times 2 \times 7}{2^4 \times 3^2 \times 7^2}} = \frac{\sqrt{19 \times 2 \times 7}}{2 \times 3 \times 7} = \frac{\sqrt{266}}{84};$$

$$\sqrt[3]{\frac{19}{504}} = \sqrt[3]{\frac{19}{2^3 \times 3^2 \times 7}} = \sqrt[3]{\frac{19 \times 3 \times 7^2}{2^3 \times 3^3 \times 7^3}} = \frac{\sqrt[3]{2793}}{2 \times 3 \times 7} = \frac{\sqrt[3]{2793}}{42}.$$

Thus the square of $\frac{19}{504}$ expressed in 84ths and the cube root in 42ds are obtained (269).

287. The square of a decimal number being the number multiplied by itself, and the cube the number taken three times as a factor, the squares and cubes of numbers are found according to the rules given for multiplication of decimals (180):

$$3.546^2 = 3.546 \times 3.546 = 12.574116;$$
$$23.7^3 = 23.7 \times 23.7 \times 23.7 = 13{,}312.053.$$

288. Since in multiplying a decimal the point is dropped and as many places pointed off in the product as the sum of the decimals in the two numbers, it follows in squaring a number the number of decimals in the square must always be even, because they are obtained by multiplying the number of places in the given number by two. In the same manner it may be shown that the cube of a decimal contains three times as many places as the given number.

289. From the rules concerning the formation of the squares and cubes of decimal numbers (287), the following conclusions may be derived:

1st. *To extract the square root of a decimal number,* drop the decimal point and proceed as though the number were whole, separating at the right of the root half as many places as there are in the given number:

$$\sqrt{54.76} = \frac{\sqrt{54\ 76}}{\sqrt{1\ 00}} = \frac{74}{10} = 7.4 \text{ (172 and 259).}$$

2d. *To extract the cube root of a decimal number,* drop the decimal point and proceed as though the number were whole, separating at the right of the root one-third as many decimal places as there are in the given number:

$$\sqrt[3]{3.048625} = \frac{\sqrt[3]{3{,}048{,}625}}{\sqrt[3]{1{,}000{,}000}} = \frac{145}{100} = 1.45.$$

290. *To obtain the square root of any number correct to a given decimal* (**175**), the number must contain twice as many decimals as are desired in the root, and if it has not that number, ciphers must be added at the right; thus, if the square is desired correct to one unit, one tenth, one hundredth, or one thousandth, etc., the given number must contain zero, two, four, or six, etc., decimals. Then dropping the decimal point the root is extracted in the usual manner (**271**), pointing off at the right of the result the required number of decimals.

Thus it is found that the square root of 247 correct to one unit is 15; that the square root of the same number to the hundredths place is $\sqrt{247.0000} = 15.71$; that of 2.5 to the hundredths place is

$$\sqrt{2.5} = \sqrt{2.5000} = 1.58;$$

that of $\frac{5}{11}$ to the thousandths place is

$$\sqrt{\frac{5}{11}} = \sqrt{0.454545\ldots} = 0.674.$$

291. Extracting the square root of 0.454545 correct to the thousandths place is the same as extracting the square root of 454545 correct to a unit (**290**) and pointing off three decimal figures at the right of the result; also the rule in (**316**) may be applied; thus, calculate $\sqrt{0.454500}$, which gives 0.674 for the root and 0.224 for the remainder, and the nearest root to the one-thousandth place is 0.675, which is slightly greater than the exact value.

292. *To obtain the cube root of any number correct to a given decimal,* operate in the same manner as when finding the square root, except that instead of taking twice as many decimals in the given number as are required in the root, three times as many are taken. The cube root of 12.5 to the hundredths place is

$$\sqrt[3]{12.500000} = 2.32\,;$$

that of 0.000012755427 to the thousandths place,

$$\sqrt[3]{0.000012755} = 0.023\,;$$

that of $\frac{71}{22}$ to the hundredths,

$$\sqrt[3]{\frac{71}{22}} = \sqrt[3]{3.227272} = 1.47.$$

293. The rule of (**316**) applies to the cube root as to the square (**291**). Thus the cube root, correct to the thousandths place, of 0.000012755427 is obtained by extracting the cube root of 0.000012000.

294. Remark. The square and cube roots obtained in (**290**) and (**292**) are slightly less than the exact values, and by increasing their last figure one unit, we still have the root correct to the required decimal place.

If the nearest value to a certain place is desired, one more decimal is used in the calculation and then dropped in the result according to the rule in (**176**).

295. *To find the square or cube root of a given whole number expressed as a fraction with a given denominator,* reduce the given number to a fraction having the square or cube of the denominator desired for a denominator, and then extract the root (**136**, **286**).

Thus, the square root of 8 expressed in sevenths:

$$\sqrt{8} = \sqrt{\frac{8 \times 7^2}{7^2}} = \frac{\sqrt{392}}{7}.$$

Since the square root of 392 falls between 19 and 20, that of 8 between $\frac{19}{7}$ and $\frac{20}{7}$, and each one of these fractions expresses the square root of 8 correct to $\frac{1}{7}$ of unity:

$$\left(\frac{19}{7}\right)^2 < 8 < \left(\frac{20}{7}\right)^2.$$

In the same manner the cube root of 5 expressed in sevenths is

$$\sqrt[3]{5} = \sqrt[3]{\frac{5 \times 7^3}{7^3}} = \frac{\sqrt[3]{1715}}{7},$$

and $\sqrt[3]{1715}$ lies between 11 and 12, that of 5 between $\frac{11}{7}$ and $\frac{12}{7}$. That is,

$$\left(\frac{11}{7}\right)^3 < 5 < \left(\frac{12}{7}\right)^3.$$

POWERS AND ROOTS OF THE *Nth* DEGREE

296. *The product of several powers of the same number is a power of that number,* of a degree equal to the sum of the degrees of the powers of the factors:

$$3^2 \times 3^2 = 3^4 = 81; \quad 3^2 \times 3^3 \times 3^4 = 3^9 = 19{,}683.$$

297. *Any power of a power of a number* is a power of that number, of a degree equal to the product of the degrees. Thus:

$$(3^2)^2 = 3^4 = 81, \quad (3^2)^3 = 3^6 = 729, \quad [(2^2)^3]^3 = 2^{18} = 262{,}144.$$

298. From the preceding article (297), it follows *that in order to extract a root whose index contains only the factors* 2 *and* 3, it suffices to extract successively, in any convenient order, as many cube and square roots as the factors 3 and 2 enter in the index of the root. Thus:

$$\sqrt[4]{81} = \sqrt{\sqrt{81}} = \sqrt{9} = 3;$$

$$\sqrt[6]{4096} = \sqrt[3]{\sqrt{4096}} = \sqrt[3]{64} = 4;$$

$$\sqrt[18]{262{,}144} = \sqrt{\sqrt[3]{\sqrt{262{,}144}}} = \sqrt[3]{\sqrt[3]{512}} = \sqrt[3]{8} = 2.$$

299. *To raise the product of several factors to the second, third, or any power,* raise each factor to the desired power:

$$(3 \times 4)^2 = 3^2 \times 4^2 = 144, \quad (2^2 \times 5)^3 = 2^6 \times 5^3 = 8000.$$

300. *Power of a quotient.* The same rule holds for any power as for square or cube. Thus,

$$\left(\frac{2}{3}\right)^5 = \frac{2^5}{3^5} = \frac{32}{243}.$$

301. *To extract a root of a product,* extract the root of each factor of the product. Thus:

$$\sqrt{4 \times 9} = \sqrt{4} \times \sqrt{9} = 2 \times 3 = 6;$$

$$\sqrt[3]{\frac{8}{27} \times 64} = \sqrt[3]{\frac{8}{27}} \times \sqrt[3]{64} = \frac{2}{3} \times 4 = \frac{8}{3}.$$

302. *Root of a quotient* (286). We have

$$\sqrt[4]{\frac{16}{81}} = \frac{\sqrt[4]{16}}{\sqrt[4]{81}} = \frac{2}{3}.$$

303. *To raise the sum or difference of several numbers to a given power,* complete the sum or difference and raise the result to the given power:

$$(3 + 4 + 5)^2 = 12^2 = 144; \qquad (9 + 2 - 5)^2 = 6^2 = 36;$$

$$\left(\frac{1}{2} + 1.4 + 3\right)^3 = (0.5 + 1.4 + 3)^3 = (4.9)^3 = 117.649.$$

304. *To extract the root of a sum or difference of several numbers,* extract the root of the result of the given operations:

$$\sqrt{87 + 57} = \sqrt{144} = 12; \qquad \sqrt{25 - 9} = \sqrt{16} = 4;$$

$$\sqrt[3]{25.17 + 49.715 + 42.764} = \sqrt[3]{117.649} = 4.9.$$

305. *The quotient obtained by dividing a power of a number by another power of that same number, is equal to that number raised to a power of a degree equal to the difference between the degrees of the dividend and divisor:*

Thus, $$\frac{3^6}{3^2} = 3^{6-2} = 3^4,$$

and $$\frac{3^2}{3^6} = 3^{2-6} = 3^{-4}.$$

As $$\frac{3^2}{3^6} = \frac{3^2}{3^2 \times 3^4} = \frac{1}{3^4},$$

we have $$\frac{1}{3^4} = 3^{-4}.$$

Special case, $$\frac{3^4}{3^4} \text{ or } 1 = 3^{4-4} = 3^0;$$

which shows that a number raised to the 0 power is equal to 1. Another special case is

$$\frac{3^5}{3^4} \text{ or } 3 = 3^{5-4} = 3^1;$$

which shows that the first power of a number is equal to the number itself. Likewise we have

$$\frac{3^4}{3^5} \text{ or } \frac{1}{3} = 3^{4-5} = 3^{-1};$$

which shows that the reciprocal, $\frac{1}{3}$, of a number, 3, is the -1 power of that number, 3^{-1} (183).

306. *A root of a power of a number is equal to the number raised to a power the degree of which is a fraction whose numerator is the degree of the original power and whose denominator is the index of the root.* Thus:

$$\sqrt[2]{3^6} = 3^{\frac{6}{2}} = 3^3,$$
$$\sqrt[6]{3^2} \text{ or } \sqrt[3]{3} = 3^{\frac{2}{6}} = 3^{\frac{1}{3}},$$
$$\sqrt[6]{\frac{1}{3}} \text{ or } \sqrt[6]{3^{-1}} = 3^{-\frac{1}{6}}$$

307. Remark. The rules given in the preceding chapters show that the extraction of the square or cube root of any number, whole, decimal, or fractional, leads to the extraction of the square or cube root of a whole number, correct to units' place (271, 278, 290, 292).

308. *Use of a table of squares of consecutive whole numbers from* 1 *to* 1000, *in shortening the process of extracting the square root of any number, whole, decimal, or fractional:* 1st, *Correct to the first whole unit;* 2d, *Correct to a decimal of a given order.*

1st. *To extract the square root of any number, correct to the first whole unit.*

The operation consisting of extracting the square root, correct to the first whole unit, of a whole number, the whole part of a decimal number or the whole part of a fraction reduced to decimals (290), it is not necessary to consider more than the whole numbers ; and there are two cases, one where the number is not greater than the greatest number in the table, 1,000,000, and one where it is.

First Case. Extract the square root, correct to one unit, of the whole number 786,545.

Looking in the column of squares,* the square 784,996 is the nearest to the given square, which is less than the given square, that is, it is the largest whole square contained in the given number; the root, 886, is found in the first column, and is the root of the given number correct to one unit. This root is slightly less than the exact; 887 is also the correct root to one unit, but is slightly larger. The difference, 1549, between the given number and the largest square which it contains, is the

* Reference may be had to almost any handbook for a table of powers and roots.

remainder which would be obtained in extracting the square root of that number, correct to one unit:

$$786{,}545 - 784{,}996 = 1549.$$

Any decimal number, 786,545.273 for example, having 786,545 for a whole part, would have 886 for its square root, to the first whole unit, with 1549.273 for a remainder.

Second Case. Extract the square root, correct to one whole unit, of the whole number 7,875,127,437.

Separate at the right of the number an even number, 4, of figures so that the part at the left will be the largest possible number less than the square of 1000; this part coming under the first case, 887 is given for its square root, to a whole unit, with 743 for a remainder. This number, 887, forms the first three left-hand figures of the required root (271), and to obtain the remaining figures, operate according to the rule of (271):

78 75 12.74.37	88 741	
78 67 69	17 744	177 481
7 43 7.4	4	1
7 09 7 6	70 976	177 481
33 9 83.7		
17 7 48 1		
16 2 35 6		

Thus at the right of the remainder, 743, write the next period 74, separate the figure 4 on the right, and divide the part at the left, 7437, by twice 887, that part of the root already obtained, which gives 4 as the next figure of the root if not too large. The correctness of 4 is proved and the work continued as per (271). Thus it is found that 88,741 is the square root and 162,356 the remainder.

It is seen that the table gives directly the first three figures of the root.

Any decimal number, 7,875,127,437.45 for example, having for a whole part the preceding number, would have the same root; the remainder being 162,356.45.

2d. *To extract the square root of any number to a given decimal place.*

From the rule in (290), it follows that these calculations are the same as those given in 1st, and that there are two cases to be considered.

First Case. Extract the square root, correct to one hundredth, of the number 78.6545273.

Retaining four decimal places, we have 78.6545; dropping the decimal point and extracting the root to one unit as in the first case of 1st, the table gives 886 for the root and 1549 for the remainder; therefore 8.86 is the required root and 0.1549273 is the remainder.

Second Case. Extract the square root, correct to one thousandth, of the number 7875.1274.

Add ciphers to complete the number to 6 decimal places; neglect the decimal point, which gives the number 7,875,127,400; find the square root of this whole number, correct to one whole unit, as in the second case of the 1st. This gives 88,741 for root and 162,319 for a remainder; pointing off the decimals, 88.741 is the required root, and 0.162319 the remainder.

309. *Use of the table of cubes of the consecutive whole numbers from* 1 *to* 1000, *to shorten the process of extracting the cube root of any number, whole, decimal, or fractional:* 1st, *Correct to a whole unit;* 2d, *Correct to a given decimal.*

1st. *Extract the cube root of any number, correct to one whole unit.*

The operation consisting of extracting the cube root, correct to the first whole unit, of a whole number, the whole part of a decimal number or the whole part of a fraction reduced to decimals (292), it is not necessary to consider more than the whole numbers; and there are two cases, one where the number is not greater than the greatest cube in the table, and one where it is.

First Case. Extract the cube root, correct to one whole unit, of the number 97,062,526.

Looking in the column of cubes,* the cube 96,702,579 is the nearest value to the given cube that does not exceed it, that is, it is the largest whole cube contained in the number; the root, 459, is found in the first column, and is the root of the given number correct to one unit. This root is slightly less than the exact value; 460 is also correct to one whole unit, but is slightly larger. The difference,

$$97{,}062{,}526 - 96{,}702{,}579 = 359{,}947,$$

between the given number and the largest square which it con-

* Reference may be had to almost any handbook for a table of powers and roots.

tains, is the remainder which would be obtained in extracting the cube root of that number, correct to one whole unit.

Any decimal number, 97,062,526.38 for example, having 97,062,526 for a whole part, would have 459 for its cube root, to one whole unit, and 359,947.38 for the remainder.

Second Case. Extract the cube root, correct to one whole unit, of the number 97,062,526,893,127.

Separate at the right of the number of figures, 6, whose multiple is 3, such that the part at the left will be the largest possible number which is less than the cube of 1000; this part comes under the first case; and from the table we have 459 as the first three figures of the root, and 359,947 as the remainder (**278**). To obtain the following figures of the root, continue the operation according to the rule in (**278**), as was done in the second case of 1st for the square root:

97 062 526.8 93.1 27	45 956	
96 702 579	63 204 300 × 5	6 334 207 500 × 6
359 947 8.93	68 850 × 5	827 100 × 6
316 365 8 75	25 × 5	36 × 6
43 582 0 18 1.27	63 273 175 × 5	6 335 034 636 × 6
38 010 2 07 8 16	316 365 875	38 010 207 816
5 571 8 10 3 11		

Thus it is found that the cube root of the given number is 45,956, and the remainder 5,571,810,311.

It is seen, that as in the case of the square root (**308**), the table gives directly the first three figures of the root. As in the first case, any decimal number having the number given in the above example would have 45,956 as its cube root, correct to one unit; and the remainder would be the same as found above, followed by the decimal part of the given number.

2d. *To extract the cube root of any number, correct to a given decimal place.*

From the rule in (**292**), it follows that these calculations are the same as those given in 1st, and that there are two cases to be considered.

First Case. Extract the cube root, correct to one hundredth, of the number 97.06252632.

Retaining six decimal places, and dropping the decimal point, we have 97,062,526; operating upon this number as in first case,

1st, the table gives the root 4̇59 and the remainder 359,947; pointing off the decimals, we have 4.59 for the root and 0.359947 for the remainder.

Second Case. Extract the cube root, correct to one thousandth, of the number 97,062.52689.

Add ciphers to complete the number to 9 decimal places, and neglecting the decimal point we have 97,062,526,890,000, the cube root of which is found precisely as in second case of 1st. This gives 45,956 as root and 5,571,807,184 as remainder, and pointing off we have 45.956 for the root and 5.571807184 for the remainder.

EXTRACTION OF SQUARE AND CUBE ROOTS BY MEANS OF SUCCESSIVE ADDITIONS

310. *Some of the properties of squares of whole numbers.* Write the three following series, one immediately beneath the other: *first,* the successive odd numbers, commencing at unity; *second,* the successive whole numbers (n); *third,* the squares (c) of these successive whole numbers:

	1	3	5	7	9	11	13	15	17	19	21	23	25...
(n)	1	2	3	4	5	6	7	8	9	10	11	12	13...
(c)	1	4	9	16	25	36	49	64	81	100	121	144	169...

1st. The square c, in the third series, of any number n, which is directly above in the second series, is equal to the sum of the first n terms of the first series (3d). Thus, the square, $c = 25$ of $n = 5$, is equal to the sum of the first five terms in the first series; which is easily proved.

2d. The first series is an arithmetical progression commencing at unity, of which the *constant difference* is 2, the nth term t.

$$t = 1 + 2(n - 1) = 2n - 1. \qquad (359)$$

Thus the whole square, 49, having 7 for its root, is the sum of the first seven terms of the first series, and the seventh term of this series is

$$t = 2 \times 7 - 1 = 13.$$

3d. The sum c of the first n terms in the first series, considered as an arithmetical progression, being equal to one-half the product of the first term plus the nth term t and the number of terms n, we have

$$c = \frac{(1 + t)\,n}{2}. \qquad (361)$$

In substituting for t the value in 2d, c becomes equal to n^2, as was stated in 1st.

The sum s of the first n terms in the second series is

$$s = \frac{(1+n)\,n}{2}; \text{ for } n = 5, \;\; s = \frac{(1+5)\,5}{2} = 15.$$

The sum S of the first n terms of the third series, that is, the squares of the first n consecutive whole numbers, is equal to twice the root, $2n$, of the largest square, plus one, divided by one-third the sum s of the roots:

$$S = (2n+1)\frac{s}{3}.$$

Substituting for s the value given above, we have

$$S = \frac{1}{6}\,n\,(n+1)\;(2n+1). \qquad \text{(Algebra, Book III.)}$$

Find the sum s of the first $n = 13$ consecutive whole numbers. According as the sum, $s = \dfrac{(n+1)\,n}{2} = \dfrac{(13+1)\,13}{2} = 91$, has or has not been calculated, the first or second expression for the value of s should be used:

$$S = (2 \times 13 + 1) \times \frac{91}{3} = 819 \quad \text{or} \quad S = \frac{1}{6} \times 13 \times 14 \times 27 = 819.$$

4th. When a series of whole consecutive squares does not commence with unity, for example, the first square is $n'^2 = c'$, and the last $n^2 = c$; the sum s' of the corresponding roots is equal to the difference $c - c'$ between the largest and smallest square, plus the sum $n + n'$ of the two square roots and the whole expression divided by 2. Thus we have

$$s_1 = \frac{c - c' + n + n'}{2}.$$

In fact, the second series considered as an arithmetical progression the first term of which is n' and the last n, the number of terms is $n - n' + 1$, giving

$$s_1 = \frac{(n'+n)\;(n - n' + 1)}{2};$$

which is the same as the preceding equation when n^2 and n'^2 are substituted for c and c'.

If the first square of the series $c' = 9$, the last $c = 64$, and $n' = 3$ and $n = 8$, then the sum of the series of roots is

$$s_1 = \frac{64 - 9 + 8 + 3}{2} = 33.$$

5th. To obtain the sum of the squares of the consecutive whole numbers of which the smallest is n' and the largest n, calculate, as in 3d, the sum S of the squares of the first n consecutive whole numbers, then the sum S' of the first $n' - 1$ consecutive whole numbers, and subtract S' from S, which will give the desired sum.

311. *Some of the properties of cubes of whole numbers* (**310**). Write the four following series one immediately beneath the other: *first*, the successive numbers forming an arithmetical progression, whose common difference is 6 and whose first term is 3; *second*, the successive whole numbers, n; *third*, the cubes c of these successive whole numbers; *fourth*, the sums of the successive whole numbers:

	3	9	15	21	27	33	39	45	51	57...
(n)	1	2	3	4	5	6	7	8	9	10...
(C)	1	8	27	64	125	216	343	512	729	1000...
	1	3	6	10	15	21	28	36	45	55...

1st. The cube C, in the third series, of any whole number, n, in the second series, is equal to one-third of the sum of the first n terms of the first series, multiplied by the number n of terms (3d). Thus, the cube $C = 125$ of $n = 5$ is equal to one-third 25, of the sum $s' = 75$ of the first five terms in the first series, multiplied by 5; which can be easily proved.

2d. The first series being an arithmetical progression, of which the first term is 3 and the common difference 6, the nth term t is

$$t = 3 + 6(n - 1) = 6n - 3. \qquad (\mathbf{359})$$

Thus the whole cube, 343, having 7 for a cube root, is a third of the first seven terms in the first series, multiplied by 7; and the seventh term of this series is

$$t = 6 \times 7 - 3 = 39.$$

3d. The sum s', of first n terms of the first series, considered as an arithmetical progression, is equal to one-half the product

of the sum of the first term 3 and the nth term t, and the number n of terms. Thus,

$$s' = \frac{(3+t)\,n}{2}. \tag{361}$$

Substituting for t the value found above,

$$s' = 3n^2, \quad \text{whence} \quad n^2 = \frac{s'}{3},$$

and therefore, in multiplying the two terms by n,

$$n^3 = C = \frac{s'n}{3}.$$

4th. Any cube, C, of a whole number, n, is equal to 6 times the sum of the first $n-1$ terms in the fourth series, plus the number n of terms. Thus,

$$n^3 = 8^3 = 6\,(1+3+6+10+15+21+28)+8 = 6\times 84+8 = 512.$$

5th. The sum, S, of the cubes of the n consecutive whole numbers, commencing at 1 or the first n terms of the third series, is equal to the square of half the sum of n^2, and n. Thus,

$$S = \left(\frac{n^2+n}{2}\right)^2. \qquad \text{(Algebra, Book III.)}$$

Putting $n = 8$ we have

$$S = \left(\frac{8^2+8}{2}\right)^2 = 36^2 = 1296.$$

6th. To obtain the sum of the cubes of the consecutive whole numbers, commencing with n' and ending with n, calculate, as in 5th, the sum s of the cubes of the first n consecutive numbers and the sum S' of the first $n'-1$ consecutive numbers, and then subtract the two sums, which will give the required sum.

312. *Extraction of the square root by successive additions.*

This method of operating rests upon the fact that the square of a whole number, n, increased by twice the number, n, and by 1, is equal to the square of the next larger whole number, $n+1$ (**270**).

The first three figures of the root may be taken from the table, as in (**308**), and the remaining figures calculated according to the method of successive squares, which will be sufficient to demonstrate the method so that the entire root could be obtained by its use.

Given the number 787,512.74 to extract the square root, cor-

rect to one hundredth. The operation is the same as (**282**, 2d case, 2d); that is, find the square root of 7,875,127,400, correct to 1 unit, and point off two places in the result.

The table gives 887 for the first three figures, the square, 786,769, of which is the greatest whole square contained in the number 787,512.

Writing 786,769 below, and proceeding according to the rule given in (**270**), we have:

The square of 8870	7,867,690
Twice the root 8870, plus 1	17,741
The sum or the square of 8871 . . .	78,694,641
Twice the root 8871, plus 1	17,743
The sum or the square of 8872 . . .	78,712,384
Twice the root 8872, plus 1	17,745
The sum or the square of 8873 . . .	78,730,129
Twice the root 8873, plus 1	17,747
The sum or the square of 8874	78,747,876
Twice the root 8874, plus 1	17,749
The sum or the square of 8875	78,765,625

The last square being greater than the number formed by the first four periods at the left of the given number, 8874 is the greatest whole square contained in the number, and 4 is the fourth figure of the root.

To calculate the 5th, operate in the same manner.

The square of 87,740	7,874,787,600
Twice the root 87,740, plus 1 . . .	177,481
The sum or the square of 88,741 .	7,874,965,081
Twice the root 88,741, plus 1 . . .	177,483
The sum or the square of 88,742 .	87,75,142,564

The last square being greater than the number formed by the first five periods at the left of the given number, 88,741 is the greatest whole square contained in the number, and 1 is the fifth figure of the root; pointing off, we have 887.41 as the required root.

The remainder is obtained by subtracting the largest square found, from the number formed by all the periods of the given number, with twice as many decimal places pointed off at the right as there are decimals in the root. The remainder in the above example is 16.2319. Noting that twice the roots plus one, which are successively added, increase by a common difference

of 2, it is seen that the extraction of the root is reduced to a series of very simple additions; and as for each figure of the root, the number of these additions averages 5 and is never greater than 9, it follows that in less than an hour the root of a number containing 60 figures could be extracted, which, according to the ordinary way, would take at least a half a day (271).

313. *The cube of a whole number n being given, required to find that of* $(n + 1)$. (276.)

$$(n + 1)^3 = n^3 + 3n^2 + 3n + 1.$$

Since $3n^2$ is equal to the sum s^1 of the first n terms in the first series (311, 3d), for example, to obtain the cube of 21, knowing that of 20, operate thus:

Cube of 20	8000
Sum of the terms $s = 3n^2$ or $\frac{3n^3}{n} = 3\times 20^2$ or $\frac{3\times 20^3}{20}$	1200
3 times the root $n = 20$	60
Unity	1
The cube of 21	9261

314. The cubes of two consecutive whole numbers, n and $n+1$, being given; to find that of the next consecutive number, $n + 2$.

Let d be the difference between the cubes $(n + 1)^3$ and n^3 (313).

$$d = 3n^2 + 3n + 1.$$

Writing (313)

$$(n + 2)^3 = (n + 1)^3 + 3\,(n + 1)^2 + 3\,(n + 1) + 1;$$

expanding

$$(n + 2)^3 = (n + 1)^3 + 3n^2 + 6n + 3 + 3n + 3 + 1;$$

substituting

$$\begin{aligned}(n + 2)^3 &= (n + 1)^3 + (3n^2 + 3n + 1) + 6\,(n + 1)\\ &= (n + 1)^3 + d + 6\,(n + 1).\end{aligned}$$

For example, having $20^3 = 8000$ and $21^3 = 9261$ given, to find the cube of 22, then of 23, etc., operate as follows:

Cube of 21 (313).	9261
Difference, $d = 21^3 - 20^3$ . . .	1261
$6\,(n + 1)$, or 6 times the root, 21	126
The sum or the cube of 22 . . .	10,648
Difference, $22^3 - 21^3$	1387
6 times the root 22	132
The sum or cube of 23	12,167

315. *Extraction of the cube root by successive additions.*

It follows from the two preceding articles that the cube root may be extracted by means of successive additions, as was the square root (312).

Let it be given to find the cube root, correct to one thousandth, of the number 97,062.52689. The operation resolves itself (309, 2d, 2d case) into the extraction of the cube root, to one whole unit, of the number 97,062,526,890,000, and separating three decimal figures at the right of the result. The table gives 459 as the first three figures of the root, the cube 96,702,579 of which is the largest whole cube contained in the three periods at the left.

The remaining figures are obtained as follows:

Cube of 4590	96,702,579,000
Three times the square of the root 4590	63,204,300
Three times the root 4590	13,770
Unity	1
The sum or cube of 4591 (313)	96,765,797,071
Difference between this cube and the preceding,	63,218,071
6 times the root 4591	27,546
Sum or cube of 4592 (314)	96,829,042,688
Difference between this cube and the preceding,	63,245,617
6 times the root 4592	27,552
Sum or cube of 4593	96,892,315,857
Difference between this cube and the preceding,	63,273,169
6 times the root 4593	27,558
Sum or cube of 4594	96,955,616,584
Difference between this and preceding cube . .	63,300,727
6 times the root 4594	27,564
Sum or cube of 4595	97,018,944,875
Difference between this and preceding cube . .	63,328,291
6 times the root 4595	27,570
Cube of 4596	97,082,300,736

The last cube being greater than the number formed by the first four periods of the given number, 4595 is the greatest whole cube contained in the number, and 5 is the fourth figure in the root. To get the fifth figure, operate as before; but it may be noted that in finding three times the square of 45,950, the cal-

culations may be greatly simplified by resolving the number into 45,900 and 50 (269); thus:

The square of 45,900 is obtained by writing four ciphers at the right of the square of 459, which is taken from the table.	2,106,810,000
45,900 × 50 × 2	4,590,000
Square of 50	2,500
Sum or square of 45,950	2,111,402,500
Multiplying by 3, we have 3 times the square . .	6,334,207,500

This method of calculating the square, or three times the square of a number formed by writing figures at the right of a number of which the square is known, shortens long and tedious operations, especially in extracting the cube root where the triple square of that part of the root already found is so often used (278, 309).

Continuing the example:

The cube of 45,950.	97,018,944,875,000
Triple square of the root 45,950	6,334,207,500
Three times the root 45,950 . . .	137,850
Unity	1
The cube of 45,951.	97,025,279,220,351
Difference, $\overline{45,951}^3 - \overline{45,950}^3$	6,334,345,351
6 times the root 45,951	275,706
The cube of 45,952.	97,031,613,841,408
Difference	6,334,621,057
6 times the root	275,712
The cube of 45,953.	97,037,948,738,177
Difference	6,334,896,769
6 times the root	275,718
Cube of 45,954	97,044,283,910,664
Difference	6,335,172,487
6 times the root	275,724
Cube of 45,955	97,050,619,358,875
Difference	6,335,448,211
6 times the root	275,730
Cube of 45,956	97,056,955,082,816

Continuing thus, it is found that the cube of 45,957 is greater than the number 97,062,526,890,000, formed by the first five

periods; therefore 6 is the fifth figure of the root, and pointing off, we have 45.956, the required root.

The remainder is found by subtracting from the number formed by all the periods the largest cube which is contained in that number, and separating at the right of the difference three times as many decimal figures as there are in the root. Thus the remainder in the given example is 5.571807184.

No matter how many figures there are in the root, they may all be calculated in the same manner as 5 and 6 in the above example.

It may be noted that the above operations are simply additions; thus the difference of two consecutive cubes is equal to the sum of the two numbers written between these cubes, and 6 times the root is obtained simply by adding 6 to the latter of these numbers.

SHORT METHODS OF CALCULATING THE SQUARE AND CUBE ROOT

316. To extract the mth root of a whole number, A, with an error less than one whole unit, it suffices to retain more than the mth part of the figures in A; which is more than half for the square root, and more than one-third for the cube root.

Since the error tends to decrease the root, it follows that in order to extract the root of a number correct to one whole unit, take $\frac{n+1}{m}$ figures at the left and complete the n figures by adding ciphers to this part; then extract the mth root, which will be correct to one whole unit and slightly larger than the exact value. Thus:

1st. The square root, 274, of the number 74,600, greater, by less than one whole unit, than the exact root, is in general the square root of any number containing 5 figures, the first 3 of which are 746. Likewise the square root, 88,742, of the number 7,875,120,000, greater, by less than one whole unit, than the exact root, is the square root of 7,875,127,400, correct to one whole unit (**308**).

2d. The cube root, 460, of the number 97,000,000, greater, by less than one whole unit, than the exact root, is the cube root of the number 97,062,526, correct to one unit. Likewise the cube root, 45,957, of the number 97,062,000,000,000, greater, by

less than one whole unit, than the exact root, is the cube root of the number 97,062,526,893,127, correct to one unit (309).

REMARK 1. That which has been said, applies equally to the extraction of the square, cube, or mth root of a number, correct to any given decimal (290, 292, 308, 309). Thus:

1st. The square root, 2.74, of the number 7.4600, greater, by less than one hundredth, than the exact root, is the square root of the number 7.467342, correct to one hundredth.

2d. The cube root, 45.957, of a number, 97,062.000000000, greater, by less than one thousandth, than the exact root, is the cube root of the number 97,062.52689, correct to one thousandth.

REMARK 2. From the above it follows that when the number, the root of which is to be found, has to be calculated, as is the case with fractions (290, 292), only those figures which are desired at the left need be obtained.

317. When in extracting the square root of a number, correct to a unit, more than half of the figures of the root have been obtained, the rest may be obtained by dividing the given number, less the square of that part of the root already obtained, that is, the number formed by the last remainder followed by the periods which have not been operated upon, by twice that part of the root already obtained.

Thus, in the example (308, 1st, 2d case), having obtained the first three figures of the root, the last two figures are found as shown here below:

743, the last remainder, followed by the periods which have not been operated upon, 7437, gives the number 7,437,437 as the dividend, and the quotient 41 is obtained by dividing this dividend by twice that part of the root already obtained, 88,700:

$$\begin{array}{r|l} 7\ 43\ 74\ 37 & \underline{17\ 74\ 00} \\ 0\ 34\ 14 & 41 \\ 16\ 40\ 37 & \end{array}$$

The square root thus obtained is equal to, greater or less than, the exact, according as the square of the quotient 41 is equal to, greater or less than, the remainder 164,037. Thus, in the above example, having $41^2 = 1681 < 164{,}037$, the root 88,741 is less than the exact root.

As may be seen, the quotient 41 may be obtained by writing only half the figures of the unused periods after 743 and divid-

ing the resulting number, 74,374, by twice the root already obtained, considered as simple units, 1774. Writing at the right of the remainder 1640 the figures which were not employed, the remainder 164,037 is obtained.

Applying simultaneously this rule and the one preceding:

$$\begin{array}{r|l} 7\ 43\ 00\ 00 & 17\ 74\ 00 \\ \cline{2-2} 33\ 40 & 41 \\ 15\ 66\ 00 & \end{array}$$

which gives 88,742 for the square root, greater, by less than one unit, than the exact root.

318. When in extracting the cube root of a number, correct to a whole unit, more than half of the figures plus one have been obtained, the rest may be obtained by dividing the given number, less the cube of that part of the root already obtained, that is, the number formed by writing the remaining unused periods after the remainder, by the triple square of the root already obtained.

Thus, in the example (311, 1st, 2d case), having obtained the first four figures of the root, the remaining figures are found as shown below:

Dividing the number 43,582,018,127 by the triple square 6,334,-207,500 of that part of the root already obtained, 45,950, the last figure, 6, of the root is obtained. Thus:

$$\begin{array}{r|l} 43\ 582\ 018\ 127 & 6\ 334\ 207\ 500 \\ \cline{2-2} 5\ 576\ 773\ 127 & 6 \end{array}$$

The cube root thus obtained is equal to, greater or less than, the exact, according as the product of 3 times that part of the root already obtained, plus the quotient 6 and the square of the quotient, is respectively equal to, greater or less than, the remainder 5,576,773,127. Thus in the given example,

$$(3 \times 45{,}950 + 6) \times 6^2 = 4{,}962{,}816 < 5{,}576{,}773{,}127,$$

the root is less than the exact root.

Analogous with the square root (317), the quotient 6 may be obtained by writing at the right of the last remainder, 43,582,018, one-third of the figures not employed, and dividing the resulting number 435,820,181 by 63,342,075. Writing the rest of the figures in the given number at the right of the remainder, we have the required remainder, 5,576,773,127.

Applying simultaneously this rule and the one preceding:

43 055 125 000	6 334 207 500
5 049 880 000	6

which gives 45,957 for the cube root, greater by less than one unit.

If the root should have six figures, as for the number 97,062,256,893,127,463 for example, after having determined the first four figures, 4595, the two others, 6 and 8, are obtained by the following divisions:

4 358 201 812	63 342 075	4 355 512 500	63 342 075
557 677 312	68	554 988 000	68
50 940 712		48 251 400	

The root is 459,569, greater than the exact root by less than one unit.

319. Remark. The rules in the two preceding articles apply also to the extraction of the square and cube roots of any number, correct to a given decimal, provided the number contains 2 or 3 times as many decimals as are required in the root (316, Remarks).

BOOK V

RATIOS, PROPORTIONS AND PROGRESSIONS

DEFINITIONS

321. *Ratio* is the result of the comparison of two numbers of the same kind. This comparison is made by taking the difference of the two quantities or dividing one by the other.

The *arithmetical ratio* is the difference of two quantities. Thus the arithmetical ratio of 6 and 18 is written

$$18 - 6,$$

and pronounced 18 to 6 or 18 less 6.

In the case where the second number is larger than the first, the difference is preceded by the negative sign $-$, which indicates that the quantity could not be subtracted (**31**). Thus:

$$6 - 18 = -12.$$

A *geometrical ratio* is the quotient obtained by dividing the first quantity by the second. Thus, the geometrical ratio of 18 and 6 is the quotient 3 (**207**). Written

$$18 : 6 \text{ or } \frac{18}{6},$$

and pronounced 18 is to 6, or 18 divided by 6, or the ratio of 18 to 6.

REMARK. When the word *ratio* is used alone, a geometrical ratio is always understood.

322. In the preceding arithmetical and geometrical ratios (**321**), 18 and 6 are the two *terms* of the ratio, the first term 18 is the *antecedent*, and the second 6 the *consequent.*

323. An arithmetical ratio being the difference of two quantities, the properties given in articles **28**, **34**, and **63** hold here. Thus, for example, *an arithmetical ratio is not altered by increasing or decreasing both its terms by the same number.*

Likewise a geometrical ratio being a quotient, the properties given in articles **71**, **72**, **73**, **74**, and **77** also apply here. Thus,

for example, *a geometrical ratio is unaltered when both its terms are multiplied or divided by the same number.*

324. Two equal arithmetical ratios form an *arithmetical proportion.* The ratio 8 to 4 being equal to 13 to 9, these numbers form a proportion, which is written

$$8 - 4 = 13 - 9,$$

and pronounced 8 is to 4 as 13 to 9, or 8 less 4 equals 13 less 9.

325. Likewise, two equal geometrical ratios form a *geometrical proportion.* Thus, the geometrical ratio 8 to 4 being equal to 12 to 6, these four numbers form a geometrical proportion, which is written

$$8 : 4 :: 12 : 6 \text{ or } 8 : 4 = 12 : 6 \text{ or } \frac{8}{4} = \frac{12}{6},$$

and is pronounced 8 is to 4 as 12 to 6, or 8 divided by 4 equals 12 divided by 6, or the ratio of 8 to 4 equals the ratio of 12 to 6.

REMARK 1. *Two incommensurable ratios are equal* when the antecedent of the first ratio contains a fraction, as small as desired, of its consequent, as many times as the antecedent of the second ratio contains the same fraction of its consequent (**162**, **213**).

REMARK 2. The word *proportion* used alone means geometrical proportion.

326. Four quantities are said to be *proportional* or *in proportion* when the ratio of the first to the second is equal to the ratio of the third to the fourth. Thus, given the four proportional quantities 8, 4, 12, 6; $8 : 4 = 12 : 6$. In this case the first two or the last two are in *direct* proportion to the two others.

If four quantities of a proportion are so related that an increase in one of the four causes a corresponding decrease in another, the two quantities are said to be *inversely proportional* to each other. Thus, in the proportion,

$$8 : 4 = 12 : 6,$$

the quantity 8 is *inversely proportional* to the quantity 6, while the quantity 8 is *directly proportional* to the quantity 12.

327. In any arithmetical or geometrical proportion, the antecedent of the first ratio, that of the second ratio, the consequent of the first ratio and that of the second, are called respectively the *first antecedent,* the *second antecedent,* the *first conse-*

quent, and the *second consequent.* The first and fourth terms of the proportion are called the *extremes,* and the second and third terms the *means.*

328. The fourth term of a proportion is called the *fourth proportional* of the other three terms (326). It is a *fourth arithmetical* or a *fourth geometrical,* according as the proportion is arithmetical or geometrical.

329. In an arithmetical proportion, such as

$$5 - 7 : 7 - 9,$$

where the means are equal, the term 7 is an *arithmetical mean* between the two others, 5 and 9, and the term 9 is the *third arithmetical* of the two, 5 and 7. Such a proportion is written

$$5 \cdot 7 \cdot 9.$$

330. Likewise, in a geometrical proportion,

$$4 : 12 = 12 : 36,$$

where the means are equal, the mean, 12, is the *mean proportional* of the two others, 4 and 36, and 36 is the *third proportional* of 4 and 12.

Such a proportion is written

$$4 : 12 : 36.$$

331. Remark. 1st, when the antecedents or the consequents of an arithmetical or geometrical proportion are equal to one another, the consequents or antecedents are equal; 2d, when two arithmetical or geometrical proportions have a common ratio, the ratios which are not common form a proportion, that is, are equal.

ARITHMETICAL PROPORTIONS

332. *In all arithmetical proportions the sum of the extremes is equal to that of the means.* Thus, having

$$9 - 4 = 13 - 8, \text{ we have } 9 + 8 = 4 + 13.$$

333. When the sum, $9 + 8$, of two numbers is equal to the sum, $4 + 13$, of two others, the four numbers form an arithmetical proportion in which the two numbers forming one of the sums are the extremes or the means, and the other two numbers forming the second sum are the means or extremes.

334. When four numbers are not in arithmetical proportion, the sum of the means does not equal the sum of the extremes.

335. An arithmetical proportion is not altered by: 1st, increasing or diminishing an extreme and a mean by the same quantity; 2d, dividing or multiplying all the terms by the same number. Thus, the preceding proportion gives:

$$(9 + 2) - (4 + 2) = 13 - 8, \quad (9 + 2) - 4 = (13 + 2) - 8, \quad \text{etc.,}$$
$$\text{and } (9 \times 2) - (4 \times 2) = (13 \times 2) - (8 \times 2), \quad \text{etc.}$$

336. *In any arithmetical proportion each extreme is equal to the sum of the means less the other extreme,* and *each mean is equal to the sum of the extremes diminished by the other mean.*

Thus, the proportion $8 - 4 = 13 - 9$ gives

$$8 = 4 + 13 - 9 \quad \text{and} \quad 13 = 8 + 9 - 4.$$

From this it follows that if three terms of an arithmetical proportion are known, the fourth is easily found.

337. *The arithmetical mean of two numbers,* 5 and 9, is half, 7, of the sum, 14, of these numbers:

$$5 - 7 = 7 - 9.$$

338. An arithmetical proportion may be transformed as much as desired so long as the equality between the sum of the means and that of the extremes is not destroyed (**333**). Thus, having $9 + 8 = 4 + 13$, the 8 following proportions may be constructed:

$$9-4=13-8, \quad 9-13=4-8, \quad 8-4=13-9, \quad 8-13=4-9,$$
$$4-9=8-13, \quad 4-8=9-13, \quad 13-9=8-4, \quad 13-8=9-4.$$

The remarks in (**345**) apply to arithmetical as well as to geometrical proportions.

GEOMETRICAL PROPORTIONS

339. *In all geometrical proportions the product of the extremes is equal to the product of the means.* Thus, in the proportion

$$8 : 4 = 12 : 6, \text{ we have } 8 \times 6 = 4 \times 12.$$

340. When the product, 8×6, of two numbers is equal to the product, 4×12, of two other numbers, the four numbers form a proportion, of which the two factors of one of the products are the extremes or the means, and the two factors of the other product the means or extremes.

341. When four numbers are not in proportion, the product of the means is not equal to that of the extremes.

342. A geometrical proportion is not altered by multiplying or dividing one of the extremes and one of the means by the same number. Thus, the preceding proportion gives

$$\frac{8 \times 2}{4 \times 2} = \frac{12}{6}, \qquad \frac{8 \times 2}{4} = \frac{12 \times 2}{6}, \text{ etc.}$$

343. *In any proportion, each extreme is equal* to the product of the means divided by the other extreme, and *each mean is equal* to the product of the extremes divided by the other mean. From this it follows that the fourth term, x, of the proportion,

$$6 : 2 = 24 : x, \text{ is } x = \frac{2 \times 24}{6} = 8.$$

344. *The geometrical mean, x, of two numbers,* 4 and 36, is the square root of the product of the two numbers (**330**). The proportion

$$4 : x = x : 36 \text{ gives } x^2 = 4 \times 36, \text{ or } x = \sqrt{4 \times 36} = 12.$$
$$4 : 12 = 12 : 36.$$

345. A proportion may be transformed as much as desired so long as the equality between the product of the means and that of the extremes is not destroyed. Thus, having $8 \times 3 = 2 \times 12$, the 8 following proportions may be constructed:

$$8 : 2 = 12 : 3, \quad 8 : 12 = 2 : 3, \quad 3 : 2 = 12 : 8, \quad 3 : 12 = 2 : 8,$$
$$2 : 8 = 3 : 12, \quad 2 : 3 = 8 : 12, \quad 12 : 8 = 3 : 2, \quad 12 : 3 = 8 : 2.$$

REMARKS: 1. The first four of the above proportions show that when four numbers are in proportion they will be in proportion when their means or extremes are transposed (**340**).

2. The last four of these proportions show that a proportion is not destroyed when the means and extremes are interchanged.

3. The first proportion, $8 : 2 = 12 : 3$, giving $8 : 12 = 2 : 3$, it follows that in any proportion the first antecedent is to the second antecedent as the first consequent is to the second.

346. A proportion is not destroyed by multiplying or dividing the four terms or only an extreme and a mean by the same number (**323**). Thus, having

$8:2 = 12:3$, we have also $8 \times 3 : 2 \times 3 = 12 \times 3 : 3 \times 3$.

347. *From this it follows that fractional terms may be reduced.* Thus, reduce the terms to the same denominator and suppress the denominator:

$$\frac{1}{2} : \frac{1}{6} = 2 : \frac{2}{3} \text{ gives } \frac{3}{6} : \frac{1}{6} = \frac{12}{6} : \frac{4}{6} \text{ or } 3 : 1 = 12 : 4.$$

When only one extreme or one mean is a fraction or one extreme and one mean, two terms are all that need be reduced to a common denominator (**323**, **340**):

$$\frac{2}{3} : 4 = 3 : 18 \text{ gives } \frac{2}{3} : \frac{12}{3} = 3 : 18 \text{ or } 2 : 12 = 3 : 18;$$

$$\frac{3}{4} : 18 = \frac{1}{3} : 8 \text{ gives } \frac{9}{12} : 18 = \frac{4}{12} : 8 \text{ or } 9 : 18 = 4 : 8.$$

The terms of a proportion may be simplified by multiplying or dividing the four terms or only an extreme and a mean by the same number:

$$9 : 3 = 36 : 12 \text{ gives } 3 : 1 = 12 : 4.$$

348. When two proportions have the same antecedents or the same consequents, their consequents or their antecedents are proportional (**331**, **327**):

$$3 : 9 = 15 : 45 \text{ and } 3 : 6 = 15 : 30 \text{ give } 9 : 45 = 6 : 30.$$

349. In any proportion, $8 : 4 = 6 : 3$, for example:

1st. The sum or difference of the first two terms is to the first or second term as the sum or difference of the last two terms is to the third or fourth. Thus,

$$(8 + 4) : 4 = (6 + 3) : 3 \text{ and } (8 + 4) : 8 = (6 + 3) : 6;$$
$$(8 - 4) : 4 = (6 - 3) : 3 \text{ and } (8 - 4) : 8 = (6 - 3) : 6.$$

2d. The sum of the first two terms is to the sum of the last two terms as the difference of the first two is to the difference of the last two:

$$(8 + 4) : (6 + 3) = (8 - 4) : (6 - 3);$$

or by interchanging the means:

$$(8 + 4) : (8 - 4) = (6 + 3) : (6 - 3).$$

3d. The sum or difference of the two antecedents is to the

second or first antecedent as the sum or difference of the consequents is to the second or first consequent:

$$(8 + 6) : 6 = (4 + 3) : 3 \quad \text{and} \quad (8 + 6) : 8 = (4 + 3) : 4;$$
$$(8 - 6) : 6 = (4 - 3) : 3 \quad \text{and} \quad (8 - 6) : 8 = (4 - 3) : 4.$$

4th. The sum of the antecedents is to that of the consequents as the difference of the antecedents is to that of the consequents:

$$(8 + 6) : (4 + 3) = (8 - 6) : (4 - 3).$$

5th. The sum or difference of the antecedents is to the sum or difference of the consequents as any antecedent is to its consequent:

$$(8 + 6) : (4 + 3) = 8 : 4 = 6 : 3,$$
$$(8 - 6) : (4 - 3) = 8 : 4 = 6 : 3.$$

350. When the terms of several proportions are multiplied together in order, the four products form a proportion.

Thus, having

$$4 : 2 = 6 : 3, \quad 7 : 5 = 14 : 10, \quad 3 : 9 = 6 : 18,$$

we have

$$4 \times 7 \times 3 : 2 \times 5 \times 9 = 6 \times 14 \times 6 : 3 \times 10 \times 18.$$

351. The quotients obtained by dividing, in order, the terms of one proportion by those of another, are in proportion:

$$\frac{4}{7} : \frac{2}{5} = \frac{6}{14} : \frac{3}{10}.$$

352. Similar powers and roots of the four terms of a proportion form a proportion. Thus, having $3 : 7 = 6 : 14$, we have also

$$3^3 : 7^3 = 6^3 : 14^3, \quad \text{and} \quad \sqrt{3} : \sqrt{7} = \sqrt{6} : \sqrt{14}.$$

353. In a series of equal ratios, the sum of any number of antecedents is to the sum of their consequents as any antecedent is to its consequent. Thus, having

$$3 : 6 = 4 : 8 = 7 : 14 = 5 : 10,$$

or

$$\frac{3}{6} = \frac{4}{8} = \frac{7}{14} = \frac{5}{10},$$

we have $\quad (3 + 4 + 7) : (6 + 8 + 14) = 3 : 6 = 5 : 10. \qquad (137)$

354. In a proportion, and in general in a series of equal ratios,

the square root of the sum of the squares of a certain number of antecedents is to the square root of the sum of the squares of their consequents as any antecedent is to its consequent. Thus, the above series gives

$$\sqrt{3^2 + 4^2 + 7^2 + 5^2} : \sqrt{6^2 + 8^2 + 14^2 + 10^2} = 3 : 6.$$

That which is true for the square root of the sum of the squares is true for any root, *m*th, of the sum of the *m*th powers:

$$\sqrt[3]{3^3 + 4^3 + 7^3} : \sqrt[3]{6^3 + 8^3 + 14^3} = 3 : 6.$$

355. In any proportion, the product of the antecedents is to the product of the consequents as the square of one antecedent is to the square of its consequent:

$$3 : 7 = 6 : 14 \text{ gives } 3 \times 6 : 7 \times 14 = 3^2 : 7^2.$$

356. In a series of equal ratios, the product of a certain number of antecedents is to the product of their consequents as any antecedent raised to a power of a degree equal to the number of antecedent factors is to its consequent raised to the same power:

$$3 : 6 = 4 : 8 = 7 : 14 = 5 : 10,$$
$$3 \times 4 \times 7 : 6 \times 8 \times 14 = 3^3 : 6^3 = 5^3 : 10^3.$$

ARITHMETICAL PROGRESSIONS

357. A series of numbers increasing or decreasing, such that the arithmetical ratio of each term to the term which immediately precedes it is constant (**321**), forms an *arithmetical progression.* These numbers are the *terms* of the progression, and the constant ratio of each term to the one immediately preceding is the *common difference.* Thus the numbers 4, 7, 10, 13, 16 form an *ascending arithmetical progression* of which the common difference is $7 - 4 = 3$. It is written

$$4 \cdot 7 \cdot 10 \cdot 13 \cdot 16, \qquad (a)$$

and pronounced, as 4 is to 7 is to 10 is to 13, etc.

REMARK. The same numbers written in the inverse order would form a *descending arithmetical progression:*

$$16 \cdot 13 \cdot 10 \cdot 7 \cdot 4. \qquad (b)$$

358. An arithmetical progression is not altered when all its terms are increased or decreased by the same quantity (**28**, 4th). A progression is not altered when all its terms are multiplied or

divided by the same number; but the common difference is multiplied or divided by that number (**34**, **63**).

359. According as an arithmetical progression is ascending or descending, *each term is equal to the first plus or minus the common difference, taken as many times as there are terms before the one under consideration.*

Thus, in the progression (*a*) the 5th term is $4 + (3 \times 4) = 16$, and in the progression (*b*) the third term is $16 - (3 \times 2) = 10$ (**310, 311**).

360. *The sum of two terms equally distant from the extremes* is equal to the sum of the extremes in the arithmetical progression. Thus, $4 \cdot 7 \cdot 10 \cdot 13 \cdot 16$ gives

$$4 + 16 = 7 + 13 = 10 + 10.$$

361. *The sum, s, of the terms of an arithmetical progression* is equal to the sum of the extremes, times the number of terms divided by 2. The progression above gives

$$s = \frac{4 + 16}{2} \times 5 = 50. \qquad \textbf{(310, 311)}$$

362. *To insert a certain number of arithmetical means between two given numbers,* determine the common difference in the desired progression thus: take the difference between the two given numbers and divide this difference by the number of means plus one. Having the common difference, add it to the first number, and then to the successive sums obtained, which sums are the means.

Given the numbers 4 and 28, required to insert three means between them:

The common difference is $\dfrac{28 - 4}{3 + 1} = \dfrac{24}{4} = 6$;

and adding 6 to 4 and successively to the sums, we have

$$4 \cdot 10 \cdot 16 \cdot 22 \cdot 28.$$

The same result is obtained by commencing with the larger number and subtracting the common difference.

363. When the number of arithmetical means to be inserted is equal to a power of 2 less 1, these arithmetical means may be found directly by taking an arithmetical mean between the two given numbers (**337**); then an arithmetical mean between each of the given numbers and the term which has been found, and so on.

Let it be required to insert $2^2 - 1 = 3$ means between 0 and 1. Taking the arithmetical mean 0.5 between 0 and 1, we have the progression $0 \cdot 0 \cdot 5 \cdot 1$; then inserting an arithmetical mean between each of the successive terms of this progression, the required progression is obtained:

$$0. \cdot 0.25 \cdot 0.5 \cdot 0.75 \cdot 1.$$

364. In inserting the same number of means between the consecutive terms of an arithmetical progression, the whole forms a new arithmetical progression. Inserting three means between the consecutive terms of the arithmetical progression $2 \cdot 14 \cdot 26$, we obtain the new progression,

$$2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 17 \cdot 20 \cdot 23 \cdot 26.$$

365. *The sums of the corresponding terms of several arithmetical progressions form an arithmetical progression* of which the common difference is the sum of the common differences of the several progressions the terms of which have been added. *In subtracting the terms of an arithmetical progression from the corresponding terms of another arithmetical progression, the remainders form an arithmetical progression* of which the common difference is the difference of the common differences of the given progressions.

GEOMETRICAL PROGRESSIONS

366. An ascending or descending series of numbers, such that the geometrical ratio of each one to the one which precedes it is constant, forms a *geometrical progression.* These numbers are the *terms* of the progression, and the constant ratio of each term to the one which precedes is called the *multiplier* (**321**).

Thus the numbers 2, 6, 18, 54, 162 form an ascending geometrical progression, of which the multiplier is 3. It is written

$$2 : 6 : 18 : 54 : 162,$$

and pronounced, as 2 is to 6 is to 18 is to 54, etc.

Remark. The same numbers written in an inverse order give a descending geometrical progression, of which the multiplier is $\frac{1}{3}$.

367. A geometrical progression is not altered when all its terms are multiplied or divided by the same number (**323**).

368. In an ascending or descending geometrical progression, *any term is equal to the first multiplied by the multiplier raised to a power of a degree equal to the number of terms which precede the*

term in question. Thus, in the preceding progression, the fifth term is equal to

$$2 \times 3^4 = 2 \times 81 = 162.$$

369. *The product of two terms equally distant from the extremes* is equal to the product of the extremes. The example of (**366**) gives

$$2 \times 162 = 6 \times 54 = 18 \times 18.$$

370. *The product, p, of the terms of a geometrical progression* is equal to the square root of the product of the extremes raised to a power of a degree equal to the number of terms in the progression. Thus, the above example gives

$$p = \sqrt{(2 \times 162)^5} = 1{,}889{,}568.$$

371. *The sum, s, of the terms of a geometrical progression* is obtained by subtracting the first term from the product of the last term and the multiplier and dividing this difference by the multiplier less one. The progression of (**366**) gives

$$s = \frac{(162 \times 3) - 2}{3 - 1} = 242.$$

If the progression were descending, the sum of the terms would be obtained by dividing the first term diminished by the product of the last term and the multiplier, by one less the multiplier. Thus, the progression 162 : 54 : 18 : 6 : 2 gives

$$s = \frac{162 - 2 \times \frac{1}{3}}{1 - \frac{1}{3}} = \frac{162 - \frac{2}{3}}{\frac{2}{3}} = \frac{162 \times 3 - 2}{2} = 242.$$

372. *To insert a certain number of geometrical means between two given numbers,* determine the multiplier of the progression which is desired thus: Divide the second of the numbers by the first, and extract the root, of an index equal to the number of means plus one, of the quotient. Now multiply the first number by the multiplier thus obtained, and the product will be the first mean, or the second term of the progression, which in turn multiplied by the multiplier will give the third term, and so on.

Let it be required to insert three geometrical means between the numbers 2 and 162. The multiplier is

$$\sqrt[4]{\frac{162}{2}} = \sqrt[4]{81} = \sqrt{\sqrt{81}} = \sqrt{9} = 3. \qquad \textbf{(298)}$$

Multiplying the first term, then the successive products, by 3, the following progression is obtained:

$$2 : 6 : 18 : 54 : 162.$$

373. When, as in the preceding example, the number of geometrical means to be inserted is equal to a power of 2 less 1, the means may be found by first finding a mean between the given numbers (**319**), then the mean between each of the given numbers and the mean already found, and so on. Let it be required to insert $2^2 - 1$ means between 2 and 162. Taking the geometrical mean $\sqrt{2 \times 162} = 18$, between 2 and 162, the progression $2 : 18 : 162$ is obtained. Inserting a geometrical mean between each of the consecutive terms of this progression 2 and 18, 18 and 162, the required progression is obtained:

$$2 : 6 : 18 : 54 : 162.$$

374. In inserting the same number of geometrical means between the consecutive terms of a geometrical progression, the whole forms a new geometrical progression. Thus, in inserting three means between each of the consecutive terms of the progression $1 : 81 : 6561$, the following progression is obtained:

$$1 : 3 : 9 : 27 : 81 : 243 : 729 : 2187 : 6561.$$

375. *The products of the corresponding terms of several geometrical progressions* form a new progression, of which the multiplier is equal to the product of the multipliers of the progressions.

In dividing the terms of a geometrical progression by the corresponding terms of another progression, the quotients form a geometrical progression, of which the multiplier is equal to the multiplier of the first progression divided by the multiplier of the second.

In raising all the terms of a progression to the same power, a new geometrical progression is obtained, of which the multiplier is equal to the multiplier of the given progression raised to the given power.

In extracting the same root of all the terms of a progression, another progression is obtained, of which the multiplier is equal to the same root of the multiplier of the given progression.

Table Giving Number of Days included between the Same Dates of Different Months

(*Following Year*)

	February	March	April	May	June	July	August	September	October	November	December	January	February	March	April	May	June	July	August	September	October	November	December
From Jan. to	31	59	90	120	151	181	212	243	273	304	334	365											
February		28	59	89	120	150	181	212	242	273	303	334	365										
March			31	61	92	122	153	184	214	245	275	306	337	365									
April				30	61	91	122	153	183	214	244	275	306	334	365								
May					31	61	92	123	153	184	214	245	276	304	335	365							
June						30	61	92	122	153	183	214	245	273	304	365	334						
July							31	62	92	123	153	184	215	243	274	304	335	365					
August								31	61	92	122	153	184	212	243	273	304	334	365				
September									30	61	91	122	153	181	212	242	273	303	334	355			
October										31	61	92	123	151	182	212	243	273	304	335	365		
November											30	61	92	120	151	181	212	242	273	304	334	365	
December												31	62	90	121	151	182	212	243	274	304	335	365

EXAMPLE 1. — *How many days between March 15 and July 15?* The square at the intersection of the row opposite March with the column under July contains the required number, 122.

2. *How many days between April 10 and September 25?* Find the number of days (153) between April 10 and September 10 and add $25 - 10 = 15$. Thus, $153 + 15 = 168$ days.

3. *Find the number of days between September 30 and May 10 of the following year.* From September 30 to May 30 (following year) we find 242 days, and subtracting $30 - 10 = 20$, we obtain $242 - 20 = 222$ days.

REMARK. — The table is for the ordinary year. When February of leap year comes in the interval, the one extra day must be taken into account.

BOOK VI

DIVERSE RULES

RULE OF THREE

376. A *rule of three* is a rule by which a problem may be solved, that is, an unknown value determined by means of several proportions (**325**).

377. The rule of three is *simple* when it consists in the determination of the fourth term of a proportion, of which three terms are known (**343**). If, on the contrary, the three terms are not given directly, but have to be determined by applying the rule of three several times, the rule is called the *compound rule of three.*

378. Any problem, which may be solved by the rule of three, contains two known quantities of the same kind, and two other quantities of the same kind only one of which is known.

A ratio can exist only between like quantities; and according as the ratio of the like quantities, one of which is unknown, is the direct or inverse of that of the other two (**326**), the rule of three is said to be *direct* or *inverse.*

379. *Simple direct rule of three.*

If 5 *workmen construct* 25 *meters of road, how many meters would* 7 *workmen construct in the same time?*

It is evident that the number of meters is directly proportional to the number of workmen which do the work; therefore, designating the number of meters constructed by 7 men, by x, we have (**326**):

$$5 : 7 = 25 : x, \quad \text{from which} \quad x = \frac{7 \times 25}{5} = 35 \text{ meters.}$$

This problem, or any problem involving the simple or composite rule of three, may be solved by the method of *reduction to unity,* using proportions. Thus, if 5 workmen do 25 meters of road, one man will do $\frac{25}{5} = 5$ meters in the same time, and 7 men will do seven times as much, or

$$\frac{7 \times 25}{5} = 35 \text{ meters.}$$

380. *The simple inverse rule of three* (**378**).

1*st Problem. If it takes* 20 *hours for* 4 *men to do a certain piece of work, how long would it take* 10 *men to do the same work?*

The number of hours being inversely proportional to the number of men, and letting x be the number of hours it takes 10 men to do it, we have

$$4 : 10 = x : 20, \text{ from which } x = \frac{20 \times 4}{10} = 8 \text{ hours}$$

Method of reduction to unity. Since it takes 4 men 20 hours, it would take one man 4×20 hours, and 10 men

$$\frac{20 \times 4}{10} = 8 \text{ hours.}$$

2*d Problem. How many yards of cloth* $\frac{3}{4}$ *of a yard wide will it take to line a piece* 45 *yards long and* $\frac{7}{6}$ *of a yard wide?*

The lengths being inversely proportional to the widths, we have:

$$\frac{3}{4} : \frac{7}{6} = 45 : x,$$

from which

$$x = \frac{45 \times \frac{7}{6}}{\frac{3}{4}} = \frac{45 \times 7 \times 4}{3 \times 6} = 5 \times 7 \times 2 = 70 \text{ yards.}$$

Method of reduction to unity. 45 yards of cloth $\frac{7}{6}$ of a yard wide is equivalent to $45 \times \frac{7}{6}$ yards, one yard wide, and $\frac{3}{4}$ of a yard wide would be

$$\frac{45 \times \frac{7}{6}}{\frac{3}{4}} = 70 \text{ yards long.}$$

381. *Examples of the compound rule of three* (**377**).

1*st Example.* 2 *men working* 3 *hours per day for* 5 *days, construct* 90 *yards of road; how many yards would* 3 *men working* 7 *hours per day for* 2 *days construct?*

Solution by proportions. Writing the knowns and the unknowns as follows:

2 men	3 hr.	5 da.	90 yds.
3 men	7 hr.	2 da.	x yds.

the problem may be solved by a series of simple rules of three or proportions; but it is more convenient to reduce the problem to a simple rule of three as follows:

2 men, working 3 hours a day, do as much as 2×3 men working one hour, and 2×3 men working 1 hour a day for 5 days, do as much as $2 \times 3 \times 5$ men working one hour.

Likewise, 3 men working 7 hours per day for 2 days do as much work as $3 \times 7 \times 2$ men working one hour. The problem is now: If $2 \times 3 \times 5$ men do 90 yards of construction, how many yards will $3 \times 7 \times 2$ men do in the same time?

This may be solved by a simple direct proportion, thus (**379**):

$$2 \times 3 \times 5 : 3 \times 7 \times 2 = 90 : x,$$

from which

$$x = \frac{90 \times 3 \times 7 \times 2}{2 \times 3 \times 5} = 18 \times 7 = 126 \text{ yards.}$$

The terms should be written with all their factors so as to facilitate cancellation.

Method of reduction to unity. Since 2 men, working 3 hours a day for 5 days, have made 90 yards, 1 man, working 1 hour a day for 1 day, would make $\frac{90}{2 \times 3 \times 5}$ yards, and therefore, 3 men working 7 hours a day for 2 days would make

$$\frac{90 \times 3 \times 7 \times 2}{2 \times 3 \times 5} = 126 \text{ yards.}$$

2d Example. 2 men, working 3 hours a day for 5 days, make 90 yards of road; how many days would 3 men, working 7 hours a day, have to work in order to do the same amount?

Solution by proportions.

2 men	3 hr.	5 da.	90 yds.
3 men	7 hr.	x da.	90 yds.

Proceeding as in the 1st example, the above is reduced to the simple inverse proportion:

2×3 men having taken 5 days to do a certain piece of work, how many days will it take 3×7 men to do the same work?

We have (380):

$$(2 \times 3) \cdot (3 \times 7) = x : 5, \text{ from which } x = \frac{5 \times 2 \times 3}{3 \times 7} \text{ days.}$$

Method of reduction to unity. From the problem it follows that 1 man working 1 hour a day would take $5 \times 2 \times 3$ days to do 90 yards of construction; therefore 3 men working 7 hours a day would take

$$\frac{5 \times 2 \times 3}{3 \times 7} \text{ days.}$$

3d *Example. If the men working* 7 *hours a day were obliged to make* 126 *yards of road instead of* 90 *yards,* for instance,

2 men	3 hr.	5 da.	90 yds.
3 men	7 hr.	x da.	126 yds.

the operation would have been divided into two parts, first finding the number of days it would take them to do 90 yards as was done above; and then we have: *A certain number of men working* $\frac{5 \times 2 \times 3}{3 \times 7}$ *days construct* 90 *yards of road; how many days will it take them to make* 126 *yards?* This is again a simple proportion (379):

$$90 : 126 = \frac{5 \times 2 \times 3}{3 \times 7} : x$$

$$x = \frac{5 \times 2 \times 3 \times 126}{3 \times 7 \times 90} = \frac{126}{7 \times 9} = \frac{14}{7} = 2 \text{ days.}$$

Method of reduction to unity. 1 man working 1 hour a day would take $\frac{5 \times 2 \times 3}{90}$ days to do 1 yard of work; therefore 3 men working 7 hours a day would make 126 yards in

$$\frac{5 \times 2 \times 3 \times 126}{3 \times 7 \times 90} = 2 \text{ days.}$$

382. *A general rule for solving a simple or a compound rule of three* (379, 380, 381).

The quantities which enter into the problem are like in pairs, and the ratio of the unknown to the known quantity of the same kind is equal to the product of the direct or inverse ratios of the others; thus, in the 3d problem (381) the ratios of the number of

workmen and the number of hours being inverse to that of the number of days, and that of the number of yards being direct, we have:

$$\frac{x}{5} = \frac{2}{3} \times \frac{3}{7} \times \frac{126}{90}, \text{ from which } x = 5 \times \frac{2 \times 3 \times 126}{3 \times 7 \times 90} = 2 \text{ days.}$$

INTEREST RULES

383. *Interest* is the sum paid for the use of money. The sum which draws the interest is called the *capital* or *principal.*

384. The interest on \$100 for one year is the *rate of interest.* Thus, when \$100 brings \$5 per year, the rate of interest is 5 *per cent,* which is written 5%.

Legal interest is interest according to a rate fixed by law. This differs in different states. If no rate is specified, legal rate is understood.

385. Interest is said to be *simple* when the principal remains the same throughout the duration of the loan.

386. Interest is *compound* when the interest is added to the principal at the end of each year or other fixed period and bears interest with it. Savings banks furnish an example of this kind of interest.

387. The solution of the various problems in interest depends upon the two following principles:

1st. *The simple interest on a principal is proportional to the time for which the loan is made* (326).

2d. *Two principals loaned at the same rate, for the same time, are directly proportional to their interests* (326).

388. *Problems in simple interest.*

Let C be the capital loaned, T the duration of the loan in years, I the simple interest on the principal C for the time T, and i the rate of interest; then from 1st, it follows that $i \times T$ is equal to the simple interest on \$100 for the time T, and from 2d we have the proportion

$$I : i \times T = C : 100;$$

from which:

1st. $I = \dfrac{C \times i \times T}{100}$;

2d. $i \times T = \dfrac{I \times 100}{C}$, or 4th, $i = \dfrac{I \times 100}{C \times T}$ and $T = \dfrac{I \times 100}{C \times i}$;

3d. $C = \dfrac{I \times 100}{i \times T}$

Time must always be expressed in years (229). Thus, 5 months $= T = \frac{5}{12}$, and 125 days $= T = \frac{125}{360}$. With the aid of the proportion, given above, or the 4 equations, all problems in simple interest may be solved (391, 395).

PROBLEM 1. *What is the interest, I, on* \$45,000, *loaned for* 4 *years at* 5%?

Substituting in formula 1,

$$I = \frac{45{,}000 \times 5 \times 4}{100} = \$9000.00,$$

which shows that in order *to find the interest on a principal loaned for a certain number of years, multiply the principal by the rate and by the number of years, and divide the product by* 100.

After 4 years, the *amount* is

$$C + I = 45{,}000 + 9000 = \$54{,}000.00.$$

The value of I and of $I + C$ may be found directly by the method of reducing to unity. Thus, in one year \$100 would bear \$5.00 interest, and \$1.00 would bear \$0.05; in 4 years, \$1.00 would bear \$0.05 $\times$ 4, and at the end of this time the amount would be $(1 + 0.05 \times 4)$ dollars; thus,

$$I = 45{,}000 \times 0.05 \times 4 = \$9000.00$$
$$C + I = 45{,}000\,(1 + 0.05 \times 4) = \$54{,}000.00.$$

PROBLEM 2. *What is the interest, I, on* \$45,000, *loaned at* 5% *for* 4 *years and* 3 *months?*

4 years and 3 months are $12 \times 4 + 3 = 51$ months or $\frac{51}{12}$ years; substituting in formula 1 (388):

$$I = \frac{45{,}000 \times 5 \times \frac{51}{12}}{100} = \frac{45{,}000 \times 5 \times 51}{100 \times 12} = \$9562.50.$$

Thus, to *obtain the interest on a principal loaned for a certain number of months, multiply the principal by the rate and by the number of months, and divide the product by* 1200.

At the end of 4 years 3 months the amount is

$$C + I = 45{,}000 + 9562.50 = \$54{,}562.50.$$

Proceeding as in Problem 1, the method of reducing to unity gives:

$$I = 45{,}000 \times 0.05 \times \frac{51}{12} = \$9562.50.$$

$$C + I = 45{,}000\left(1 + 0.05 + \frac{51}{12}\right) = \$54{,}562.50.$$

PROBLEM 3. *What is the interest, I, on \$45,000, loaned at 5% for 48 days?*

One day is equal to $\frac{1}{360}$ of a year, and therefore 48 days is equal to $\frac{48}{360}$ years; and substituting in formula 1 (388):

$$I = \frac{45{,}000 \times 5 \times \frac{48}{360}}{100} = \frac{45{,}000 \times 5 \times 48}{36{,}000} = \frac{45{,}000 \times 48}{7200} = \$300.$$

The expression, $\frac{45{,}000 \times 5 \times 48}{36{,}000}$, shows that in order *to calculate the interest on a loaned principal for a certain number of days, multiply the principal by the rate and by the number of days, and divide the product by* 36,000.

The expression, $\frac{45{,}000 \times 48}{7200}$, shows that *when the rate is* 5% *the interest may be obtained by multiplying the principal by the number of days and dividing the product by* 7200.

At the end of 48 days the amount is:

$$45{,}000 + I = 45{,}000 + 300 = \$45{,}300.00.$$

The method of reduction to unity (Problems 1 and 2) gives:

$$I = 45{,}000 \times 0.05 \times \frac{48}{360} = \$300.00.$$

$$C + I = 45{,}000\left(1 + 0.05 \times \frac{48}{360}\right) = \$45{,}300.00.$$

In commercial calculations of interest, the quotient, $\frac{36{,}000}{5}$,

obtained in dividing 36,000 by the rate, is called the *constant divisor*. If the rate were 6%,

$$I = \frac{45{,}000 \times 6 \times 48}{36{,}000} = \frac{45{,}000 \times 48}{6000} = \$360.00,$$

which shows that the interest is obtained by substituting the constant divisor, 6000, for 7200.

Table of Constant Divisors for the Rates in Most Common Use

Rate.	Divisor.	Rate.	Divisor.	Rate.	Divisor.	Rate.	Divisor.	Rate.	Divisor.
1	36,000	3.25	11,077	5.50	6,545	7.75	4,645	10	3,600
1.25	28,800	3.50	10,286	5.75	6,261	8	4,500	10.25	3,512
1.50	24,000	3.75	9,600	6	6,000	8.25	4,364	10.50	3,429
1.75	20,571	4	9,000	6.25	5,760	8.50	4,235	10.75	3,349
2	18,000	4.25	8,470	6.50	5,538	8.75	4,114	11	3,273
2.25	16,000	4.50	8,000	6.75	5,333	9.	4,000	11.25	3,200
2.50	14,400	4.75	7,579	7	5,143	9.25	3,892	11.50	3,130
2.75	13,091	5	7,200	7.25	4,965	9.50	3,789	11.75	3,064
3	12,000	5.25	6,857	7.50	4,800	9.75	3,692	12	3,000

In obtaining the interest, instead of dividing the product of the principal and the number of days by the constant divisor, this product may be multiplied by the reciprocal of the constant divisor, which is called the *constant multiplier*. Thus, in the preceding example:

$$\begin{aligned} I &= \frac{45{,}000 \times 48}{6000} = 45{,}000 \times 48 \times \frac{1}{6000} \\ &= 45{,}000 \times 48 \times 0.00016666\ldots = \$360.00. \end{aligned}$$

This method has been and is still used to a certain extent, but the best method is that of aliquot parts, which involves the following steps:

1st. Take one hundredth of the principal, which is equal to the interest at 6% for 60 days. The interest on \$2400.00 at 6% for 60 days is

$$I = \frac{2400 \times 60}{6000} = \frac{2400}{100} = \$24.00.$$

2d. By the method of aliquot parts, find the interest for the given number of days, knowing it for 60 days.

3d. From this interest found for 6% subtract

$$\frac{1}{6}, \quad \frac{1}{4}, \quad \frac{1}{3}, \quad \frac{1}{2},$$

according as the given rate is

5, 4.5, 4, 3.

Thus, to obtain the interest on \$2400 for 175 days at 4.5%:

Interest at 6% for	60 days	=	\$24.00
" " 6% "	60 "	=	24.00
" " 6% "	30 "	=	12.00
" " 6% "	20 "	=	8.00
" " 6% "	5 "	=	2.00
	175 "		\$70.00
One fourth of 70.			17.50
The required interest . . .			\$52.50

The quotient obtained in dividing 360 by the rate $\frac{360}{6} = 60$ is called the *base*, and expresses the number of days which the principal must be loaned in order that the interest equal one hundredth of the principal. For the following rates:

	6,	5,	4.5,	4,	3,
it is	60,	72,	80,	90,	120.

Instead of commencing with the base, 60, as above, which has the advantage of having a large number of aliquot parts, the base which corresponds to the rate given in the problem may be used. Thus, find the interest on \$2400 at 4.5% for 175 days.

Interest for 80	\$24.00
" " 80	24.00
" " 10	3.00
" " 5	1.50
Required interest	\$52.50

PROBLEM 4. *If the interest on \$45,000, placed for 4 years 3 months, is \$9562.50, what is the rate?*

Substituting in formula (2) (**388**):

$$i = \frac{9562.50 \times 100}{45{,}000 \times \frac{51}{12}} = \frac{9562.50 \times 100 \times 12}{45{,}000 \times 51} = 5\%.$$

Using the method of reduction to unity, the interest on \$1.00 for 4 years 3 months being $\frac{9562.50}{45{,}000}$ dollars, that on \$100.00 for the same time would be $\frac{9562.50 \times 100}{45{,}000}$, and for 1 year

$$\frac{9562.50 \times 100}{45{,}000} \times \frac{12}{51} = \$5.00, \text{ which is } 5\%.$$

PROBLEM 5. *For how long will the principal, \$45,000, have to be loaned at 5% in order that the interest be \$9562.50?*

Substituting in formula (2) (**388**):

$$T = \frac{9562.50 \times 100}{45{,}000 \times 5} = 4.25 \text{ yrs., or 4 yrs., 3 mos. } (\mathbf{229}).$$

PROBLEM 6. *What principal loaned for 4 years 3 months at 5% will bring \$9562.50 interest?*

Substituting in formula (3) (**388**):

$$C = \frac{9562.50 \times 100}{5 \times \frac{51}{12}} = \frac{9562.50 \times 100 \times 12}{5 \times 51} = \$45{,}000.00.$$

PROBLEM 7. *What principal must be placed at 5% to amount to \$54,562.50 in 4 years 3 months?*

In 4 years 3 months \$1.00 would bring (formula 1):

$$I = \frac{1 \times 5 \times \frac{51}{12}}{100} = \frac{5 \times 51}{1200} = \$0.2125.$$

Therefore the amount of \$1.00 placed for 4 years 3 months is \$1.2125, and the required principal is

$$\frac{54{,}562.50}{1.2125} = \$45{,}000.00.$$

389. *Problems in compound interest* (361, 365).

Problem 1. *What would be the amount of* \$45,000 *loaned for* 4 *years at* 5% *compound interest?*

At the end of one year the amount of \$1.00 would be \$1.05, and that of \$45,000,

$$45{,}000 \times 1.05.$$

This, taken as a new principal, at the end of the second year would give

$$45{,}000 \times 1.05 \times 1.05 = 45{,}000 \times \overline{1.05}^2.$$

In like manner, at the end of the third year the amount would be

$$45{,}000 \times \overline{1.05}^2 \times 1.05 = 45{,}000 \times \overline{1.05}^3,$$

and so on. From this it follows that *the amount of a principal, at the end of a whole number of years at compound interest, is equal to the principal multiplied by the amount of* \$1.00 *at the end of* 1 *year raised to a power the degree of which is equal to the number of years.* Thus, at the end of 4 years the principal \$45,000 would be

$$45{,}000 \times \overline{1.05}^4 = 45{,}000 \times 1.215506 = \$54{,}697.77.$$

If the rate had been 4.5, for example, the number 1.05 would have been replaced by 1.045.

The table given on the following pages contains, in column *a*, the successive powers of these numbers up to the 60th for the different rates of interest, that is, the successive amounts of \$1.00 from 1 to 60 years at compound interest.

To solve the foregoing problem, find the value of \$1.00 at the end of 4 years at 5%, then multiply 45,000 by that number.

Problem 2. *What principal must be placed at compound interest of* 5% *for* 4 *years in order that the amount be* \$54,697.77?

If \$1.00 amounts to $\$\overline{1.05}^4$ or \$1.215506 at the end of 4 years, then it would take as many dollars in the principal as 1.215506 is contained in the given amount, thus:

$$\frac{54{,}697.77}{1.215506} = \$45{,}000.$$

In column *b* of the tables, the principals, for different amounts at different rates and covering a period of 60 years, are given.

Thus, in the above, the principal corresponding to 4 years and 5% is 0.822703. Therefore the required principal is

$$54{,}697.77 \times 0.822703 = \$45{,}000.$$

PROBLEM 3. *What is the amount of* \$45,000 *loaned at* 5% *compound interest for* 4 *years* 3 *months?*

First find the amount at the end of 4 years as in Problem 1. Then find the simple interest at 5% for that amount, 54,697.77, taken as principal for 3 months (PROBLEM 2, **388**):

$$54{,}697.77\left(1 + 0.05 \times \frac{3}{12}\right) = \$55{,}381.49.$$

PROBLEM 4. *What principal must be placed at* 5% *compound interest for* 4 *years* 3 *months to give* \$55,381.49 *as the amount?*

At the end of 4 years \$1.00 becomes $(1.05)^4$; and at the end of 4 years 3 months \$1.00 becomes

$$\overline{1.05}^4\left(1 + 0.05 \times \frac{3}{12}\right) = \$1.2307.$$

Therefore the principal is the quotient obtained in dividing the amount 55,381.49 by the value of \$1.00 at the end of 4 years 3 months:

$$\frac{55{,}381.49}{1.2307} = \$45{,}000.$$

This problem may also be solved by using the table. Let x be the principal placed for 3 months which will give \$1.00 as the amount:

$$\$1.00 = x\left(1 + 0.05 \times \frac{3}{12}\right) = x \times 1.0125,$$

$$x = \frac{1}{1.0125}.$$

From the column b of the table, and corresponding to 5% and 4 years, the principal which will give \$1.00 as amount is found, and then the principal for 4 years 3 months is $0.822703 \times \frac{1}{1.0125}$, and the principal which will give \$55,381.49 is:

$$\frac{55{,}381.49 \times 0.822703}{1.0125} = \$45{,}000.$$

PROBLEM 5. *How long must* \$45,000 *be placed at* 5% *compound interest, in order to obtain an amount equal to* \$55,381.49 ?

The problem consists in finding how long \$1.00 would have to be placed in order to obtain the amount:

$$\frac{55{,}381.49}{45{,}000} = \$1.2307.$$

Calculating, as in Problem 1, the value of \$1.00 at the end of the first, second, third, etc., years, it is found that the duration of the loan is between 4 and 5 years. This may also be taken directly from the tables, column *a*.

At the end of 4 years \$1.00 becomes \$1.215506, and now it must be found how long it will take \$1.215506 to bear 1.2307 − 1.215506 = \$0.015194, which is done as in Problem 5 (363). The time is

$$T = \frac{0.015194 \times 100}{1.215506 \times 5} = 0.25 \text{ years or 3 months.}$$

Therefore the total duration is 4 years 3 months.

390. *Interest Tables.* The following compound interest tables contain:

1st. Column *a*, the amount of \$1.00 at the end of each year of the loan. Each value is equal to the value of \$1.00 at the end of 1 year raised to a power with an exponent equal to the duration of the loan. Thus, at the end of 4 years, at 5%, the value is $\$\overline{1.05}^4 = \$1.215506$ (PROBLEM 1, 389).

2d. Column *b*, the principal which will produce an amount equal to \$1.00 in 1, 2, 3, etc., years. For example, the principal which will produce an amount equal to \$1.00 in 7 years, at 5%, is equal to $\frac{1}{\overline{1.05}^7} = 0.710681$, that is, the value of \$1.00 divided by its value at the end of 1 year raised to the power the exponent of which is equal to the number of years (PROBLEM 2, 389).

3d. Column *c*, the amount at the end of each year where there is a yearly deposit of \$1.00. It is to be noted that the amount at the end of 5 years, at 5%, is equal to the sum 5.801913 of the first 5 values in column *a*.

4th. Column *d*, the principal which will produce a yearly income of \$1.00 per year payable during 1, 2, . . . 60 years.

Years.	3 %.				Years.	3½ %.			
	a	*b*	*c*	*d*		*a*	*b*	*c*	*d*
1	1.03	0.970874	1.03	0.970874	1	1.035	0.966184	1.035	0.966184
2	1.060900	0.942596	2.090900	1.913470	2	1.071225	0.933511	2.106225	1.899694
3	1.092727	0.915142	3.183627	2.828611	3	1.108718	0.901943	3.214943	2.801637
4	1.125509	0.888487	4.309136	3.717098	4	1.147523	0.871442	4.362466	3.673079
5	1.159274	0.862609	5.468410	4.579707	5	1.187686	0.841973	5.550152	4.515052
6	1.194052	0.837484	6.662462	5.417191	6	1.229255	0.813501	6.779408	5.328553
7	1.229874	0.813092	7.892336	6.230283	7	1.272279	0.785991	8.051687	6.114544
8	1.266770	0.789409	9.159106	7.019692	8	1.316809	0.759412	9.368496	6.873956
9	1.304773	0.766417	10.463879	7.786109	9	1.362897	0.733731	10.731393	7.607687
10	1.343916	0.744094	11.807796	8.530203	10	1.410599	0.708919	12.141992	8.316605
11	1.384234	0.722421	13.192030	9.252624	11	1.459970	0.684946	13.601962	9.001551
12	1.425761	0.701380	14.617790	9.954004	12	1.511069	0.661783	15.113030	9.663334
13	1.468534	0.680951	16.086324	10.634955	13	1.563956	0.639404	16.676986	10.302739
14	1.512590	0.661118	17.598914	11.296073	14	1.618695	0.617782	18.295681	10.920520
15	1.557967	0.641862	19.156881	11.937935	15	1.675349	0.596891	19.971030	11.517411
16	1.604706	0.623167	20.761588	12.561102	16	1.733986	0.576706	21.705016	12.094117
17	1.652848	0.605016	22.414435	13.166119	17	1.794676	0.557204	23.499691	12.651321
18	1.702433	0.587395	24.116868	13.753513	18	1.857489	0.538361	25.357180	13.189682
19	1.753506	0.570286	25.870374	14.323799	19	1.922501	0.520156	27.279682	13.709837
20	1.806111	0.553676	27.676486	14.877475	20	1.989789	0.502566	29.269471	14.212403
21	1.860295	0.537549	29.536780	15.415024	21	2.059431	0.485571	31.328902	14.697974
22	1.916103	0.521893	31.452884	15.936917	22	2.131512	0.469151	33.460414	15.167125
23	1.973587	0.506692	33.426470	16.443608	23	2.206114	0.453286	35.666528	15.620411
24	2.032794	0.491934	35.459264	16.935542	24	2.283328	0.437957	37.949857	16.058368
25	2.093778	0.477606	37.553042	17.413148	25	2.363245	0.423147	40.313102	16.481515
26	2.156591	0.463695	39.709634	17.876842	26	2.445959	0.408838	42.759060	16.890352
27	2.221289	0.450189	41.930923	18.327032	27	2.531567	0.395012	45.290627	17.285365
28	2.287928	0.437077	44.218850	18.764108	28	2.620172	0.381654	47.910799	17.667019
29	2.356566	0.424346	46.575416	19.188455	29	2.711878	0.368748	50.622677	18.035767
30	2.427262	0.411987	49.002678	19.600441	30	2.806794	0.356278	53.429471	18.392045
31	2.500080	0.399987	51.50276	20.000429	31	2.905031	0.344230	56.33450	18.736276
32	2.575083	0.388337	54.07784	20.388766	32	3.006708	0.332590	59.34121	19.068866
33	2.652335	0.377026	56.73018	20.765792	33	3.111942	0.321343	62.45315	19.390208
34	2.731905	0.366045	59.46208	21.131837	34	3.220860	0.310476	65.67401	19.700684
35	2.813862	0.355383	62.27594	21.487220	35	3.333590	0.299977	69.00760	20.000661
36	2.898278	0.345032	65.17422	21.832253	36	3.450266	0.289833	72.45787	20.290494
37	2.985227	0.334983	68.15945	22.167235	37	3.571025	0.280032	76.02890	20.570525
38	3.074783	0.325226	71.23423	22.492462	38	3.696011	0.270562	79.72491	20.841087
39	3.167027	0.315754	74.40126	22.808215	39	3.825372	0.261413	83.55028	21.102500
40	3.262038	0.306557	77.66330	23.114772	40	3.959260	0.252573	87.50954	21.355072
41	3.359899	0.297628	81.02320	23.412400	41	4.097834	0.244031	91.60737	21.599104
42	3.460696	0.288959	84.48389	23.701359	42	4.241258	0.235779	95.84863	21.834883
43	3.564517	0.280543	88.04841	23.981902	43	4.389702	0.227806	100.23833	22.062689
44	3.671452	0.272372	91.71986	24.254274	44	4.543342	0.220102	104.78167	22.282791
45	3.781596	0.264439	95.50146	24.518713	45	4.702359	0.212659	109.48403	22.495450
46	3.895044	0.256737	99.39650	24.775449	46	4.866941	0.205468	114.35097	22.700918
47	4.011895	0.249259	103.40840	25.024708	47	5.037284	0.198520	119.38826	22.899438
48	4.132252	0.241999	107.54065	25.266707	48	5.213589	0.191807	124.60185	23.091244
49	4.256219	0.234950	111.79687	25.501657	49	5.396065	0.185320	129.99791	23.276565
50	4.383906	0.228107	116.18077	25.729764	50	5.584927	0.179053	135.58284	23.455618
51	4.515423	0.221463	120.69620	25.951227	51	5.780399	0.172998	141.36324	23.628616
52	4.650886	0.215013	125.34708	26.166240	52	5.982713	0.167148	147.34595	23.795765
53	4.790412	0.208750	130.13749	26.374990	53	6.192108	0.161496	153.53806	23.957260
54	4.934125	0.202670	135.07162	26.577661	54	6.408832	0.156035	159.94689	24.113295
55	5.082149	0.196767	140.15377	26.774428	55	6.633141	0.150758	166.58003	24.264053
56	5.234613	0.191036	145.38838	26.965464	56	6.865301	0.145660	173.44533	24.409713
57	5.391651	0.185472	150.78003	27.150936	57	7.105587	0.140734	180.55092	24.550448
58	5.553401	0.180070	156.33343	27.331006	58	7.354282	0.135975	187.90520	24.686423
59	5.720003	0.174825	162.05344	27.505831	59	7.631682	0.131377	195.51688	24.817800
60	5.891603	0.169733	167.94504	27.675564	60	7.878091	0.126934	203.39497	24.944734

Years.	4%. a	4%. b	4%. c	4%. d	Years.	4½%. a	4½%. b	4½%. c	4½%. d
1	1.04	0.961539	1.04	0.961539	1	1.045	0.956938	1.045	0.956938
2	1.081600	0.924556	2.121600	1.886095	2	1.092025	0.915730	2.137025	1.872668
3	1.124864	0.888996	3.246464	2.775091	3	1.141166	0.876297	3.278191	2.748964
4	1.169859	0.854804	4.416323	3.629895	4	1.192519	0.838561	4.470710	3.587526
5	1.216653	0.821927	5.632975	4.451822	5	1.246182	0.802451	5.716892	4.389977
6	1.265319	0.790315	6.898294	5.242137	6	1.302260	0.767896	7.019152	5.157873
7	1.315932	0.759918	8.214226	6.002055	7	1.360862	0.734829	8.380014	5.892701
8	1.368569	0.730690	9.582795	6.732745	8	1 422101	0.703185	9.802114	6.595886
9	1.423312	0.702587	11.006107	7.435332	9	1.486095	0.672904	11.288209	7.268791
10	1.480244	0.675564	12.486351	8.110896	10	1.552969	0.643928	12.841179	7.912718
11	1.539454	0.649581	14.025805	8.760477	11	1.622853	0.616199	14.464032	8.528917
12	1.601032	0.624597	15.626838	9.385074	12	1.695881	0.589664	16.159913	9.118581
13	1.665074	0.600574	17.291911	9.985648	13	1.772196	0.564272	17.932109	9.682852
14	1.731676	0.577475	19.023588	10.563123	14	1.851945	0.539973	19.784054	10.222825
15	1.800944	0.555265	20.824531	11.118387	15	1.935282	0.516720	21.719337	10.739546
16	1.872981	0.533908	22.697512	11.652296	16	2.022370	0.494469	23.741707	11.234015
17	1.947900	0.513373	24.645413	12.165669	17	2.113377	0.473176	25.855084	11.707191
18	2.025817	0.493628	26.671229	12.659297	18	2.208479	0.452800	28.063562	12.159992
19	2.106849	0.474642	28.778079	13.133939	19	2.307860	0.433302	30.371423	12.593294
20	2.191123	0.456387	30.969202	13.590326	20	2.411714	0.414643	32.783137	13.007937
21	2.278768	0.438834	33.247970	14.029160	21	2.520241	0.396787	35.303378	13.404724
22	2.369919	0.421955	35.617889	14.451115	22	2.633652	0.379701	37.937030	13.784425
23	2.464716	0.405726	38.082604	14.856842	23	2.752166	0.363350	40.689196	14.147775
24	2.563304	0.390122	40.645908	15.246963	24	2.876014	0.347704	43.565210	14.495478
25	2.665836	0.375117	43.311745	15.622080	25	3.005434	0.332731	46.570645	14.828209
26	2.772470	0.360689	46.084214	15.982769	26	3.140679	0.318403	49.711324	15.146611
27	2.883369	0.346817	48.967583	16.329586	27	3.282010	0.304691	52.993333	15.451303
28	2.998703	0.333478	51.966286	16.663063	28	3.429700	0.291571	56.423033	15.742874
29	3.118651	0.320651	55.084938	16.983715	29	3.584036	0.279015	60.007070	16.021889
30	3.243398	0.308319	58.328335	17.292033	30	3.745318	0.267000	63.752388	16.288889
31	3.373133	0.296460	61.70147	17.588494	31	3.913857	0.255502	67.66625	16.544391
32	3.508059	0.285058	65.20953	17.873552	32	4.089981	0.244500	71.75623	16.788891
33	3.648381	0.274094	68.85791	18.147646	33	4.274030	0.233971	76.03026	17.022862
34	3.794316	0.263552	72.65223	18.411198	34	4.466362	0.223896	80.49662	17.246758
35	3.946089	0.253416	76.59831	18.664613	35	4.667348	0.214254	85.16397	17.461012
36	4.103933	0.243669	80.70225	18.908282	36	4.877378	0.205028	90.04134	17.666041
37	4.268090	0.234297	84.97034	19.142579	37	5.096860	0.196199	95.13821	17.862240
38	4.438813	0.225285	89.40915	19.367864	38	5.326219	0.187750	100.46442	18.049990
39	4.616366	0.216621	94.02552	19.584485	39	5.565899	0.179666	106.03032	18.229656
40	4.801021	0.208289	98.82654	19.792774	40	5.816365	0.171929	111.84669	18.401584
41	4.993061	0.200278	103.81960	19.993052	41	6.078101	0.164525	117.92479	18.566110
42	5.192784	0.192575	109.01238	20.185627	42	6.351615	0.157440	124.27640	18.723550
43	5.400495	0.185168	114.41288	20.370795	43	6.637438	0.150661	130.91384	18.874210
44	5.616515	0.178046	120.02939	20.548841	44	6.936123	0.144173	137.84997	19.018383
45	5.841176	0.171198	125.87057	20.720040	45	7.248248	0.137964	145.09821	19.156347
46	6.074823	0.164614	131.94539	20.884654	46	7.574420	0.132023	152.67263	19.288371
47	6.317816	0.158283	138.26321	21.042936	47	7.915268	0.126338	160.58790	19.414709
48	6.570528	0.152195	144.83373	21.195131	48	8.271456	0.120898	168.85936	19.535607
49	6.833349	0.146341	151.66708	21.341472	49	8.643671	0.115692	177.50303	19.651298
50	7.106683	0.140713	158.77377	21.482185	50	9.032636	0.110710	186.53567	19.762008
51	7.390951	0.135301	166.16472	21.617485	51	9.439105	0.105942	195.97477	19.867950
52	7.686589	0.130097	173.85131	21.747582	52	9.863865	0.101380	205.83863	19.969330
53	7.994052	0.125093	181.84536	21.872675	53	10.307739	0.097015	216.14637	20.066345
54	8.313814	0.120282	190.15917	21.992957	54	10.771587	0.092837	226.91796	20.159182
55	8.646367	0.115656	198.80554	22.108612	55	11.256308	0.088839	238.17427	20.248021
56	8.992222	0.111207	207.79776	22.219819	56	11.762842	0.085014	249.93711	20.333034
57	9.351910	0.106930	217.14967	22.326749	57	12.292170	0.081353	262.22928	20.414387
58	9.725987	0.102817	226.87566	22.429567	58	12.845318	0.077849	275.07460	20.492236
59	10.115026	0.098863	236.99069	22.528430	59	13.423357	0.074497	288.49795	20.566733
60	10.519627	0.095060	247.51031	22.623490	60	14.027408	0.071289	302.52536	20.638022

Years.	5%.				Years.	6%.			
	a	b	c	d		a	b	c	d
1	1.05	0.952381	1.05	0.952381	1	1.06	0.943396	1.06	0.943396
2	1.102500	0.907030	2.152500	1.859410	2	1.123600	0.889996	2.183600	1.833393
3	1.157625	0.863838	3.310125	2.723248	3	1.191016	0.839619	3.374616	2.673012
4	1.215506	0.822703	4.525631	3.545951	4	1.262477	0.792094	4.637093	3.465106
5	1.276282	0.783526	5.801913	4.329477	5	1.338226	0.747258	5.975319	4.212364
6	1.340096	0.746215	7.142008	5.075692	6	1.418519	0.704961	7.393838	4.917324
7	1.407100	0.710681	8.549109	5.786373	7	1.503630	0.665057	8.897468	5.582381
8	1.477455	0.676839	10.026564	6.463213	8	1.593848	0.627412	10.491316	6.209794
9	1.551328	0.644609	11.577893	7.107822	9	1.689479	0.591899	12.180795	6.801692
10	1.628895	0.613913	13.206787	7.721735	10	1.790848	0.558395	13.971643	7.360087
11	1.710339	0.584679	14.917127	8.306414	11	1.898299	0.526788	15.869941	7.886875
12	1.795856	0.556837	16.712983	8.863252	12	2.012196	0.496969	17.882138	8.383844
13	1.885649	0.530321	18.598632	9.393573	13	2.132928	0.468839	20.015066	8.852683
14	1.979932	0.505068	20.578564	9.898641	14	2.260904	0.442301	22.275970	9.294984
15	2.078928	0.481017	22.657492	10.379658	15	2.396558	0.417265	24.672528	9.712249
16	2.182875	0.458112	24.840366	10.837770	16	2.540352	0.393646	27.212880	10.105895
17	2.292018	0.436297	27.132385	11.274066	17	2.692773	0.371364	29.905653	10.477260
18	2.406619	0.415521	29.539004	11.689587	18	2.854339	0.350344	32.759992	10.827604
19	2.526950	0.395734	32.065954	12.085321	19	3.025600	0.330513	35.785591	11.158117
20	2.653298	0.376890	34.719252	12.462210	20	3.207135	0.311805	38.992727	11.469921
21	2.785963	0.358942	37.505214	12.821153	21	3.399564	0.294155	42.392290	11.764077
22	2.925261	0.341850	40.430475	13.163003	22	3.603537	0.277505	45.995828	12.041582
23	3.071524	0.325571	43.501999	13.488574	23	3.819750	0.261797	49.815577	12.303379
24	3.225100	0.310068	46.727099	13.798642	24	4.048935	0.246979	53.864512	12.550358
25	3.386355	0.295303	50.113454	14.093945	25	4.291871	0.232999	58.156383	12.783356
26	3.555673	0.281241	53.669126	14.375185	26	4.549383	0.219810	62.705766	13.003166
27	3.733456	0.267848	57.402583	14.643034	27	4.822346	0.207368	67.528112	13.210534
28	3.920129	0.255094	61.322712	14.898127	28	5.111687	0.195630	72.639798	13.406164
29	4.116136	0.242946	65.438848	15.141074	29	5.418388	0.184557	78.058186	13.590721
30	4.321942	0.231377	69.760790	15.372451	30	5.743491	0.174110	83.801677	13.764831
31	4.538039	0.220360	74.29883	15.592811	31	6.088101	0.164255	89.88978	13.929086
32	4.764941	0.209866	79.06377	15.802677	32	6.453387	0.154957	96.34317	14.084043
33	5.003189	0.199873	84.06696	16.002549	33	6.840590	0.146186	103.18376	14.230230
34	5.253348	0.190355	89.32031	16.192904	34	7.251025	0.137912	110.43478	14.368141
35	5.516015	0.181290	94.83632	16.374194	35	7.686087	0.130105	118.12087	14.498246
36	5.791816	0.172657	100.62814	16.546852	36	8.147252	0.122741	126.26812	14.620987
37	6.081407	0.164436	106.70955	16.711287	37	8.636087	0.115793	134.90421	14.736780
38	6.385477	0.156605	113.09502	16.867893	38	9.154252	0.109239	144.05846	14.846019
39	6.704751	0.149148	119.79977	17.017041	39	9.703507	0.103056	153.76197	14.949075
40	7.039989	0.142046	126.83976	17.159086	40	10.285718	0.097222	164.04768	15.046297
41	7.391988	0.135282	134.23175	17.294368	41	10.902861	0.091719	174.95055	15.138016
42	7.761588	0.128840	141.99334	17.423208	42	11.557033	0.086527	186.50758	15.224543
43	8.149667	0.122704	150.14301	17.545912	43	12.250455	0.081630	198.75803	15.306173
44	8.557150	0.116861	158.70016	17.662773	44	12.985482	0.077009	211.74351	15.383182
45	8.985008	0.111297	167.68516	17.774070	45	13.764611	0.072650	225.50813	15.455832
46	9.434258	0.105997	177.11942	17.880067	46	14.590487	0.068538	240.09861	15.524370
47	9.905971	0.100949	187.02539	17.981016	47	15.465917	0.064658	255.56453	15.589028
48	10.401270	0.096142	197.42666	18.077158	48	16.393872	0.060998	271.95840	15.650027
49	10.921333	0.091564	208.34800	18.168722	49	17.377504	0.057546	289.33591	15.707572
50	11.467400	0.087204	219.81540	18.255926	50	18.420154	0.054288	307.75606	15.761861
51	12.040770	0.083051	231.85617	18.338977	51	19.525364	0.051215	327.28142	15.813076
52	12.642808	0.079096	244.49897	18.418073	52	20.696885	0.048316	347.97831	15.861393
53	13.274949	0.075330	257.77392	18.493403	53	21.938698	0.045582	369.91701	15.906974
54	13.938696	0.071743	271.71262	18.565146	54	23.255020	0.043002	393.17203	15.949976
55	14.635631	0.068326	286.34825	18.633472	55	24.650322	0.040567	417.82235	15.990543
56	15.367412	0.065073	301.71566	18.698545	56	26.129341	0.038271	443.95169	16.028814
57	16.135783	0.061974	317.85144	18.760519	57	27.697101	0.036105	471.64879	16.064919
58	16.942572	0.059023	334.79402	18.819542	58	29.358927	0.034061	501.00772	16.098980
59	17.789701	0.056212	352.58372	18.875754	59	31.120463	0.032133	532.12818	16.131113
60	18.679186	0.053536	371.26290	18.929290	60	32.987691	0.030314	565.11587	16.161428

BOOK VII

LOGARITHMS

391. *Definition.* When two progressions,

$$\cdots \frac{1}{81} : \frac{1}{27} : \frac{1}{9} : \frac{1}{3} : 1 : 3 : 9 : 27 : 81 \cdots$$
$$\cdots -8 \cdot -6 \cdot -4 \cdot -2 \cdot 0 \cdot 2 \cdot 4 \cdot 6 \cdot 8 \cdots$$

one, geometrical and containing the term 1; and the other arithmetical and containing the term 0, are written one beneath the other so that the terms 0 and 1 come in the same column (**332** and **341**), then each term of the arithmetical progression is the logarithm of the corresponding term of the geometrical progression. Thus the *logarithm* of 27, which is written *log* 27, is equal to 6 or log 27 = 6.

392. The multiplier of the geometrical progression is the *base* of the system of logarithms.

393. Instead of considering logarithms as the terms of a progression, they may be considered as degrees of a power of a constant number. This constant number is the base of the system, and any power of this base has the degree of the power for its logarithm. Thus, $3^2 = 9$, $3^3 = 27$, $3^0 = 1$, $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$ (**305**), have respectively 2, 3, 0, and -2 for logarithms in the system whose base is 3.

394. *Common logarithms.* The base of this system is 10. The system was first published by Henry Briggs, and is sometimes called the Briggs system. In this system the two progressions of (**391**) are replaced by

$$\cdots \frac{1}{10{,}000} : \frac{1}{1000} : \frac{1}{100} : \frac{1}{10} : 1 : 10 : 100 : 1000 : 10{,}000 : 100{,}000 \cdots$$
$$\cdots -4 \cdot -3 \cdot -2 \cdot -1 \cdot 0 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots$$

Considering the logarithms as exponents as in (**393**), we have

$$\cdots 10^{-4} \quad 10^{-3} \quad 10^{-2} \quad 10^{-1} \quad 10^{0} \quad 10^{1} \quad 10^{2} \quad 10^{3} \quad 10^{4} \quad 10^{5} \cdots$$

which means, according to the definition (**391**),

$$\log 1 = 0; \quad \log 10 = 1; \quad \log 100 = 2; \quad \log 1000 = 3, \text{ etc.}$$

$$\log \frac{1}{10} = -1; \qquad \log \frac{1}{100} = -2; \qquad \log \frac{1}{1000} = -3, \text{ etc.}$$

395. *How the two fundamental progressions can give the logarithms of all the numbers.*

This series of powers infinitely prolonged in both directions, or the two progressions continued in the same manner, give only the numbers which have whole, positive, or negative numbers for logarithms; but as many geometrical means may be inserted between the terms of the geometrical progression as desired, and in this manner, by inserting an equal number of arithmetical means between the terms of the arithmetical progression, the terms of the new arithmetical progression are the logarithms of the corresponding terms of the geometrical progression. Thus the logarithms of any number may be found (263 and 273).

Likewise, numbers, which differ from one another by an infinitely small amount, may be taken as exponents in the preceding series, and the successive powers will differ from one another also by an infinitely small amount.

Thus it is seen that any given number may be a term of the geometrical progression or one of the powers in the series given above, and that its logarithm is the corresponding term of the arithmetical progression, or the exponent of the power. Likewise any given number may be a term of the arithmetical series or an exponent of a power, and is the logarithm of the corresponding term of the geometrical progression or of the power.

Thus any positive number has a logarithm, and any number, positive or negative, is the logarithm of a positive number.

It is evident that a table cannot be constructed which contains all the numbers, neither as numbers nor as logarithms, but there are tables which contain enough so that the differences between the successive numbers are so small that the values obtained may be considered exact.

396. *The properties of a system of logarithms.* The properties given below for the common system hold true for any system when the base of the given system is substituted for the base 10. Considering the two progressions or the powers of the base (394), we have:

1st. The logarithm of the base 10 is unity.

2d. The logarithm of unity is zero.

3d. The logarithm of a number greater than unity is positive.

4th. The logarithm of a number less than unity is negative.

5th. A negative number has no logarithm.

6th. The logarithm of the product of several factors, $10^{-2} = \frac{1}{100}$, $10^1 = 10$, and $10^4 = 10{,}000$, is equal to the sum, $-2+1+4=3$, of the logarithms of the factors:

$\log(10^{-2} \times 10^1 \times 10^4) = \log 10^{-2+1+4} = \log 10^3 = -2+1+4=3$ (296).

The logarithm 3 corresponds to $10^3 = 1000$, that is, 1000 is the product of the factors $\frac{1}{100}$, 10 and 10,000.

Thus, multiplication is accomplished by aid of addition.

7th. The logarithm of a power, $(10^2)^3$, of a number, $10^2 = 100$, is equal to the logarithm 2 of the number multiplied by the degree 3 of the power:

$$\log(10^2)^3 = \log 10^{2\times 3} = 2 \times 3 = 6. \quad (297)$$

The logarithm 6 corresponds to $10^6 = 1{,}000{,}000$, that is, $100^3 = 1{,}000{,}000$.

Therefore a number may be raised to any power by a simple multiplication.

8th. The logarithm of the quotient obtained by dividing one number, $10^5 = 100{,}000$, by another, $10^2 = 100$, is the logarithm 5 of the dividend less the logarithm 2 of the divisor:

$$\log \frac{10^5}{10^2} = \log 10^{5-2} = 5 - 2 = 3. \quad (305)$$

3 being the logarithm of 1000, $1000 = \frac{100{,}000}{100}$, and it is seen that *a division may be performed by means of a subtraction.*

9th. The logarithm of a root of a number, 10^6, is equal to the logarithm 6 of the number divided by the index 2 of the root:

$$\log \sqrt{10^6} = \log 10^{\frac{6}{2}} = \log 10^3 = \frac{6}{2} = 3. \quad (306)$$

The logarithm 3 corresponds to 1000, that is,

$$\sqrt[2]{1{,}000{,}000} = 1000.$$

Therefore roots may be extracted by means of a simple division.

10th. According as a number lies between 1 and 10, 10 and 100, 100 and 1000, etc., its logarithm lies respectively between

0 and 1, 1 and 2, 2 and 3, etc.; from which it follows that *since the logarithms are expressed in decimals, the whole part of the logarithm of a whole number or a decimal number greater than unity, contains as many units less one as there are figures in the whole part of the given number.* Thus the whole part is 3 for the number 4725, and 2 for the number 827.34.

Likewise, for a number lying between 1 and 0.1, 0.1 and 0.01, 0.01 and 0.001, etc., whose logarithm lies between 0 and -1, -1 and -2, -2 and -3, etc., *the whole part of a negative logarithm of a decimal number less than unity, contains as many units as there are ciphers between the decimal point and the first significative figure in the given number.*

Thus the whole part is 0 for the number 0.236 and -2 for the number 0.00326.

397. The whole part of a positive or negative logarithm is called the *characteristic*, and the decimal part is called the *mantissa.*

398. The *logarithm of a number multiplied or divided by a power of* 10. From (**396**) it follows that knowing the logarithm of a number, in order to find the logarithm of a product or quotient of the given number and unity followed by several ciphers, it suffices to increase or decrease the given logarithm by as many units as there are ciphers at the right of the 1.

Thus, having

$$\log 68 = 1.8325089, \text{ we have } \log 6800 = 3.8325089,$$

and having

$$\log 5657 = 3.7525862, \text{ we have } \log 5.657 = 0.7525862.$$

In fact (**396**, 6th and 8th):

$$\log (68 \times 100) = \log 68 + \log 100 = \log 68 + 2,$$

$$\log \frac{5657}{1000} = \log 5657 - \log 1000 = \log 5657 - 3.$$

Thus it is seen that when the logarithm is increased or diminished by one or several units, the result is the logarithm of the product or the quotient of the given number and a power of 10 of a degree equal to the number of units by which the given logarithm has been increased or diminished.

It is also seen that the logarithms, of the products or quotients of a certain number and the different powers of 10, differ only in the characteristic, which is increased or decreased by as many

units as there are units in the exponents of the powers of 10; the mantissa remains the same.

399. From what was said in (**398**) it follows that in order *to determine the logarithm of a decimal number*, neglect the decimal point and take the logarithm of the number, and subtract as many units from characteristic as there are decimal figures in the given number. Thus, having $18.27 = \frac{1827}{100}$ (**396**, 8th), we have:

$$\log 18.27 = \log 1827 - 2 = 3.2617385 - 2 = 1.2617385.$$

Likewise, having $0.826 = \frac{826}{1000}$, we have

$$\log 0.826 = \log 826 - 3 = 2.91698005 - 3.$$

400. *Logarithm of which the characteristic alone is negative.* The logarithm of 826 being less than 3, it is seen, as was shown in (**396**), that *the logarithm of* 0.826, *and in general of any number less than one, is negative.* To express the value of the logarithm of 0.826, subtract 2.91698005 from 3 and place the negative sign $-$ before the result. Thus:

$$\log 0.826 = -(3 - 2.91698005) = -0.08301995.$$

It is convenient not to have the mantissa negative (**405**). In order to obtain this, subtract only the characteristics 2 and 3, and take 1 for the characteristic and write the negative sign above it to indicate that it alone is negative. Thus:

$$\log 0.826 = \bar{1}.91698005.$$

Likewise,

$$\log 0.0826 = \bar{2}.91698005, \text{ and } \log 0.00826 = \bar{3}.91698005.$$

Thus the number of negative units in the characteristic is equal to the order of the first significative figure after the decimal point.

401. *The complement of a positive number* is that number which, if added to the given number, would give a whole number equal to unity followed by as many ciphers as there are figures in the whole part of the given number.

Thus we have:

$$c^t\ 375.8762 = 1000 - 375.8762 = 624.1238.$$

The complement of a positive number is easily obtained: subtract each of the significative figures except the last from 9,

and the last from 10, and place as many ciphers at the right of the number obtained as there are at the right of the given number:

$$c^t\ 587{,}300 = 412{,}700.$$

As the whole part of a logarithm generally does not contain more than one figure, *the complement of a positive logarithm* is the result obtained in subtracting the logarithm from 10. Thus,

$$c^t \log 826 = 10 - 2.91698005 = 7.08301995.$$

Since it is so easy to obtain the complement, in operations where there is a logarithm to be subtracted, add it to its complement and subtract 10 from the result. Thus:

Having $\dfrac{127 \times 39}{826}$, instead of writing

$$\begin{aligned}\log \frac{127 \times 39}{826} &= \log 127 + \log 39 - \log 826 \\ &= 2.10380372 + 1.59106461 - 2.91698005 \\ &= 0.77788828\end{aligned}$$

it is written thus:

$$\begin{aligned}\log 127 &= 2.10380372 \\ \log\ \ 39 &= 1.59106461 \\ c^t \log 826 &= \underline{7.08301995} \\ &\ 0.77788828\end{aligned}$$

The required result is the number 5.9964, corresponding to the logarithm 0.77788828 (see Rule 31).

402. *Logarithmic tables.* There are many logarithmic tables. The smaller ones give the logarithms of all the whole numbers up to 10,000; the larger ones up to 108,000. Often the characteristics are omitted, as they are easily supplied (**397**, 10th).

The logarithms of the numbers between 1 and 10, 10 and 100, etc., being incommensurable, it is impossible to put their exact values in the tables. In Callet's tables the values are given to 8 decimal places for the whole numbers less than 1200 and those between 100,000 and 108,000, and to 7 decimal places for the numbers between 1200 and 100,000 (**176**). The tables by Jerome Lalande give the logarithms of all the whole numbers up to 10,000, correct to 5 decimal places. M. Marie has carried this table to 8 decimals for the numbers up to 990 and from there to 10,000 to 7 places. The tables have the numbers in the first column, the logarithms in the second, and the difference of the consecutive logarithms in the third.

Supposing that we have a large table of logarithms at our

disposal, that of Lalande for example, we will solve the following problems:

403. PROBLEM 1. *Find the logarithm of a given number:*

1st. Of a whole number, 847, which may be found in the table, that is less than 10,000. Looking in the first column, the number 847 is found; then in the same horizontal line in the second column will be found the logarithm 292,788,341.

2d. *Of a whole number,* 487,346, *which is not found in the table.* Separate on the right of the number just enough decimal figures so that the part on the left will be the largest possible number less than 10,000, the upper limit of the table. Thus, having 487,346 = 4873.46 × 100, we have (**398** and **399**):

$$\log 487{,}346 = \log 4873.46 + \log 100 = \log 4873.46 + 2,$$

which reduces to finding the logarithm of 4873.46. The number 4873.46 lies between 4873 and 4874, and therefore its logarithm lies between the tabular values 3.6877964 and 3.6878855. To obtain the quantity x which must be added to the log 4873 in order to get that of 4873.46, take the difference 0.0000891 between the logarithms of 4873 and 4874, as found in the third column; this difference represents a difference of unity in the numbers; therefore for the difference 4873.46 − 4873 = 0.46, assuming that the differences of the logarithms are proportional to the differences of the numbers, for such small values, we have

$$x = 0.0000891 \times 0.46 = 0.0000410.$$

Therefore $\log 4873.46 = 3.677964 + 0.0000410 = 3.6878374,$

and $\log 487346 = 5.6878374.$

In this manner the logarithm of any number may be obtained.

Callet's table gives, besides the differences, the nearest approximate values of the products of this difference and the first 9 multiples of 0.1, retaining 7 decimals, which greatly shortens the calculation of x. Thus, to obtain the product of 891 ten millionths and 0.46, since 891 × 0.46 = 891 × 0.4 + 891 × 0.06 (**33**), taking 356 ten millionths in the column under 891 and at the right of 4 as the product of 891 and 0.4, and then 535 ten millionths opposite 6 as the product of 891 and 0.6 or 54 ten millionths as the product of 891 and 0.06, $x = 0.0000356 + 0.0000054 = 0.0000410$.

	891
1	89
2	178
3	267
4	356
5	445
6	535
7	624
8	713
9	802

The calculations for the preceding example are written as follows:

Number 487,346

$$\begin{array}{lll} \log 4873 & = & 3.6877964 \\ \text{for} \quad 0.4 & & 356 \\ \text{for} \quad 0.06 & & 54 \\ \hline \log 4873.46 & = & 3.6878374 \\ \log 487\,346 & = & 5.6878374 \end{array}$$

Assuming proportionality between the increments of the numbers and the logarithms does not permit of the use of more than two decimals, and even these two are not exact.

3d. *Of a fraction* $\frac{7}{4}$. According to (**396**, 8th), we have:

$$\log \frac{7}{4} = \log 7 - \log 4 = 0.84509804 - 0.60205999 = 0.24303805.$$

If the fraction was less than unity, the logarithm of its denominator would be larger than that of its numerator, therefore the sign would be negative. Thus, according to (**400**),

$$\log \frac{24}{47} = \log 24 - \log 47 = 1.38021124 - 1.67209786 = -0.29188662,$$

or $$-1 + 1 - 0.29188662 = \bar{1}.70811338.$$

4th. *Of a decimal.* A decimal number may be considered as a fraction whose numerator is the given number, omitting the decimal point, and whose denominator is unity followed by as many ciphers as there are decimal figures in the given number. The rule given in (**399**) is deduced from Problem 1, 3d. Thus we have,

$$\log 4.873 = \log 4873 - 3 = 3.6877964 - 3 = 0.6877964.$$

Likewise,

$$\log 0.0487346 = \log 487{,}346 - 7 = 5.6878374 - 7 = \bar{2}.6878374.$$

404. Problem 2. *To find the number corresponding to a given logarithm.*

1st. *When the given logarithm can be found in the table,* the corresponding number is found in the column at the left. Thus the number which has 1.91907809 for a logarithm is 83.

2d. *When a logarithm differs only in the characteristic from a logarithm given in the table,* multiply or divide the corresponding number by 1 followed by as many ciphers as the number of units in the given logarithm exceeds or is exceeded by that in the logarithm found in the table. Thus, to find the number whose logarithm is 4.91907809, we find 8300 in the table whose logarithm is 3.91907809, and multiplying by 10 we have 83,000 whose

logarithm is 4.91907809. The same result would have been obtained if the log of 830 or 83 had been found, which are respectively 2.91907809 and 1.91907809.

3d. *When the given logarithm cannot be found in the tables,* and its characteristic is the largest in the table, as, for example, 3.2733127, find between what logarithms the given logarithm lies, in this case, between 3.2732328 and 3.2734643, and the number corresponding to the given logarithm lies between 1876 and 1877. Evidently the whole part of this number is 1876; to obtain the decimal part x, take the difference 0.0002315, given in the third column, between the logarithms of 1876 and 1877; then find the difference between 3.2733127 − 3.2732328 = 0.0000799, the given logarithm and the next lower found in the table. The difference of the numbers being 1 for 0.0002315, for a difference of 0.0000799 it will be,

$$x = \frac{0.0000799}{0.0002315} = \frac{799}{2315} = 0.345.$$

The number whose logarithm is 3.2733127 is therefore 1876.345.

The products of the difference 2315 and the first 9 multiples of 0.1, given in Callet's table (403, 2d), may be used to shorten the above operation. Thus, in taking 694, the largest difference which is not greater than 799, the figure 3 at the left is the tenths figure of the required number. Taking the difference 799 − 694 = 105, the product 926 × 0.1 = 92.6 being the largest difference contained in 105, the figure 4 is the hundredths figure in the required number. Now taking the difference 105 − 92.6 = 12, the product 1157 × 0.01 = 11.57 is the largest difference contained in 12, and gives 5 as the thousandths figure.

	2315
1	231
2	463
3	694
4	926
5	1157
6	1389
7	1620
8	1852
9	2083

Therefore, $x = 0.345$.

The calculations may be tabulated thus:

log	3.2733127	
for	3.2732328	1876
1st remainder	799	
for	694	0.3
2d remainder .	105	
for	93	0.04
3d remainder .	12	
for	12	0.005
Number . .		1876.345

Assuming proportionality between the increments of the logarithms and the numbers, only two decimals can be taken as exact and the third as an approximation. If the table gives 5 decimals, then not more than one should be counted on in the above calculation.

4th. *When the given logarithm cannot be found in the table, and its characteristic is not the largest in the table,* reduce the characteristic to 3, the largest in the table, by adding or subtracting the proper number of units, and proceed as in the preceding 3d example. The characteristic is reduced to 3 so as to have the largest number of figures possible. The decimal point in the number found is moved to the right or left as many places as there were units subtracted from or added to the given logarithm. Thus, to find the number whose logarithm is 1.2733127, reduce the characteristic to 3 by adding 2, and proceeding as in 3d we have the corresponding number 1876.345; dividing this by 100, we have 18.76345, or the number corresponding to the given logarithm.

5th. *When the given logarithm is entirely negative,* add enough units to make it entirely positive, and to give it the largest characteristic 3 in the table. Find the number corresponding to the resulting logarithm, and move the decimal point to the left as many places as there were units added to the characteristic of the given logarithm. Thus, to find the number whose logarithm is -2.3121626, add 6 units to this logarithm, which gives 3.6878374. The number corresponding to the latter is 4873.46, therefore the number corresponding to the given logarithm is 0.00487346.

6th. *When only the characteristic of the given logarithm is negative,* add enough units to the characteristic to make it positive and equal to the largest characteristic 3 in the table; find the number corresponding to the resulting logarithm, and move the decimal point as many places to the left as there were units added to the given characteristic, and the number thus obtained will correspond to the given logarithm.

Thus, to find the number corresponding to the logarithm $\bar{2}.6878374$, add 5 units to the characteristic -2, which gives 3.6878374, and the corresponding number is 4873.46; moving the decimal point 5 places to the left, we have the number 0.0487346, corresponding to the given logarithm.

405. *The use of logarithms.*

1st. *To multiply* 5736 by 743 (**396**, 6th).

$$\log (5736 \times 743) = \log 5736 + \log 743 = 3.7586091 + 2.8709888 = 6.6295979.$$

The number 4,261,848 which corresponds to this logarithm is the required product.

2d. *To divide* 4,261,848 by 743 (**396**, 8th):

$$\log \left(\frac{4{,}261{,}848}{743}\right) = \log 4{,}261{,}848 - \log 743$$
$$= 6.6295979 - 2.8709888 = 3.7586091.$$

The number 5736 which corresponds to this logarithm is the required quotient.

3d. *Raise a number* 17 *to the third power* (**396**, 7th).

$$\log (17^3) = 3 (\log 17) = 3 \times 1.23044892 = 3.69134676.$$

The number 4913 which corresponds to this logarithm is the cube of 17.

Calculate the cube of $\dfrac{0.042}{0.529}$.

$$\log \left(\frac{0.042}{0.529}\right)^3 = (\log 0.042 - \log 0.529) \times 3$$
$$= (\bar{2}.6232493 - \bar{1}.7234557) \times 3 = \bar{2}.8997936 \times 3 = \bar{4}.6993808;$$

then

$$\left(\frac{0.042}{0.529}\right)^3 = 0.00050047.$$

In this example the logarithm $\bar{2}.8997936$ is multiplied by 3. Multiply the decimal part separately and add the 2 units to the product $3 \times \bar{2} = \bar{6}$, which gives $2 + \bar{6} = \bar{4}$ for the characteristic of the required logarithm (31).

Instead of operating as above, reduce the logarithm to an entirely negative logarithm and multiply by 3, thus (**400**):

$$\bar{2}.8997936 \times 3 = -\ 1.1002064 \times 3 = -\ 3.3006192 = \bar{4}.6993808,$$

which is not as convenient as the first method.

4th. *Extract the fifth root of* 243 (**396**, 9th).

$$\log \sqrt[5]{243} = \frac{\log 243}{5} = \frac{2.38560627}{5} = 0.47712125.$$

The number 3 which corresponds to this logarithm is the required root.

Calculate the cube root of $\frac{0.042}{0.529}$.

$$\log \sqrt[3]{\frac{0.042}{0.529}} = \frac{\log 0.042 - \log 0.529}{3} = \frac{\bar{2}.6232493 - \bar{1}.7234557}{3}$$

$$= \frac{\bar{2}.8997936}{3} = \bar{1}.6332645\,;$$

then

$$\sqrt[3]{\frac{0.042}{0.529}} = 0.4298.$$

In this example the logarithm $\bar{2}.8997936$ is divided by 3. Reduce the characteristic to a multiple of 3 by adding $\bar{1}$, which gives $\bar{3}$, and this is compensated for by adding 1 to the decimal part. This is all done without writing anything, and continuing one-third of $\bar{3}$ is $\bar{1}$, of 18 is 6, of 9 is 3, etc. As in the multiplication (3d), the logarithm may be reduced to an entirely negative logarithm.

406. *From 3d and 4th in the preceding article, it is seen that any power or root of any number may be found with the aid of logarithms.*

Let it be required to raise 125 to the $\frac{1}{3}$ power.

$$\log\left(125^{\frac{1}{3}}\right) = \frac{1}{3}(\log 125) = \frac{2.09691001}{3} = 0.69897000.$$

The number 5, corresponding to this logarithm, is the $\frac{1}{3}$ power of 125.

Thus it is seen that raising a number to the $\frac{1}{3}$ power is the same as taking the cube root of it (306).

In general, to raise a number to a fractional power, extract the root whose index is the reciprocal of the degree of the power; and, conversely, to extract a fractional root, raise the number to the power the degree of which is the reciprocal of the index of the root. Thus,

$$\log \sqrt[\frac{2}{3}]{64} = \log\left(64^{\frac{3}{2}}\right) = \frac{3}{2} \times 1.80617997 = 2.70926996.$$

The number 512, corresponding to this logarithm, is the $\frac{2}{3}$ root or the $\frac{3}{2}$ power of 64.

This example shows that in order to raise a given number to a fractional power, the $\frac{3}{2}$ power for instance, raise the number to the power 3 equal to the numerator, and extract the root indicated by the denominator of the power obtained. It is also seen that in order to extract a fractional root, the $\frac{2}{3}$ for instance, extract the root of the number indicated by the numerator, and raise this root to the power 3 indicated by the denominator; which is the same as raising the given number to $\frac{3}{2}$ power, that is, cubing the number and then extracting the square root of the cube.

407. *Naperian or hyperbolic logarithms.* This system was invented by the Scottish baron John Napier and published by him in 1614. The base of the system is the number 2.718281828459 . . . The common logarithms are better adapted to ordinary numerical calculations, but the hyperbolic or natural logarithms are used in higher mathematics (see Part V).

408. The logarithms $\log A$ and $\log_e A$, of the same number A, in two systems which have respectively b and b' for their base, are inversely proportional to the logarithms of these bases taken in any system. Thus, taking, for example, the logarithms b and b' in the system $\log A$,

$$\frac{\log A}{\log_e A} = \frac{\log b'}{\log b},$$

whence

$$\log A = \log_e A \, \frac{\log b'}{\log b} \text{ and } \log_e A = \log A \, \frac{\log b}{\log b'},$$

or, noting that $\log b = 1$ (396, 1st),

$$\log A = \log_e A \times \log b', \text{and } \log_e A = \log A \times \frac{1}{\log b'}.$$

The above makes it possible to change the logarithm of any number A in a system to a logarithm of this same number in another system.

For example, the hyperbolic log $\log_e A = 6.6106960$ of the number $A = 743$ being given; find the common logarithm of the same number A.

The base $b' = 2.7182818$ of the natural system has for common logarithm $\log b' = 0.4342945$; therefore,

$$\log 743 = 6.6106960 \times 0.4342945 = 2.8709888.$$

Thus *the product of the natural logarithm of a number and* 0.4342945 *is the common logarithm of the number.*

We have also,

$$\log_e A \text{ or } 6.6106960 = \frac{\log A}{\log b'} = \frac{2.8709888}{0.4342945} = 2.8709888 \times 2.302585.$$

The natural logarithm of a number is equal to the quotient obtained by dividing the common logarithm of the number by 0.4342945, *or the product of the common logarithm and* 2.302585, *or* 2.3026.

$$\log_e 10 = 2.302585.$$

Table of the first 9 multiples of the log b' and of the $\frac{1}{\log b'}$, to 10 decimals:

	$\log b'$
1	0.4342944819
2	0.8685889638
3	1.3028834457
4	1.7371779276
5	2.1714724095
6	2.6057668914
7	3.0400613733
8	3.4748558552
9	3.9086503371

	$\frac{1}{\log b'}$
1	2.3025850930
2	4.6051701860
3	6.9077552790
4	9.2103403720
5	11.5129254650
6	13.8155105580
7	16.1180956510
8	18.4206807440
9	20.7232658369

409. *A general formula for the calculation of compound interest.* The calculation of compound interest was given in (389). The general formula is developed as follows: Let r be the interest on \$1.00 for one year. After one year,

\$1 is worth $1 + r = v_1$,
\$2 are worth $(1 + r)\,2 \ldots$ etc.

If v_1 is taken as a new principal placed at simple interest for the second year, at the end of the second year the principal v_1 will be,

$$v_2 = (1 + r)(1 + r) = (1 + r)^2.$$

Likewise, if v_2 is taken as a new principal for the next year, at the end of the third year

$$v_3 = (1 + r)^2 (1 + r) = (1 + r)^3$$

and so on. Thus the principal of \$1.00 placed for n years will become

$$v_n = (1 + r)^n$$

at the end of the nth year. Therefore, a principal C placed at

compound interest at the rate r for n years would at the end of the nth year amount to

$$V = C\,(1 + r)^n,$$

from which, taking the logarithms (1),

$$\log V = \log C + n \log (1 + r).$$

By the aid of the formula (1) the diverse problems of compound interest may be solved.

EXAMPLE 1. What principal must be placed at 4.5% compound interest in order that the amount be \$290,818.00 after 40 years?

Solution. The formula (1) gives:

$$C = \frac{V}{(1 + r)^n}$$

or

$$C = \frac{290{,}818}{(1 + 0.045)^{40}}$$

whence $\log C = \log 290{,}818 + C^t\,40 \log (1.045).$

The logarithmic calculations :

$$\begin{aligned}
40 \log\ (1.045) &= 0.7646516\\
C^t\,40 \log\ (1.045) &= 9.2353484\\
\log 290{,}818 &= 5.4989700\\
C^t\,40 \log\ (1.045) &= 9.2353484\\
&\ -\underline{10.0000000}\\
\log C &= 4.6989700\\
C &= \$50{,}000.
\end{aligned}$$

EXAMPLE 2. How many years must \$50,000.00 be placed at 4.5% compound interest in order that the amount equal \$290,-818.00?

Solution. Substituting in formula (1):

$$\log V = \log C + n \cdot \log (1 + r),$$

$$n = \frac{\log V - \log C}{\log (1 + r)},$$

$$n = \frac{5.4636216 - 4.6989700}{0.0191163} = 40 \text{ years.}$$

EXAMPLE 3. How many years will it take for a certain principal to double itself when placed at 5% compound interest?

Solution. According to the statement of the problem, $V = 2\,C$; then substituting in the formula (1):

$$2\,C = C\,(1 + r)^n;$$

dividing by C,

$$2 = (1 + r)^n;$$

taking the logarithms of the two numbers,

$$\log 2 = n \log (1 + r),$$

for $r = 0.05$,

$$n = \frac{\log 2}{\log 1.05} = \frac{0.3010300}{0.0211893}$$

$$n = 14 \text{ years, } 207;$$

or reducing to days,

$$n = 14 \text{ years, } 75 \text{ days.}$$

The preceding calculation presupposes that the compounding holds for fractions of a year, which is not the case. Therefore the number of years is all that should be used; and to calculate the number of days, find the value of \$1.00 after 14 years, thus:

$$(1.05)^{14} = \$1.9799;$$

then find how many days this amount must be placed at 5% simple interest to become equal to \$2.00 or to give the interest $2 - 1.9799 = \$0.0201$.

\$1.00 brings in 360 days	\$0.05
and in 1 day	$\frac{0.05}{360}$
in n days	$\frac{0.05 \cdot n}{360}$.

Therefore, \$1.9799 after n days will amount to

$$\frac{0.05 \times 1.9799n}{360} = 0.0201$$

$$n = 73 \text{ days.}$$

It is seen that the two results differ but little, and therefore it is generally sufficiently accurate to use the general rule for compound interest even for fractions of a year.

410. *General formula for annuity.* The general formula is developed below: The capital C is loaned at compound interest and must be fully repaid at the end of n years, paying a constant sum each year, called an annuity.

Let r be the interest on \$1.00 for 1 year.

According to article (407), the final value of C is

$$V = C(1 + r)^n.$$

The sum of the final values of the different payments A is equal to the final value V.

The first payment can be placed at compound interest for $n - 1$ years; therefore, this payment represents a final value of:

$$v_1 = a(1 + r)^{n-1};$$

likewise the second payment represents a final value

$$v_2 = a(1 + r)^{n-2};$$

the third,

$$v_3 = a(1 + r)^{n-3};$$

the next to the last,

$$v_{n-1} = a(1 + r);$$

and finally the last,

$$v_n = a.$$

Summing these different final values, the final value V of C is obtained:

$$a + a(1 + r) + a(1 + r)^2 + \ldots a(1 + r)^{n-1} = C(1 + r)^n.$$

The first member:

$$a[1 + (1 + r) + (1 + r)^2 + \ldots (1 + r)^{n-1}].$$

Writing it in this manner, we see that the annuity is multiplied by the sum of the terms of a geometrical progression whose first term is 1, whose multiplier is $(1 + r)$, and whose last term is $(1 + r)^{n-1}$, and according to article (371) the sum is

$$\frac{(1+r)^{n-1}(1+r)-1}{(1+r)-1} = \frac{(1+r)^n-1}{r}$$

$$a\left[\frac{(1+r)^n-1}{r}\right] = C(1+r)^n$$

$$a = \frac{r \cdot C(1+r)^n}{(1+r)^n-1}. \tag{1}$$

This is the value of the annuity. This formula cannot be calculated by logarithms. In using logarithms, commence with the term

$$(1 + r)^n,$$

writing

$$(1 + r)^n = V$$

$$\log V = n \log(1 + r),$$

then $V - 1$ is the denominator in (1), giving

$$a = \frac{r \cdot cV}{V - 1},$$

which may be calculated by logarithms.

If the annuity a, the rate r, and the number of years n, are given, *the capital C is found* by substituting in the formula (1),

$$C = \frac{a\,(1 + r)^n - a}{r\,(1 + r)^n}. \tag{2}$$

The determination of the number of years n, when the capital C, the rate r, and the annuity are given, from the formula (2),

$$Cr\,(1 + r)^n = a\,(1 + r)^n - a;$$

transposing,

$$a = a = (1 + r)^n\,(a - Cr);$$

and taking the logarithms,

$$\log a = n \cdot \log\,(1 + r) + \log\,(a - Cr)$$

$$n = \frac{\log a - \log\,(a - Cr)}{\log\,(1 + r)}.$$

In order that the problem be possible, it is necessary that the difference $(a - Cr)$ be positive, because a negative number has no logarithm. Thus the annuity a should always be greater than Cr the simple interest on the capital. It is possible to find a fractional number of years, $15\frac{2}{3}$ years for example, then take either 15 or 16 years and calculate the corresponding annuity, which is a practical solution of the problem.

Determine the rate when the capital C, the annuity a, and the number of years n are given.

Solution. Write the formula (1):

$$a = \frac{r \cdot C\,(1 + r)^n}{(1 + r)^n - 1};$$

transposing,

$$a\,(1 + r)^n - a = r \cdot C\,(1 + r)^n$$

$$Cr\,(1 + r)^n = a\,(1 + r)^n - a$$

$$r = \frac{a}{C} - \frac{a}{C\,(1 + r)^n}. \tag{3}$$

r can only be calculated by a method of successive approximations.

SINKING FUNDS

411. A sinking fund is a sum set aside annually at compound

interest to liquidate a debt, or replace an equipment which has a limited life.

Let

The debt $= C$.
The rate of interest $= r$.
The sum set aside $= S$.
The number of years $= n$.

Then we have the following relations:

Sum at the end of the first year $= S$.
Sum at the end of the second year $= S + s\,(1 + r)$.
Sum at the end of the third year $= s + s\,(1 + r) + s\,(1 + r)^2$.
Sum at the end of the nth year $= s + s(1 + r) + \ldots s(1 + r)^{n-1}$.

Summing this series (371), we have

$$C = \frac{s\,[(1 + r)^n - 1]}{r}.$$

EXAMPLE 1. If a government owes \$500,000, what sum must be set aside annually as a sinking fund to liquidate the debt at the end of 10 years, money being worth 5%?

$$S = \frac{Cr}{(1 + r)^n - 1} = \frac{500{,}000 \cdot 0.05}{(1.05)^{10} - 1} = \frac{25{,}000}{0.628} = \$39{,}800.$$

EXAMPLE 2. If \$10,000 is set aside each year as a sinking fund with which to renew a \$110,000 equipment, how long will it take to accumulate the required sum, money being worth 5%?

Putting $C =$ \$110,000, $S =$ \$10,000, $r = 0.05$, and $n =$ the number of years, we have,

$$C = \frac{s\,[(1 + r)^n - 1]}{r},$$

$$Cr = s\,(1 + r)^n - s,$$

$$\frac{Cr + s}{s} = (1 + r)^n,$$

$$\log \frac{Cr + s}{s} = \log\,(Cr + s) - \log s = n \log\,(1 + r),$$

$$n = \frac{\log\,(Cr + s) - \log s}{\log\,(1 + r)}$$

$$= \frac{\log\,(110{,}000 \cdot 0.05 + 10{,}000) - \log 10{,}000}{\log 1.05}$$

$$= \frac{\log 15{,}500 - \log 10{,}000}{\log 1.05}$$

$$= \frac{4.1903 - 4.0000}{0.0212} = \frac{0.1903}{0.0212} = 9 \text{ years}.$$

STOCKS AND BONDS

412. A *corporation* is an association of individuals transacting business as a single person under rights and limitations granted by *statute* or *charter*.

413. The *capital stock* of a corporation is the amount of money invested, and is represented by a certain number of equal *shares;* each share generally represents $100.

414. A *stock certificate* is a written evidence of the holder's title to a described share or interest in stock.

415. The *gross earnings* are the total receipts from the business, and deducting the expenses from these the *net earnings* are obtained.

416. A *dividend* is an apportionment of a certain part of the earnings, and is generally declared at a certain per cent.

417. An *assessment* is a sum levied upon the stock to meet expenses.

418. The *face value* of the stock is called the *par value;* and when the company is prosperous and declares large dividends, its stock is quoted *above par;* and on the other hand, when the company must levy an assessment, it is not considered prosperous, and its stock falls *below par.*

419. *Market value* is the selling price of the stock.

420. *Preferred stock* is stock that does not share in the general dividends, but is entitled to its share of the profits before the regular stock.

421. *Watered stock* is the inflation of the capital stock by the issue of stock for which no payment is made.

422. *Bonds* are written agreements under seal to pay a specified amount on or before a specified date.

423. *Coupon bonds* are bonds which have coupons or certificates of interest attached.

424. *Government bonds* are bonds issued by the government. They usually take their name from the rate and date they bear; thus, $4\frac{1}{2}$'s of '91 means $4\frac{1}{2}\%$ bonds payable in 1891.

425. Persons who buy and sell stocks and bonds are called *stock brokers.* They receive a commission called brokerage, which is reckoned on the par value of the stock.

426. *In operations with stocks, let*

$$\begin{array}{ll}
\text{The par value . . .} & = C. \\
\left.\begin{array}{l}\text{Per cent premium}\\ \text{Per cent discount}\\ \text{Per cent assessment}\\ \text{Per cent dividend}\end{array}\right\} & = r. \\
\left.\begin{array}{l}\text{Premium}\\ \text{Discount}\\ \text{Assessment}\\ \text{Dividend}\end{array}\right\} \text{ . .} & = I. \\
\text{Market value . .} & = A. \\
\text{Number of shares .} & = n.
\end{array}$$

Then the relations between these various quantities are expressed by the following formulas:

$$nCr = I \quad \text{and} \quad nC \pm I = A.$$

With the aid of these formulas any problem in stocks can be performed, providing the brokerage is deducted, always bearing in mind that brokerage is computed upon the par value of the stock.

427. EXAMPLES:

1. *A business man meets an assessment of* \$83.25 *levied at* $2\frac{1}{4}\%$ *on his stock. How many shares has he?*

Putting shares $= n$, assessment $= I$, per cent assessment $= r$, we obtain,

$$nCr = I \text{ or } n = \frac{I}{Cr} = \frac{83.25}{100 \cdot 0.0225} = 37 \text{ shares.}$$

2. *If a* 7% *dividend is declared upon* 50 *shares Chicago City R. R. stock, what is the amount of the dividend?*

Putting $n = 50$, $r = 7\%$, dividend $= I$, we have,

$$I = nCr = 50.100.0.07 = \$350.$$

3. *A broker bought stock for a party at* $124\frac{5}{8}$ *and immediately sold the same for* $143\frac{1}{4}$, *remitting* \$1341 *as net proceeds. How many shares did he buy, the brokerage being* $\frac{1}{8}\%$?

Putting $A_1 = 124\frac{5}{8}$, $A_2 = 143\frac{1}{4}$, $n =$ number of shares, then the

total brokerage is,

$$2 \cdot n \cdot C \cdot 0.00\tfrac{1}{8} = \text{brokerage},$$

and the net proceeds, \$1341.

$$\begin{aligned}\$1331 &= nA_2 - nA_1 - 2 \cdot n \cdot C \cdot 0.00\tfrac{1}{8} \\ &= n[A_2 - (A_1 + 2\,C\,0.00\tfrac{1}{8})] \\ &= n\left[143\tfrac{1}{4} - \left(124\tfrac{3}{8} + \frac{2}{8}\right)\right]. \\ n &= \frac{1341}{18.625} = 72 \text{ shares.}\end{aligned}$$

428. *In operations with bonds, let*

Market price $= C$.
Years yet to run $= n$.
Rate of interest $= r$.
Face of bond $= C'$.
Current rate of interest $= r'$.
Rate of interest on investment $= x$.

Then (**409**)

$$C(1 + x)^n$$

is the value of the purchase money at the end of n years (**409**); and if the interest received on the bond is put immediately at compound interest at $r'\%$, the amount of money received is (**371**),

$$\begin{aligned}&C'r\,(1 + r')^{n-1} + C'r\,(1 + r')^{n-2} + \cdots\cdot C'r + C' \\ &= C' + \frac{C'r\,[(1 + r)^n - 1]}{r'}.\end{aligned}$$

Therefore,

$$C\,(1 + x)^n = C' + \frac{C'r\,[(1 + r)^n - 1]}{r'},$$

$$\begin{aligned}1 + x &= \left(\frac{C'}{C} + \frac{C'r\,[(1 + r)^n - 1]}{Cr'}\right)^{\frac{1}{n}} \\ &= \left(\frac{C'r' + C'r\,(1 + r)^n - C'r}{Cr'}\right)^{\frac{1}{n}}.\end{aligned}$$

Example. *At what price must 7% bonds, running 12 years with interest payable semi-annually, be bought in order that the purchaser may receive 5% on his investment semi-annually, which is the current rate of interest?*

Putting $C' = 100$, and since the interest is paid semi-annually $r' = 0.025$, $r = 0.035$, $n = 24$, and $x = 0.025$.

Substituting these values in the above formulas,

$$C(1+x)^n = \frac{C'r' + C'r(1+r')^n - C'r}{r'},$$

$$C = \frac{C'r' + C'r(1+r')^n - C'r}{r'(1+x)^n},$$

we obtain,

$$C = \frac{2.5 + 3.5\,(1.025)^{24} - 3.5}{0.025\,(1.025)^{24}},$$

which, when solved by logarithms, gives

$$C = 118.$$

BANK DISCOUNT

429. A *bank* is an institution for the deposit, discount, or circulation of money.

430. A *note* is a written evidence of debt coupled with a promise to pay.

431. The *maker* is the one who promises to pay, and the *payee* is the one to whom the promise is made.

432. A *draft* is an order on one person to pay another.

The party who writes the draft is the *drawer*, the one to whom it is given is the *payee*, and the one on whom it is drawn is the *drawee*.

433. Writing on the back of commercial paper constitutes an *indorsement*.

If the draft is acknowledged by the drawee, it is said to be *accepted*.

434. *Bank discount* is simple interest computed upon the sum due at a future date and paid in advance.

435. The sum named in the note is the *face*, and the face less the discount is the *proceeds*.

436. The time from the date of discount to the date of maturity is called the *term of discount*.

In non-interest-bearing notes, the face is the sum to be discounted. In interest-bearing notes, the face plus the interest due at maturity is the sum to be discounted.

437. The operations with notes and relations between the different factors are expressed by the following formulas:

Let Face of note $= C$.
Face plus interest due at maturity $= C'$.
Rate of interest $= r'$.
Rate of discount $= r$.
Discount $= I$.
Proceeds $= A$.
Term $= T$.
Interest $= I'$.
Then $C' = C + I'$.
CrT or $C'rT = I$.
$C - I$ or $C' - I = A$.

438. EXAMPLE:

$890.00 NEW YORK, N. Y., MAY 18, 1896.
Five months after date, we promise to pay Harper Bros. and Co. Eight Hundred Ninety dollars. Value received with interest at 6% per annum. BROWN, SMITH & Co.

Find the proceeds, discounted on the date of issue at 7%.

Referring to table (p. 126), we find the term of discount, which is 153 days, and adding the three days' grace we have 156 days.

Now substituting in the formulas,

$$C' = (C + I') = C + Cr'T = 890 + 890 \cdot 0.06 \cdot \frac{156}{365} = \$912.82.$$

$$A = C' = C'rT = 912.82 - 912.82 \cdot 0.07 \frac{156}{365} = 912.82 - 27.36 = \$885.52.$$

INSURANCE

439. *Insurance* is an indemnity in case of loss.

The insurance business is carried on by corporations regulated by state laws. They may be mutual companies, stock companies, or both.

The contract between the insured and the insurer is called the *policy*.

The sum paid for the insurance is called the *premium*.

440. *Life insurance* is an agreement of a company to pay a certain sum of money in event of the death of a person or at the expiration of a term of years.

441. A *straight life policy* continues during the life of the insured.

442. An *endowment policy* is payable to the insured at the expiration of a term of years, or to his estate if he dies sooner.

443. The *expectation of life* is the probability of life as deduced from the mortality tables compiled from statistics.

444. The *rate of life insurance* is expressed as a given sum on each $1000, and is determined by the expectation of life which the insured has at the time of taking out the policy. Thus, referring to the table we see that a man of a certain age has an expectation of life of n years; then, letting the premium be c, the rate of interest be r, and the face of the policy be A, we have,

$$A = \frac{c[(1+r)^n - 1]}{(1+r) - 1} = \frac{c[(1+r)^n - 1]}{r}, \tag{371}$$

and $$c = \frac{Ar}{(1+r)^n - 1}.$$

Of course, in practice, charges have to be added to cover expenses, etc., but the above formula forms a basis of comparison, and illustrates the principle upon which life insurance is grounded.

Expectation Table

Constructed from the American Experience Table of Mortality.

AGE.	EXPECTATION, YEARS.	AGE.	EXPECTATION, YEARS.	AGE.	EXPECTATION, YEARS.
10	48.7	37	30.4	64	11.7
11	48.1	38	29.6	65	11.1
12	47.4	39	28.9	66	10.5
13	46.8	40	28.2	67	10.0
14	46.2	41	27.5	68	9.5
15	45.5	42	26.7	69	9.0
16	44.9	43	26.0	70	8.5
17	44.2	44	25.3	71	8.0
18	43.5	45	24.5	72	7.6
19	42.9	46	23.8	73	7.1
20	42.2	47	23.1	74	6.7
21	41.5	48	22.4	75	6.3
22	40.9	49	21.6	76	5.9
23	40.2	50	20.9	77	5.5
24	39.5	51	20.2	78	5.1
25	38.8	52	19.5	79	4.8
26	38.1	53	18.8	80	4.4
27	37.4	54	18.1	81	4.1
28	36.7	55	17 4	82	3.7
29	36.0	56	16.7	83	3.4
30	35.3	57	16.1	84	3.1
31	34.6	58	15.4	85	2.8
32	33.9	59	14.7	86	2.5
33	33.2	60	14.1	87	2.2
34	32.5	61	13.5	88	1.9
35	31.8	62	12.9	89	1.7
36	31.1	63	12.3	90	1.4

PART II

ALGEBRA

DEFINITIONS AND PRINCIPLES

445. *Algebra* is a generalized arithmetic. In algebraic operations the result of a certain problem is not desired, but a general solution which may be applied to all analogous propositions (**2**, **3**, **18**).

In algebra, known and unknown quantities are expressed by means of letters, and the relations which exist between them by signs. Having written a number of such quantities and expressed the relations between them, they are transformed to simpler forms and each unknown expressed in terms of the known quantities. Such a general expression is called a *formula* (**503**), and the value of the unknown quantities is obtained by substituting the values of the known quantities in the formula and performing the arithmetical operations as indicated.

446. Characters and signs used in algebra are:

1st. The *letters of the alphabet*, which are used to represent quantities. Ordinarily the first letters of the alphabet are used to represent known quantities, and the last letters unknown quantities.

The notations a', a'', a''', etc., are pronounced a prime, a double prime, a third, etc.; and a_1, a_2, a_3, etc., are pronounced a sub one, a sub two, a sub three, etc.; both are used to express analogous quantities of different values in the same proposition.

2d. The *signs* given in Art. **24**, Part I, are the same in algebra as in arithmetic; thus,

$$a + b - c = d \times e - \frac{b}{g}$$

reads, a plus b minus c equals d times e minus b divided by g.

Generally the product of several letters a, b, c^2, is indicated by writing simply abc^2 instead of $a \times b \times c^2$. This is also the

case when one of the factors is a number, and the number is always placed first. Thus, $a \times b \times c^2 \times 5$ is written $5\,abc^2$.

$\frac{a}{b}$ is read, a divided by b, a *over* b, or a is to b.

3d. The *coefficient* is the number written at the left of a quantity, and serves as a multiplier. Thus, in the following,

$$3\,a = 3 \times a = a + a + a, \text{ and } \frac{2}{5}\,a = a \times \frac{2}{5},$$

3 and $\frac{2}{5}$ are the coefficients, and are read, three a and two-fifths a. A quantity which has no number written before it has 1 for its coefficient, but it is never written.

A coefficient may also be expressed by letters, as will be seen later on.

4th. The *exponent* has the same meaning as in arithmetic (**88**). Thus, $a^5 = aaaaa$, and is read, a to the 5th power. All quantities which have no exponent written above them have 1 for an exponent (**305**).

5th. The *radical* $\sqrt[n]{\ }$ indicates, as in arithmetic (**264**), that a root is to be extracted; and the *index* above and at the left indicates the degree of the root. Thus:

$\sqrt{ab}$ indicates the square root of the product of a and b.

$\sqrt[3]{a^2 + b^3}$ indicates the cube root of the sum of the square of a and the cube of b.

447. An *algebraic quantity* is represented by an *algebraic expression* which consists of one or more symbols connected by signs of operation.

A quantity is said to be *rational* when it does not contain a radical :

$$5\,ab^2 - \frac{3\,a + b}{c} + 2\,bc.$$

A quantity is *irrational* when it contains one or more radicals:

$$4\,a^2b - \sqrt{ab^3}.$$

A quantity is *whole* when it contains neither radicals nor signs of division:

$$4\,a^2b^3 + 5\,ac - 3\,c^4.$$

A quantity is a *fraction* when it contains the sign of division:

$$2\,ab^2 + \frac{a - 3\,b}{2}.$$

448. 1st. A *term* is an algebraic quantity, the parts of which are not separated by the sign of addition or subtraction.

2d. *Monomial*, is an algebraic quantity of but a single term: $3\,ab^2$.

3d. *Binomial*, is an algebraic quantity of two terms: $a + b^2c^3$.

4th. *Trinomial*, is an algebraic quantity of three terms: $\frac{4}{3}a^4c + b^2c^2 + 3\,c^5$, etc.

5th. *Polynomial*, is an algebraic quantity of several terms: $a^2 + b^2$, $ab + b^2c^2 + c^4$, $4\,a^3 - b^2c - \sqrt[3]{a+b}$.

449. A term is positive or negative according as it is preceded by the plus + or minus − sign. When the first term of a polynomial is positive, the sign is not written. Thus, instead of writing $+\,3\,a^3 + b^2c^2$, write simply $3\,a^3 + b^2c^2$.

The + sign is never placed before a monomial. Two terms which have, one the sign + and the other the sign −, are said to have *unlike signs*. Such are $3\,ab$ and $-\,cd$.

450. The *absolute value* of a quantity is its value, neglecting the sign which precedes it. The *relative* or *algebraic value* is the value of the quantity, having regard for the signs.

451. The *numerical value of an algebraic expression or quantity* is the number obtained in substituting the value of each letter in numbers and performing the operations as indicated.

Let $a = 2$, $b = 3$, and $c = 4$; then substituting in the following expression, we have the numerical value:

$$a^2 - ab + b^2c - c^2 = 2^2 - 2 \times 3 + 3^2 \times 4 - 4^2 = 18.$$

452. Remark 1. The numerical value of a polynomial is equal to the sum of the positive terms less that of the negative terms:

$$a^2 - ab + b^2c - c^2 = a^2 + b^2c - (ab + c^2) = 4 + 36 - (6 + 16) = 18.$$

Remark 2. The numerical value of a polynomial is not changed by changing the order of the terms so long as the signs remain the same:

$$a^2 - ab + b^2c - c^2 = b^2c - ab - c^2 + a^2.$$

453. *The degree of a monomial or of a term with reference to one of its letters* is the exponent of that letter, and *its degree with reference to several letters* is the sum of the exponents of those

several letters. Thus, the monomial $7\,a^2b^3$ is of the second degree with reference to a, of the third with reference to b, and of the fifth with reference to both a and b.

When it is a question of the *degree of a monomial* with no other qualification, it is understood to be the degree of the monomial with reference to all the letters in the term. Thus the monomial $7\,ab^3c^2$ is of the 6th degree.

454. *The degree of a polynomial with reference to one or several of its letters* is the largest exponent of the one letter or the largest sum of the letters in one term of the polynomial. Thus, the polynomial $5\,ab^3 + 6\,a^2b^5 - 6\,a^4b^2$ is of the 4th degree with reference to a, of the 5th with reference to b, and of the 7th with reference to a and b. When a polynomial or monomial does not contain a letter, it is of the zero degree with reference to this letter (**483**).

455. A polynomial is *homogeneous* with reference to one or several of its letters when all its terms are of the same degree with reference to this or these letters. Thus, the polynomial $5\,a^2b^3c + 6\,a^2b^2c^2 - a^2bc^3$ is homogeneous and of the 2d degree with reference to a, and is homogeneous and of the 4th degree with reference to the letters b and c.

When a *polynomial is homogeneous*, without any other qualification, it is understood that it is homogeneous with reference to all its letters, that is, all of its terms are of the same degree. Thus, the polynomial $3\,a^2b^2c^2 - 5\,a^2b^2c^2 = a^4b^2$ is homogeneous and of the 6th degree. The polynomial $3\,a^3bc^2 - 5\,a^2bc^2 + 2a^3bc^2$ is not homogeneous.

BOOK I

THE FOUR FUNDAMENTAL ALGEBRAIC OPERATIONS

THE REDUCTION OF LIKE TERMS

456. Terms which contain the same letters having the same exponents are said to be *like terms*. Thus, ab and $4\,ab$ are like terms; $5\,a^2b^3$ and $-\,2\,a^2b^3$ are also like terms; but ab^2 and ab^3 are not like terms. Like terms can differ only in coefficient and sign.

457. *To reduce the like terms of a polynomial*, reduce each group of like terms to a single term.

458. In *reducing the like terms of a polynomial*, replace the groups of like terms by one single like term, having a coefficient equal to the difference of the sum of positive and the sum of the negative coefficients, and preceded by the sign of the largest sum.

Thus, having

$$3\,ab^2 - 4\,a^2c + 3\,a^2c - ab^2 - 5\,a^2c + 7\,bc,$$

which may be written,

$$3\,ab^2 - ab^2 + 3\,a^2c - 4\,a^2c - 5\,a^2c + 7\,bc, \qquad \textbf{(452)}$$

$3\,ab^2 - ab^2$ reduces to $2\,ab^2$; $3\,a^2c - 4\,a^2c - 5\,a^2c$ reduces to $3\,a^2c - 9\,a^2c$ or $-\,6\,a^2c$, and therefore the given polynomial reduces to

$$2\,a^2b - 6\,a^2c + 7\,bc.$$

Sometimes the coefficients of the same term are written in parentheses, each preceded by its sign; this is the case when the coefficients are represented by letters (**471**).

The polynomial

$$7\,x + ax - abx + ay^2 - cy^2 - cdy^2$$

may be reduced thus:

$$(7 + a - ab)\,x + (a - c - cd)\,y^2.$$

The reduction of like terms is frequently employed in algebraic operations.

ADDITION

459. The four fundamental operations on the algebraic quantities are analogous to those in arithmetic, and therefore need not be defined again (**24**, **27**, **32**, **51**).

460. *To add several algebraic quantities, monomials or polynomials,* write them one after the other, each preceded by its sign, and reduce the like quantities (**458**). Thus, the sum of the algebraic quantities $3\,a^2 + 4\,ab$, $6\,ab - a^2$, $5\,b^3 - 3\,ab$, and $-2\,bc$ is

$$3\,a^2 + 4\,ab + 6\,ab - a^2 + 5\,b^3 - 3\,ab - 2\,bc;$$

reducing,

$$2\,a^2 + 7\,ab + 5\,b^2 - 2\,bc.$$

In practice, the quantities to be added are written one under the other, as shown below; reduce the like terms as though the quantities were written one after the other, and write the results of the reduction with their respective signs below:

$$\begin{array}{l}
4\,a^3 \;\; + 5\,a^2b + c \\
2\,a^3 - 7\,a^2b - 4c \\
\underline{6\,a^2b + c + bc + 25\qquad} \\
6\,a^3 \;\; + 4\,a^2b - 2c + bc + 25
\end{array}$$

Remark. According as 7 or -7 is added to a quantity is that quantity increased or decreased by 7; therefore an algebraic addition is not necessarily an augmentation.

SUBTRACTION

461. *To subtract one algebraic quantity from another,* write the quantity to be subtracted at the right of the other and change all its signs; then reduce the like terms if there are any. Thus, subtracting $3\,a^2 - 2\,ab + bc - b^2$ from $7\,a^2 - 2\,ab$, we have,

$$7\,a^2 - 2\,ab - 3\,a^2 + 2\,ab - bc + b^2;$$

reducing,

$$4\,a^2 - bc + b^2.$$

To facilitate the operation, write the quantities one beneath

the other, putting like terms in the same column; then changing the signs of the subtrahend, proceed as in addition.

Thus, to subtract $2a + 3b^2c - 7$ from $8a - 5b^2c - 4$, operate thus:

$$\begin{array}{lr} & 8a - 5b^2c - 4 \\ & -2a - 3b^2c + 7 \\ \hline \text{remainder} & 6a - 8b^2c + 3 \end{array}$$

When it is not necessary to write the result in the form of a single polynomial, write the quantity to be subtracted in parentheses and at the right of the other quantity, placing a minus sign before the parenthesis. Thus the preceding example is written,

$$8a - 5b^2c - 4 - (2a + 3b^2c - 7).$$

If, having written the result as above, it is desired to reduce it to a single polynomial, reduce the like terms, changing all the signs of the quantities within the parentheses. Thus we obtain $6a - 8bc^2 + 3$, as in the first case.

REMARK. According as $+7$ or -7 is subtracted from a quantity, that quantity is decreased or increased by 7; and therefore an algebraic subtraction does not necessarily signify a diminution.

MULTIPLICATION

462. *In multiplying a monomial by a monomial,* there are 4 distinct laws to be considered:

1st. *The law of signs.* The product of two monomials having like signs has the sign $+$; the product of two monomials having unlike signs has the sign $-$. Thus either $+$ times $+$ or $-$ times $-$ gives $+$ for the product, and either $+$ times $-$ or $-$ times $+$ gives $-$ for the product.

2d. *The law of coefficients.* The coefficient of the product is equal to the product of the coefficients of the factors.

3d. *The law of letters.* All letters which enter in one or both of the factors appear once in the product.

4th. *The law of exponents.* The exponent of each letter in the product is equal to the sum of the exponents of that letter in the factors. A letter which has no exponent is supposed to have 1 for an exponent (**446**). A letter which does not appear in one of the factors has 0 for an exponent in that factor (**482**).

Applications of the rules:

$$3\,a^m \times a = 3\,a^{m+1};$$
$$-\,2\,a \times -\,3\,a^2b^2 = 6\,a^3b^2;$$
$$4\,a^2b^3c \times -\,b^2c = -\,4\,a^2b^5c^2;$$
$$-\,2\,b^2cd \times +\,4\,c^2d^3e^2 = -\,8\,b^2c^3d^4e^2.$$

463. The product of several monomials is obtained by multiplying the first two monomials together, this product by the third, and so on until the last monomial has been employed as multiplier. From this rule and (**427**) we have the following laws:

1st. The product has the sign + when the number of negative factors is even, and the sign − when it is odd.

2d. The coefficient of the product is equal to the product of the coefficients of the factors.

3d. Each letter found in any of the factors is written once in the product.

4th. The exponent of each letter in the product is equal to the sum of the exponents of that letter in the factors.

Thus we have:

$$2\,a \times 3\,a^2b \times -\,b^3c^2 \times -\,5 = 30\,a^3b^4c^2;$$
$$3\,a^2 \times -\,2\,ab^2 \times -\,5\,a^4bc^3 \times -\,5\,c = -\,150\,a^7b^3c^4.$$

464. *The product of several monomials changes or does not change its sign*, according as the sign of an odd or even number of factors is changed (**463**).

465. *To square a monomial* (**87**), square the coefficient and multiply the exponent of each letter by 2. The sign of a square is always +:

$$(3\,a^2b^3c)^2 = 9\,a^4b^6c^2, \qquad (-\,3\,a^2b^3c)^2 = 9\,a^4b^6c^2. \qquad (462)$$

To cube a monomial, cube the coefficient and multiply the exponent of each letter by 3. The cube has the same sign as the given number:

$$(3\,a^2b^3c)^3 = 27\,a^6b^9c^3, \qquad (-\,3\,ab^3c)^3 = -\,27\,a^6b^9c^3. \quad (463)$$

466. The degree of the product of several monomials is equal to the sum of the degrees of the factors (**453**, **463**).

The degree of the square or cube of a monomial is respectively equal to two or three times the degree of the given monomial (**465**).

467. *To multiply a polynomial by a monomial,* multiply successively each term of the polynomial by the monomial, following the rules given for the multiplication of monomials (**462**).

EXAMPLE:

$$\begin{array}{l} 3\,ab^2 \;+\; 4\,ab \;-\; b^2c \\ \underline{2\,ab^2 \qquad\qquad\qquad\quad} \\ 6\,a^3b^2 + 8\,a^2b^3 - 2\,ab^4c \end{array}$$

To indicate the multiplication of a polynomial by a monomial, write the polynomial in parentheses and consider it as a monomial. Thus, to indicate that $3\,a^2 + 4\,ab - b^2c$ is multiplied by $2\,ab^2$, write:

$$(3\,a^2 + 4\,ab - b^2c) \times 2\,ab^2.$$

When the monomial is positive, the sign $\times$ may be omitted. It may also be omitted when it is negative, but then the monomial is placed before the parenthesis. Thus, $-\,a\,(a - b)$ is the same as $-\,a \times (a - b)$ or $(a - b) \times -\,a$ (**470**).

468. *To multiply a polynomial by a polynomial,* multiply the multiplicand polynomial successively by each term of the multiplier (**467**), and add the partial products (**460** and **472**). Example:

$$\begin{array}{lrrrrrr}
\text{Multiplicand} & 4\,a^3 + & 2\,a^2b - & 5\,ab^2 - & 2\,b^2 & & \\
\text{Multiplier} & 2\,a^2 - & 3\,ab + & b^2 & & & \\ \hline
\text{1st partial product} & 8\,a^5 + & 4\,a^4b - & 10\,a^3b^2 - & 4\,a^2b^3 & & \\
\text{2d partial product} & & -\,12\,a^4b - & 6\,a^3b^2 + & 15\,a^2b^3 + & 6\,ab^4 & \\
\text{3d partial product} & & & +\;\;4\,a^3b^2 + & 2\,a^2b^3 - & 5\,ab^4 - & 2\,b^5 \\ \hline
\text{Product} & 8\,a^5 - & 8\,a^4b - & 12\,a^3b^2 + & 13\,a^2b^3 + & ab^4 - & 2\,b^5
\end{array}$$

To indicate the multiplication of one polynomial by another, write them in parentheses and consider them as monomials. Thus:

$$(4\,a^3 + 2\,a^2b - 5\,ab^2 - 2\,b^3) \times (2\,a^2 - 3\,ab + b^2)$$

or

$$(4\,a^3 + 2\,a^2b - 5\,ab^2 - 2\,b^3) \quad (2\,a^2 - 3\,ab + b^2).$$

469. *The product of several polynomials* is obtained by multiplying the first two together, the product of these by the third, and so on until all the polynomials have been used as multipliers. This rule also applies where there are some monomial factors.

470. *The product of several algebraic quantities, polynomials or monomials, is not altered by changing the order of the factors* (**41**).

471. *To arrange a polynomial according to the powers of some*

letter, write the terms in such an order that the exponents of that letter either descend or ascend in order of magnitude.

The polynomial $ab^4 + 3\,a^3b - 5\,a^2b^2 + a^4$, arranged according to the ascending powers of a, gives:

$$ab^4 - 5\,a^2b^2 + 3\,a^3b + a^4;$$

and according to the descending powers of a,

$$a^4 + 3\,a^3b - 5\,a^2b^2 + ab^4.$$

In this example it is seen that the polynomial is also arranged according to the powers of b.

The letter according to which a polynomial is arranged is called the *principal letter.*

When several terms of a polynomial contain the same power of the principal letter, write this power of the letter only once, and at the left of it write the multipliers either in parentheses or in a column. Thus, the polynomial

$$3\,a^2b + 5\,ab^2 + 2\,b^3 - a^2 + 4\,ab - 3\,b^2 - ac,$$

arranged according to the descending powers of the letter a, is:

$$(3\,b - 1)\,a^2 + (5\,b^2 + 4\,b - c)\,a + 2\,b^3 - 3\,b^2,$$

or

$$\begin{array}{r|l|l} 3\,b & a^2 + 5\,b^2 & a + 2\,b^3 \\ -\,1 & \quad\;\; + 4\,b & \quad - 3\,b^2 \\ & \quad\;\; - \quad c & \end{array}$$

It is well to arrange the polynomial multipliers of the different powers of the principal letter a according to the powers of another letter b, as was done in the above example.

472. The reduction of like terms in the multiplication of polynomials is greatly facilitated by arranging the polynomials according to the powers of some one letter. This is what was done in (**468**), and is shown again in the example which follows.

Multiply

$$(3\,a - b)\,x^2 + (5\,a^2 - 4\,a + b)\,x + 2\,a^3 - 3\,a^2$$

by

$$(6\,a + b)\,x - 2\,a^2 - b.$$

The coefficients of the principal letter x, not being numbers, but polynomials, the multiplication is a little more complicated. Ordinarily in this case the expression is arranged according to the second method (**471**), and the multiplication performed according to the general rule. Thus, all the terms of the multipli-

cand are multiplied at first by the first term of the multiplier $6\,ax$, then by the second bx, the third $2\,a^2$, and so on until the last has been used, and then the like terms in each column of partial products are reduced.

$$
\begin{array}{l r|l|l|l}
\text{Multiplicand} & 3\,a & x^2 + \ \ 5\,a^2 & x + \ \ 2\,a^3 & \\
 & -\ b & \quad\ -\ \ 4\,a & \quad\ -\ 3\,a^2 & \\
 & & \quad\ +\ \ \ b & & \\
\text{Multiplier} & 6\,a & x\ \ -\ \ 2\,a^2 & & \\
 & +\ \ b & \quad\ -\ \ \ b & & \\
\hline
 & 18\,a^2 & x^3 + 30\,a^3 & x^2 + 12\,a^4 & x - 4\,a^5 \\
 & -\ 6\,ab & \quad\ -\ 24\,a^2 & \quad\ -\ 18\,a^3 & \quad + 6\,a^4 \\
 & +\ 3\,ab & \quad\ +\ \ 6\,ab & \quad\ +\ \ 2\,a^3b & \quad - 2\,a^3b \\
 & -\ \ b^2 & \quad\ +\ \ 5\,a^2b & \quad\ -\ \ 3\,a^2b & \quad + 3\,a^2b \\
 & & \quad\ -\ \ 4\,ab & \quad\ -\ 10\,a^4 & \\
 & & \quad\ +\ \ \ b^2 & \quad\ +\ \ 8\,a^3 & \\
 & & \quad\ -\ \ 6\,a^3 & \quad\ -\ \ 2\,a^2b & \\
 & & \quad\ +\ \ 2\,a^2b & \quad\ -\ \ 5\,a^2b & \\
 & & \quad\ -\ \ 3\,ab & \quad\ +\ \ 4\,ab & \\
 & & \quad\ +\ \ \ b^2 & \quad\ -\ \ \ b^2 & \\
\hline
\text{Product} & 18\,a^2 & x^3 + 24\,a^3 & x^2 + \ \ 2\,a^4 & x - 4\,a^5 \\
 & -\ 3\,ab & \quad\ -\ 24\,a^2 & \quad\ -\ 10\,a^3 & \quad + 6\,a^4 \\
 & -\ \ b^2 & \quad\ +\ \ 7\,a^2b & \quad\ +\ \ 2\,a^3b & \quad - 2\,a^3b \\
 & & \quad\ -\ \ \ ab & \quad\ -\ 10\,a^2b & \quad + 3\,a^2b \\
 & & \quad\ +\ \ 2\,b^2 & \quad\ +\ \ 4\,ab & \\
 & & & \quad\ -\ \ \ b^2 & \\
\end{array}
$$

473. An arranged polynomial is said to be *complete* or *incomplete* according as it does or does not contain all the powers of the principal letter, from the first to the largest power given in the expression. Thus, the polynomial $a^4 + 3\,a^3b - 5\,a^2b^2 + ab^4$ is complete with reference to a, but is incomplete with reference to b, since it does not contain b^3.

474. The product of a polynomial, arranged according to the powers of a certain letter, and a monomial, is a polynomial arranged according to the powers of the same letter.

475. When two polynomials and their product are arranged according to the powers of the same letter, the first term of the product is equal to the product of the first terms of the factors, and the last term is the product of the last terms of the factors (468). Therefore, this product cannot have less than two terms. The greatest possible number of terms is equal to the product of the number of terms in the multiplicand and the number in the multiplier.

476. When an homogeneous polynomial (**455**) containing only two letters is arranged according to the ascending or descending powers of one of the letters, it is also arranged according to the descending or ascending powers of the other letter:

$$4\,a^3 + 7\,a^2b - ab^2 + 3\,b^3.$$

477. The product of two or any number of homogeneous polynomials is an homogeneous polynomial of a degree equal to the sum of the degrees of the factors (**455**). If all the factors are not homogeneous, the product is not homogeneous (**469**).

Likewise the product of one or several monomials and one or several homogeneous polynomials is an homogeneous polynomial of a degree equal to the sum of the degrees of the factors. If all the polynomial factors are not homogeneous, the product is not homogeneous (**466**).

478. When each letter of a monomial or of an homogeneous polynomial of the mth degree is multiplied by a factor k with the exponent of each letter, the monomial or polynomial is multiplied by k^m:

$$5\,a^2k^2 \times b^3k^3 \times ck = 5\,a^2b^3c \times k^6;$$
$$5\,a^2k^2 \times b^3k^3 \times ck + 6\,a^3k^3 \times b^2k^2 \times ck - ak \times b^3k^3 \times c^2k^2$$
$$= (5\,a^2b^3c + 6\,a^3b^2c - ab^3c^2)k^6.$$

479. *The square of the sum of two quantities is composed of* (**87, 269**): 1st, the square of the first quantity; 2d, plus twice the product of the first and the second; 3d, plus the square of the second. Thus:

$$(a + b)^2 = a^2 + 2\,ab + b^2. \qquad (\mathbf{468})$$

480. *The square of the difference of two quantities is composed of*: 1st, the square of the first quantity; 2d, minus twice the first by the second; 3d, plus the square of the second. Thus:

$$(2\,a^2b - bc)^2 = 4\,a^4b^2 - 4\,a^2b^2c + b^2c^2. \qquad (\mathbf{468})$$

481. The square of the sum of two quantities less the square of their difference is equal to 4 times the product of the quantities (**461, 479, 486**):

$$(a + b)^2 - (a - b)^2 = a^2 + 2\,ab + b^2 - a^2 + 2\,ab - b^2 = 4\,ab.$$

482. *The cube of the sum of two quantities is composed of* (**87, 276**): 1st, the cube of the first quantity; 2d, plus the triple product of the square of the first and the second; 3d, plus the triple

product of the first and the square of the second; 4th, plus the cube of the second:

$$(a + b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3. \qquad \textbf{(468)}$$

483. *The cube of the difference of two quantities is composed of:* 1st, the cube of the first quantity; 2d, minus the triple product of the square of the first and the second; 3d, plus the triple product of the first and the square of the second; 4th, minus the cube of the second:

$$(a - b)^3 = a^3 - 3\,a^2b + 3\,ab^2 - b^3. \qquad \textbf{(468)}$$

484. *The product of the sum of two quantities and their difference* is equal to the difference of the squares of the quantities:

$$(a + b) \times (a - b) = a^2 - b^2;$$
$$(2\,ab + 3\,b^2c) \times (2\,ab - 3\,b^2c) = 4\,a^2b^2 - 9\,b^4c^2. \qquad \textbf{(465, 468)}$$

485. *The square of any polynomial is composed of:* the square of the first term, twice the product of the first term and the second; the square of the second, twice the products of each of the first two terms and the third; the square of the third, twice the products of each of the first three terms by the fourth; the square of the fourth, etc. Thus we have (**465**, **468**):

$$(a + b - c)^2 = a^2 + 2\,ab + b^2 - 2\,ac - 2\,bc + c^2;$$
$$(a + bx + cx^2 + dx^3)^2 = a^2 + 2\,abx + b^2x^2 + 2\,acx^2 + 2\,bcx^3 + c^2x^4$$
$$+ 2\,adx^3 + 2\,bdx^4 + 2\,cdx^5 + d^2x^6.$$

DIVISION

486. An algebraic quantity is *divisible* by another when the quotient obtained is a whole quantity (**447**).

487. *In dividing one monomial by another*, there are 4 laws, as in multiplication (**462**), to be observed:

1st. *The sign of the quotient* is + or − according as the dividend and divisor have like or unlike signs. Thus, + divided by + or − divided by − gives + for the quotient, and + divided by − or − divided by + gives − for the quotient.

2d. *The coefficient of the quotient* is obtained by dividing the coefficient of the dividend by that of the divisor.

3d. *All letters in the dividend and divisor appear once in the quotient.*

4th. *The exponent of each letter of the quotient* is equal to the

exponent of that letter in the dividend minus the exponent of the same letter in the divisor.

From these laws we have,

$$\frac{24\,a^3b^3c^2d}{6\,ab^2c} = 4\,a^2bcd, \qquad \frac{15\,a^5b^8cd^3}{-\,3\,a^3b^5d^2} = -\,5\,a^2b^3cd.$$

REMARK 1. One monomial is divisible by another when the coefficient of the dividend is divisible by the coefficient of the divisor, and each letter of the divisor is found in the dividend with an exponent which is not less than the exponent of that letter in the divisor.

REMARK 2. In case of divisibility, the degree of the quotient is equal to the degree of the dividend less that of the divisor (453, 466).

488. *Special cases:*

1st. When the coefficient of the dividend is not exactly divisible by that of the divisors, the coefficient of the quotient is written in the form of a fraction reduced to its lowest terms (146). Thus:

$$\frac{6\,a^4b^2}{9\,a^2b} = \frac{2}{3}\,a^2b.$$

2d. When a letter has the same exponent in both dividend and divisor, the law of exponents (487, 4th) gives the exponent 0 in the quotient. Thus:

$$\frac{a^2}{a^2} = a^0.$$

Evidently $\frac{a^2}{a^2} = 1$ and $a^0 = 1$. Therefore letters having the same exponent in both dividend and divisor can be canceled.

From the law of exponents, $\frac{a^3}{a^2} = a^1$, and $\frac{a^3}{a^2} = \frac{a \times a \times a}{a \times a} = a$, and therefore $a^1 = a$ (305).

3d. When a letter has a larger exponent in the divisor than in the dividend, from the law of exponents (487, 4th) the exponent of this letter in the quotient is negative. Thus:

$$\frac{a^2}{a^5} = a^{2-5} = a^{-3}$$

4th. When a letter is found in the divisor which is not in the dividend, from the law of exponents we may suppose that letter

to be in the dividend with the exponent 0 (2d). In the quotient the letter will have a negative exponent equal to that in the divisor. Thus:

$$\frac{a^4}{a^2b} = \frac{a^4b^0}{a^2b} = a^2b^{-1}.$$

It is seen that the negative exponents make the rules in (**487**) of general application. Thus we have,

$$\frac{-12\,a^4b^2cde}{-8\,a^2b^3df^2} = \frac{3}{2}\,a^2bc^{-2}ef^{-2}.$$

489. *Although the method of using negative exponents is very convenient in many cases, it will not be used at first.*

In the cases shown above (**488**), excepting the 2d, the quotient may be written in the form of a fraction, the numerator being the dividend, and the denominator the divisor.

A fraction is reduced to its lowest terms: 1st, by dividing the two coefficients by their greatest common divisor; 2d, by canceling the letters which have the same exponent in both terms of the fraction; 3d, by subtracting the smaller exponent from the larger and writing the letter with an exponent equal to the difference in the term which had the larger exponent; 4th, by writing the letters not common to the two terms of the fraction, with their respective exponents, in the terms where they appear. Thus we have,

$$\frac{-12\,a^4b^2cde}{-8\,a^2bc^3df^2} = \frac{3\,a^2be}{2\,c^2f^2},\quad \frac{48\,a^3b^2cd^3}{36\,a^2b^3c^2de} = \frac{4\,ad^2}{3\,bce},\quad \frac{7\,ab^3c^5d}{3\,a^3bc^4d^3}$$
$$= \frac{7\,b^2c}{3\,a^2d^2},\quad \frac{7\,a^2b}{21\,a^4b^2} = \frac{1}{3\,a^2b}.$$

490. *To divide a polynomial by a monomial,* divide successively each term of the dividend by the divisor (**487**):

$$\frac{4\,a^5b + 2\,a^4b^2c - 5\,a^3b^3c^2}{a^2b} = 4\,a^3 + 2\,a^2bc - 5\,ab^2c^2.$$

A polynomial is divisible by a monomial when each term of the polynomial is divisible by the monomial (**487**). If the dividend is arranged according to the powers of some letter, the quotient will also be arranged according to the same letter (**471**).

In case some of the terms are not exactly divisible by the divisor, the division of the entire polynomial by the monomial

must be indicated, or only the non-divisible terms may be written as fractions:

$$\frac{4a^5b+3a^4b^2c-5ab^3}{2a^2b}=\frac{4a^5b}{2a^2b}+\frac{3a^4b^2b}{2a^2b}-\frac{5ab^3}{2a^2b}=2a^3+\frac{3a^2bc}{2}-\frac{5b^2}{2a}.$$

491. *To divide a polynomial by another* (see below, EXAMPLE 1), arrange both dividend and divisor according to the descending powers of the same letter a (**471**), divide the first term a^5 at the left of the dividend by the first term, a^3, at the left of the divisor, which gives the first term, a^2, of the quotient; multiply the divisor by this term and subtract the product $+a^5-3a^4b$ from the given dividend. Then divide the first term, $-2a^4b$, at the left of the remainder, by the first term, a^3, at the left of the divisor, which gives the second term, $-2ab$, of the quotient; multiply the divisor by the second term and subtract the product from the first remainder, which gives the second remainder. The operation is continued in the same manner until a remainder 0 or a remainder, the first term of which is not divisible by the first term of the divisor, is obtained.

In subtracting the products of the divisor and the terms of the quotient from the dividend and the successive remainders, the products are written under the remainders and their signs changed; thus each subtraction is performed by means of an addition, that is, by reducing the like terms (**461**).

EXAMPLE 1. Divide $5a^3b^2+3a^2b^3-5a^4b+a^5$ by a^3-3a^2b; according to the preceding rule we have:

$$\begin{array}{llll}
\text{Dividend} & a^5-5a^4b+5a^3b^2+3a^2b^3 & \Big| \; a^3-3a^2b & \text{divisor.} \\
 & \underline{-a^5+3a^4b\qquad\qquad\qquad} & \; a^2-2ab-b^2 & \text{quotient.} \\
\text{1st remainder} & -2a^4b+5a^3b^2+3a^2b^3 & & \\
 & \underline{+2a^4b-6a^3b^2\qquad\quad} & & \\
\text{2d remainder} & -a^3b^2+3a^2b^3 & & \\
 & \underline{+\;a^3b^2-3a^2b^3} & & \\
\text{Remainder of the division} & 0 & &
\end{array}$$

EXAMPLE 2.

$$\begin{array}{ll}
10a^4-48a^3b+51a^2b^2+4ab^3-15b^4+3b^5+c & \Big| \; -5a^2+4ab+3b^2 \\
\underline{-10a^4+8a^3b+6a^2b^2\qquad\qquad\qquad\qquad\qquad} & \; -2a^2+8ab-5b^2 \\
-40a^3b+57a^2b^2+4ab^3-15b^4+3b^5+c & \\
\underline{+40a^3b-32a^2b^2-24ab^3\qquad\qquad\qquad} & \\
+25a^2b^2-20ab^3-15b^4+3b^5+c & \\
\underline{-25a^2b^2+20ab^3+15b^4\qquad\qquad} & \\
\text{Remainder of the division} \qquad +3b^5+c &
\end{array}$$

EXAMPLE 3. Divide $x^4 - a^4$ by $x - a$.

$$\begin{array}{l|l}
x^4 - \quad a^4 & x - a \\
\underline{- \quad x^4 + ax^3} & \overline{x^3 + ax^2 + a^2x + a^3} \\
\quad ax^3 - a^4 & \\
\underline{- ax^3 + a^2x^2} & \\
\quad a^2x^2 - a^4 & \\
\underline{- a^2x^2 + a^3x} & \\
\quad a^3x - a^4 & \\
\underline{- a^3x + a^4} & \\
\quad 0 &
\end{array}$$

492. In the last example it is seen that the exponents of x diminish by 1 and those of a increase by 1 in the successive partial remainders and quotients.

Thus $x^m - a^m$ is exactly divisible by $x - a$, and we have:

$$\frac{x^m - a^m}{x - a} = x^{m-1} + ax^{m-2} + a^2x^{m-3} + \cdots + a^{m-2}x + a^{m-1}.$$

When $a = 1$ we have:

$$\frac{x^m - 1}{x - 1} = x^{m-1} + x^{m-2} + x^{m-3} + \cdots + x + 1.$$

$x^m + a^m$ is not divisible by $x - a$, the remainder is $2\,a^m$; thus we have:

$$\frac{x^m + a^m}{x - a} = x^{m-1} + ax^{m-2} + a^2x^{m-3} + \cdots + a^{m-2}x + a^{m-1} + \frac{2\,a^m}{x - a}\,.$$

$x^m - a^m$ is or is not divisible by $x + a$, according as m is even or odd, and we have respectively:

$$\frac{x^m - a^m}{x + a} = x^{m-1} - ax^{m-2} + ax^{m-3} - \cdots \pm a^{m-2}x \mp a^{m-1} + \frac{\pm a^m - a^m}{x + a}\,.$$

When m is even, the remainder $+ a^m - a^m = 0$, and when m is odd, the remainder $- a^m - a^m = - 2\,a^m$. $x^m + a^m$ is or is not divisible by $x + a$, according as m is odd or even, and we have respectively:

$$\frac{x^m + a^m}{x + a} = x^{m-1} - ax^{m-2} + a^2x^{m-3} - \cdots \mp a^{m-2}x \pm a^{m-1} + \frac{\mp a^m + a^m}{x + a}.$$

When m is odd, the remainder $- a^m + a^m = 0$, and when m is even, it is $+ a^m + a^m = 2\,a^m$.

493. When the principal letter in the polynomials to be divided has polynomials for coefficients, these coefficients are ar-

ranged as in multiplication (472), and the division performed according to the general rule (456):

$$
\begin{array}{r|r|r|r|}
\begin{array}{r} 18\,a^2 \\ -\ 3\,ab \\ -\ b^2 \\ \cdots\cdots \end{array} &
\begin{array}{r} x^3 + 24\,a^3 \\ -24\,a^2 \\ +\ 7\,a^2b \\ -\ ab \\ +\ 2\ b^2 \\ \cdots\cdots \\ +\ 6\,a^3 \\ +\ 3\,ab \\ -\ 2\,a^2b \\ -\ b^2 \\ \hline +30\,a^3 \\ -24\,a^2 \\ +\ 5\,a^2b \\ +\ 2\,ab \\ +\ b^2 \\ \cdots\cdots \end{array} &
\begin{array}{r} x^2 +\ 2\,a^4 \\ -10\,a^3 \\ +\ 2\,a^3b \\ -10\,a^2b \\ +\ 4\,ab \\ -\ b^2 \\ \cdots\cdots \\ +10\,a^4 \\ +\ 5\,a^2b \\ -\ 8\,a^3 \\ -\ 4\,ab \\ +2\,a^2b \\ +\ b^2 \\ \hline +12\,a^4 \\ -18\,a^3 \\ +\ 2\,a^3b \\ -\ 3\,a^2b \\ \cdots\cdots \\ -12\,a^4 \\ -\ 2\,a^3b \\ +18\,a^3 \\ +\ 3\,a^2b \\ \hline 0 \end{array} &
\begin{array}{r} x - 4\,a^5 \\ +6\,a^4 \\ -2\,a^3b \\ +3\,a^2b \\ \cdots\cdots \\ +4\,a^5 \\ +2\,a^3b \\ -6\,a^4 \\ -3\,a^2b \\ \hline 0 \end{array}
\end{array}
\begin{array}{r|r}
\begin{array}{r} 6\,a \\ +\ b \end{array} & \begin{array}{r} x - 2\,a^2 \\ -\ b \end{array} \\ \hline
\begin{array}{r} 3\,a \\ -\ b \end{array} & \begin{array}{r|r} \begin{array}{r} x^2 + 5\,a^2 \\ -4\,a \\ +\ b \end{array} & \begin{array}{r} x + 2a^3 \\ -3a^2 \end{array} \end{array}
\end{array}
$$

1st Partial Division

$$
\begin{array}{l|l}
18\,a^2 - 3\,ab - b^2 & 6\,a + b \\ \cline{2-2}
\underline{-\ 3\,ab\qquad} & 3\,a - b \\
\underline{-\ 6\,ab - b^2} & \\
\quad 0 &
\end{array}
$$

3d Partial Division

$$
\begin{array}{l|l}
12a^4 - 18a^3 + 2a^3b - 3a^2b & 6a + b \\ \cline{2-2}
\underline{\quad +18a^3 \qquad\qquad +3a^2b} & 2a^3 - 3a^2 \\
\qquad\qquad 0 &
\end{array}
$$

2d Partial Division

$$
\begin{array}{r|l}
30\,a^3 - 24\,a^2 + 5\,a^2b + 2\,ab + b^2 & 6\,a\ + b \\ \cline{2-2}
-\ 24\,a^2 \qquad\qquad + 2\,ab + b^2 & 5\,a^2 - 4\,a + b \\
\underline{+\ 4\,ab\qquad} & \\
\underline{6\,ab + b^2} & \\
0 &
\end{array}
$$

At one side divide the first term of the dividend by the first term of divisor, that is, the coefficient $18\,a^2 - 3\,ab - b^2$ by $6\,a + b$ and x^3 by x, which gives $(3\,a - b)\,x^2$ for the first term of the quotient. Multiply the divisor by this first term and subtract the product from the dividend. Divide, at one side, the first term $(30\,a^3 - 24\,a^2 + 5\,a^2b + 2\,ab + b^2)\,x^2$ of the remainder by the first term of the divisor, which gives the second term $(5\,a^2 - 4\,a + b)\,x$ of the quotient. Multiply the divisor by this second term and subtract the product from the first remainder. In the same manner the first term $(12\,a^4 - 18\,a^3 + 2\,a^3b - 3\,a^2b)\,x$

of the second remainder is divided by the first term of the divisor which gives the other terms, $2a^3$ and $-3a^2$, of the quotient which terms are independent of x in this particular example Multiplying the divisor by the expression $2a^3 - 3a^2$ and subtracting the product from the second remainder, the remainder of the division is obtained, which in this case is 0.

494. When the dividend and divisor are homogeneous (**455**) the quotient and the successive remainders are homogeneous Furthermore, the degree of the quotient is equal to that of the dividend less that of the divisor, and all the remainders are of the same degree as the dividend.

When the dividend is homogeneous and the divisor is not, the quotient has no end.

495. *The proofs of the four operations on algebraic quantities* are the same as in Arithmetic (**26**, **30**, **48**, **65**).

ALGEBRAIC FRACTIONS

496. An *algebraic fraction* is the quotient expressed by two quantities to be divided. Thus,

$$\frac{a}{b}, \qquad \frac{a^3 + b^4}{a + b},$$

pronounced *a over b* and $a^3 + b^4$ *over* $a + b$, or *a divided by b* and $a^3 + b^4$ *divided by* $a + b$, are algebraic fractions (**446**, 2d).

The dividend is the *numerator* of the fraction, the divisor is the *denominator*, and the numerator and denominator are the *terms* (**130**).

497. All that was said concerning numerical fractions applies to algebraic fractions as well. Thus we have:

$$\text{1st.} \quad a = \frac{ab}{b}; \tag{136}$$

$$\text{2d.} \quad \frac{a}{b} \times c = \frac{ac}{b} = \frac{a}{b : c}; \tag{140}$$

$$\text{3d.} \quad \frac{a}{b} : c = \frac{a}{bc} = \frac{a : c}{b}; \tag{141}$$

$$\text{4th.} \quad \frac{a}{b} = \frac{ac}{bc} = \frac{a : c}{b : c}. \tag{142}$$

To reduce a fraction to its simplest or lowest terms, divide the two terms by their common factors:

$$\frac{ac}{bc} = \frac{a}{b}, \quad \text{and} \quad \frac{12\,ab^3c^4}{3\,b^2c^6} = \frac{4\,ab}{c^2}. \tag{389}$$

It does not alter the value of a fraction to change the sign of both its terms, since that is to multiply both terms by -1:

$$\frac{a}{b} = \frac{-a}{-b}, \quad \text{and} \quad \frac{a+b-3\,c}{2\,a-d+e} = \frac{-a-b+3\,c}{-2\,a+d-e}.$$

5th. *The rules for reducing fractions to the same common denominator are the same as* (151):

6th. $\dfrac{a}{b} \pm \dfrac{c}{d} = \dfrac{ad \pm bc}{bd}$, and $a \pm \dfrac{b}{c} = \dfrac{ac \pm b}{c}$; (152, 153, 155, 156)

7th.
$$\left\{\begin{aligned} &a \times \frac{b}{c} = \frac{ab}{c}, \quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}, \\ &\left(a + \frac{b}{c}\right) \times \left(m - \frac{p}{q}\right) = \frac{(ac+b)(mq-p)}{qc}; \end{aligned}\right\} \tag{159, 160}$$

8th.
$$\left\{\begin{aligned} &a : \frac{b}{c} = \frac{ac}{b}, \quad \frac{a}{b} : \frac{c}{d} = \frac{ad}{bc} = \frac{a:c}{b:d}, \\ &\left(a + \frac{b}{c}\right) : \left(m - \frac{p}{q}\right) = \frac{(ac+b)\,q}{(mq-p)\,c}, \\ &1 : \frac{a}{b} = \frac{b}{a}, \quad \frac{a}{b} : \frac{c}{b} = \frac{a}{c}, \quad \frac{a}{b} : \frac{a}{c} = \frac{c}{b}. \end{aligned}\right\} \tag{164, 166}$$

9th. The sum of the terms of several equal fractions gives a fraction equal to any one of those fractions:

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f} = \frac{a+b+c}{d+e+f}. \tag{353}$$

$$\frac{a}{d} = \frac{b}{e} = \frac{c\sqrt{a^2+b^2+c^2}}{f\sqrt{d^2+e^2+f^2}} = \sqrt{\frac{a^2+b^2+c^2}{d^2+e^2+f^2}} = \sqrt[m]{\frac{a^m+b^m+c^m}{d^m+e^m+f^m}}; \tag{354}$$

10th. Let p be the period, of n figures, of a simple periodic decimal number. Then letting $x = 0.ppp\ldots$ represent the fraction, and multiplying by 10^n, we have $10^n x = p.ppp\ldots$ Subtracting the value of x, we now have $(10^n - 1)\,x = p$, and therefore

$$x = \frac{p}{10^n - 1}. \quad \text{If } n = 3, \text{ we have } x = \frac{p}{999},$$

which confirms what was said in Arithmetic (195).

BOOK II

EQUATIONS OF THE FIRST DEGREE

EQUATIONS OF THE FIRST DEGREE INVOLVING ONE UNKNOWN

498. Two equal expressions joined by the sign = constitute an *equation.* These expressions are the two *members* or *sides* of the equation, the one at the left being the *first member,* and the one at the right the *second member.* Such are:

$$3 + x = 7 \text{ and } x + y = \frac{a}{b}.$$

499. Equations which hold true only for particular values of the symbols involved are called *equations* of *condition.*

500. Equations which hold true for all values of the symbols involved are called *identical equations* or *identities.* Such are the equations:

$$2x+4=2x+4, \quad (a-b)^2=a^2-2ab+b^2 \text{ and } (a+b)(a-b)=a^2-b^2.$$

When the two members of an equation are the same, or when, as in the last two examples (**445**, **449**), one member is nothing but the result of the calculations indicated in the other, the equation is an *identity,* and the members should be connected by the *sign of identity,* $\equiv$. Thus,

$$(x + y)^2 \equiv x^2 + 2xy + y^2.$$

501. *Any equation should become an identity* when the numerical values are substituted for the unknowns.

502. An *equation is numerical* when it contains no letters except the unknowns; it is *algebraic* or *literal* when the knowns are represented by letters.

503. When one member of the equation contains only the unknown and the other the knowns, the equation is called a *formula* (**445**). Thus, in the following,

$$x = a^2 + 4\frac{b}{c},$$

the second member is the expression of the value of the unknown.

504. Two quantities which vary simultaneously, in such a

manner that the variation of one causes a variation of the other, are said to be *functions* of each other. The area s of a circle varies with the radius r; it is a function of the radius. This relation is represented in a general way by $s = f(r)$ or $\phi(r)$. (See Geometry.)

Likewise the distance which a body falls is a function of the time, and conversely the time is a function of the distance.

Ordinarily, one of the quantities is considered as varying in an arbitrary manner, and is called the *independent variable*, while for the other the variation is determined by that of the first, and this one is called the *function* or the *dependent variable*.

When the relation which exists between several variables can be expressed by an equation containing only algebraic quantities (**447**, **499**), the *function* is said to be *algebraic;* but if the relation between the function and the independent variable cannot be expressed by the signs $+$, $-$, $\times$, $\div$, $\sqrt{\ }$, and exponents, the function is said to be *transcendental.* Thus the logarithm of a number is a transcendental function of the number. Trigonometric functions are also transcendental. (See Trigonometry.)

505. The *root of an equation or system of equations* is each value of the unknown or each system of values of the unknowns which renders the equation or system of equations identical (**501**). Thus the value 3 of x is the root of the equation

$$5x = 15.$$

506. *To solve an equation or system of equations* is to find all the roots of the equation or system of equations.

507. Two equations are *equivalent* when they have the same roots and the same number of roots. Such are:

$$5x = 15 \quad \text{and} \quad x + 7 = 10.$$

508. *To alter an equation or a system of equations* is to transform them so as to change the roots or the number of roots.

509. *The solution of equations and systems of equations rests upon the following principles:*

1st. An equation is not altered by increasing or diminishing both its members by the same quantity. Thus, an equation may be simplified by canceling the terms common to both members.

2d. An equation is not altered by *transposing* a term, that is,

transferring a term from one member to the other and changing its sign, which is the same as adding this term to both members or subtracting it from them according as the sign is $-$ or $+$. From this it follows that the signs of all of the terms of an equation may be changed without altering the equation.

3d. An equation is not altered when both members are multiplied or divided by the same quantity, which cannot be zero nor contain any unknowns. If the quantity contained unknowns, the new equation would not be of the same degree as the first and would not be equivalent to it; if the equation is multiplied, in addition to the root of the first, it would have that of the equation obtained by putting the quantity used as multiplier equal to 0. Thus, multiplying the two members of the equation

$$x - 5 = 0$$

by $x - 3$, we have

$$(x - 5)(x - 3) = 0;$$

besides the root $x = 5$ of the first equation, the new equation contains the root $x = 3$ of the equation $x - 3 = 0$.

Dividing by a quantity which contains an unknown reduces the number of roots of the equation.

According to the above, 3d, an equation may be simplified by canceling the factors or common divisors of the two members.

4th. In eliminating the denominators, which is done by reducing all the terms of the equation to the same denominator, and then leaving off this denominator (**497**, 5th), a new equation is obtained which is equivalent to the first.

For simplicity the common denominator should be as small as possible, and therefore it should be the least common multiple of the denominators of the given equation. Eliminating the denominators from the equation:

$$2 + \frac{9}{6+x} = x \quad \text{or} \quad \frac{12 + 2x + 9}{6+x} = \frac{6x + x^2}{6+x} \quad \text{or} \quad \frac{x^2 + 4x - 21}{6+x} = 0,$$

we have

$$x^2 + 4x - 21 = 0, \quad \text{whence (538)}, \quad x = \begin{cases} 3. \\ -7 \end{cases}$$

These two roots satisfy the given equation. Operating in the same manner on the equation:

$$1 + \frac{1}{x-1} = \frac{x^2}{x-1} - 6 \text{ or } \frac{x^2 - 7x + 6}{x-1} = 0,$$

we have $$x^2 - 7x + 6 = 0, \text{ whence } x = \begin{cases} 6 \\ 1 \end{cases}.$$

The root, $x = 6$, satisfies the given equation, and the root, $x = 1$, since it makes the denominator equal to 0, gives

$$\frac{x^2 - 7x + 6}{x - 1} = \frac{0}{0};$$

which expression is meaningless, and indicates that $x - 1 = 0$ is a common factor of the numerator and denominator, and should be canceled (526), which gives $x - \frac{6}{1} = 0$, of which the only root is $x = 6$.

General Rule. When the denominators are eliminated, if one or several roots render the common denominator equal to 0, these roots should be neglected. Thus, in the preceding example $x = 1$ would be rejected and $x = 6$ retained.

5th. An equation is not altered by any modifications of its members which do not change their value. Thus, for example, the operations indicated by the signs may be performed without changing the value.

6th. A system of several equations is not altered when one of them is replaced by an equation obtained by adding or subtracting the members of the given equations.

7th. When the two members of an equation are squared

$$x = 5,$$

the equation,

$$x^2 = 25,$$

which results, has, besides the root of the equation $x - 5 = 0$ or $x = 5$, the root of the equation $x + 5 = 0$ or $x = -5$.

Which follows from

$$x^2 - 25 = (x + 5)(x - 5) = 0.$$

510. The *degree* of an equation is the greatest sum of the exponents of the unknowns in any one term of the equation. Thus the equations

$$2x - y = 7, \quad 3xy = 18, \quad y^2x^5 = 1 - x^6,$$

are respectively of the 1st, 2d, and 7th degree. This method of determining the degree of an equation assumes that there are no

unknowns in the denominator. When there are unknowns in the denominator, eliminate the denominators (509, 4th), and determine the degree as shown in the preceding case.

The equation $a + \frac{bx}{b+y} = y$, which appears to be of the first degree, is of the second, because in reducing all the terms to the same common denominator, $b + y$, and then neglecting it, the equation becomes

$$ab + ay + bx = by + y^2, \quad \text{or} \quad y^2 + (b - a)\, y - bx - ab = 0.$$

If the common denominator, $b + y$, was a factor of the first member of the equation in its final form, it would be necessary to divide it out before determining the degree. In solving the equation without eliminating the common denominator as factor of the first member, and neglecting the roots which make the denominator equal to 0, roots of the given equation are obtained which are of the true degree (526).

511. *General rule for the solution of an equation of the first degree involving one unknown:*

1st. Eliminate the denominators if there are any (509).

2d. Transpose the terms, that is, transpose to one member, generally the first, all the terms which contain the unknown, and to the other member all the knowns.

3d. Reduce the like terms (458), that is, the algebraic sum of all the coefficients of the unknown is taken as its coefficient, which reduces the first member to one term; then the operations indicated by the signs in this coefficient and in the second member of the equation are performed.

4th. Finally the second or known member is divided by the coefficient of the first, which gives the value of the unknown as quotient.

Example 1.

$$6\,x - 2 = 2\,x + 6.$$

Transposing:

$$6\,x - 2\,x = 6 + 2.$$

Reducing:

$$(6 - 2)\,x = 8 \quad \text{or} \quad 4\,x = 8, \quad \text{therefore} \quad x = \frac{8}{4} = \mathbf{2}.$$

Example 2.

$$\frac{ax}{b} + \frac{x}{c} - 2 = 8 - \frac{x}{d}.$$

Eliminating the denominators (**474**, 4th):

$$acdx + bdx - 2\,bcd = 8\,bcd - bcx.$$

Transposing and reducing:

$$(acd + bd + bc)\,x = 8\,bcd + 2\,bcd = 10\,bcd.$$

Therefore

$$x = \frac{10\,bcd}{acd + bd + bc}.$$

512. From that which precedes, it is seen that an equation of the first degree involving one unknown, can always be reduced to the general form, $ax = b$, from which $x = \frac{b}{a}$, wherein b and a are known quantities.

513. *The solution of an algebraic problem is composed of three parts:*

1st. *The writing in the form of equations,* which consists in expressing algebraically the conditions of the problem considering it as solved. This amounts to indicating, by means of algebraic signs (**446**), the operations which would have to be performed upon the unknown values to prove that they satisfy the conditions of the problem; therefore, to put a problem in the form of equations, simply indicate its proof.

2d. *The solution of the equations,* which consists in determining the values of the unknowns in such a manner that only known quantities enter in these values (**511**).

3d. The *proof* that the values of the unknowns satisfy the conditions of the problem (**501**).

514. Example:

$\frac{1}{2}$ *plus* $\frac{1}{3}$ *plus* $\frac{1}{4}$ *of a certain number* x *plus* 45 *gives* 448 *as the sum. What is the number?*

1st. Writing in the form of an equation:

$$\frac{1}{2}x + \frac{1}{3}x + \frac{1}{4}x + 45 = 448.$$

2d. Solution of the equation (**511**):

Eliminating the denominators,

$$6\,x + 4\,x + 3\,x + 12 \times 45 = 12 \times 448,$$

or $\quad 13\,x = 5376 - 540 = 4836, \quad$ whence $\quad x = \frac{4836}{13} = 372.$

3d. Proof:

$$\frac{372}{2} + \frac{372}{3} + \frac{372}{4} + 45 = 448,$$

or

$$186 + 124 + 93 + 45 = 448.$$

The first member being equal to the second, 372 is the correct solution of the problem.

EQUATIONS OF THE FIRST DEGREE INVOLVING SEVERAL UNKNOWNS

515. *When an equation involves several unknowns, it may have an infinite number of solutions.* Thus, assuming arbitrary values for all of the unknowns except one, and solving the equation, the value of the one unknown, together with the assumed values of the others, forms a solution; and it is seen that since an infinite number of arbitrary values may be assumed, there is an infinite number of solutions.

516. *In general, to be able to determine all the unknown quantities there must be as many equations as there are unknowns.*

517. When the number of equations is greater than the number of unknowns by a number m, the given system of equations has no solution except when the m equations of condition between the numbers and the constants which enter in the system, can be satisfied.

518. In solving a problem involving several unknowns, there must be as many equations as there are unknowns, and this collection of equations is called a *system of simultaneous equations.* Equations which are satisfied by the same values of the unknowns are called *simultaneous equations.*

519. *To eliminate an unknown from a system of m equations,* deduce from the given system a system of $m - 1$ equations which do not contain the unknown.

By whatever method a system of simultaneous equations is solved, it is always by elimination.

520. *There are three methods of solving two simultaneous equations of the first degree involving two unknowns:*

1st. *The method of substitution.*

Having two simultaneous equations of the first degree, which involve two unknowns, x and y, given, to find the value of one of the unknowns in terms of the other, for example, the value of

y in terms of x (511), substitute this value of y in the other equation, which gives an equation of the first degree involving only x; solving for the value of x, and substituting that value in the first equation, the value of y is found.

Let $x + y = c$ and $x - y = c'$ be given.

From the second, y in terms of x is:

$$y = x - c' \qquad (1)$$

Substituting this value in the first:

$$x + x - c' = c,$$

and

$$2\,x = c + c', \quad \text{or} \quad x = \frac{c + c'}{2}$$

Substituting this value of x in equation (1):

$$y = \frac{c + c'}{2} - c' = \frac{c + c' - 2\,c'}{2} = \frac{c - c'}{2}.$$

For $c = 12$ and $c' = 6$ we have:

$$x = \frac{12 + 6}{2} = 9, \quad \text{and} \quad y = \frac{12 - 6}{2} = 3.$$

Proof:

$$x + y \quad \text{or} \quad 9 + 3 = 12,$$

and

$$x - y \quad \text{or} \quad - 3 = 6.$$

2d. *The method of comparison.*

The value of one of the unknowns, y for example, is expressed in terms of the other in each of the given equations, and then these two expressions, being both equal to the same quantity, y, may be taken as members of a new equation, which contains only one unknown, x; solving for x, and substituting this value in the given equations, we may solve for y.

Given:

$$x + y = c,$$
$$x - y = c'.$$

Then

$$y = c - x \quad \text{and} \quad y = x - c', \qquad (2)$$

and

$$c - x = x - c' = y, \text{ from which } x = \frac{c + c'}{2}.$$

Substituting this value of x in one of the equations (2), we have:

$$y = \frac{c + c'}{2} - c' = \frac{c - c'}{2}.$$

3d. *The method of addition or subtraction.* By multiplying or dividing the terms of one of the equations by a certain number, the coefficient of one of the unknowns is made to equal that of the same unknown in the other equation. Then the members of the two equations which have the same coefficients are either added or subtracted, according as the signs of the equal coefficients are unlike or like, and thus the resulting equation contains only one unknown and may be solved. Having found the value of one, the value of the other may be found by substituting the value of the first in one of the given equations.

EXAMPLE 1. $$\begin{cases} x + y = c, \\ x - y = c'. \end{cases}$$

Considering the unknown y, we see that it has the same coefficient in both equations, and since it has unlike signs in the two equations, it may be eliminated by adding the members of the two equations. Thus:

$$2x = c + c' \quad \text{or} \quad x = \frac{c + c'}{2}.$$

Likewise considering the unknown x, we see that it also has the same coefficient in both equations, and since it has like signs, the elimination is accomplished by subtracting the members of the two equations. Thus:

$$2y = c - c' \quad \text{or} \quad y = \frac{c - c'}{2}.$$

EXAMPLE 2. $$\begin{cases} ax + y = c, & (1) \\ a'x - b'y = c'. & (2) \end{cases}$$

Considering the unknown y, it is seen that the terms of the first equation must be multiplied by b' in order that it have the same coefficient in both equations. Thus:

$$ab'x + b'y = cb'. \qquad (3)$$

Adding (2) and (3),

$$(a' + ab')x = eb' + c' \quad \text{and} \quad x = \frac{cb' + c'}{a' + ab'}.$$

Considering the unknown x, it is seen that the terms of equation (1) must be multiplied by a' and those of (2) by a in order to obtain two equations with the same coefficients of x. Thus:

$$aa'x + a'y = ca', \qquad (4)$$
$$aa'x - ab'y = c'a. \qquad (5)$$

Subtracting (5) from (4),

$$(a' + ab')\,y = ca' - ac', \text{ and } y = \frac{ca' - ac'}{a' + ab'}.$$

521. From the foregoing it is seen that any system of two simultaneous equations of the first degree involving two unknowns may be reduced to the general form (**512**),

$$ax + by = c,$$
$$a'x + b'y = c'.$$

From which,

$$x = \frac{cb' - bc'}{ab' - ba'} \text{ and } y = \frac{ac' - ca'}{ab' - ba'}.$$

522. PROBLEM. A man has some \$2.00 and \$5.00 bills; he must pay a bill of \$26.00 with 10 of these bills; how many of each kind will he use?

Let x = the number of twos and y = the number of fives.

1st. Writing in the form of an equation (**513**):

$$x + y = 10 \text{ bills},$$
$$2\,x + 5\,y = 26 \text{ dollars}.$$

2d. Solving by any one of the methods of (**520**):

$$x = 8 \quad \text{and} \quad y = 2.$$

3d. Proof:

$$x + y \quad \text{or} \quad 8 + 2 = 10 \text{ bills},$$
$$2\,x + 5\,y \quad \text{or} \quad 16 + 10 = 26 \text{ dollars}.$$

523. *To solve a system of three simultaneous equations of the first degree involving three unknowns*, such as the following, for example, which is the general form of any system of three simultaneous equations of the first degree involving three unknowns,

$$ax + by + cz = d, \tag{1}$$
$$a'x + b'y + c'z = d', \tag{2}$$
$$a''x + b''y + c''z = d'', \tag{3}$$

by the aid of the three methods in (**520**) one of the unknowns, z, for instance, may be eliminated.

1st. Between the equations (1) and (2):

$$(ac' - ca')\,x + (bc' - cb')\,y = dc' - cd'; \tag{4}$$

2d. Between the equations (2) and (3):

$$(a'c'' - c'a'')\,x + (b'c'' - c'c'')\,y = d'c'' - c'd''. \tag{5}$$

Thus two equations, (4) and (5), of the first degree, involving two unknowns, are obtained; eliminating y between them, we have:

$$x = \frac{db'c'' - dc'b'' + cd'b'' - bd'c'' + bc'd'' - cb'd''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}.$$

In the same manner x and z may be eliminated, and we have:

$$y = \frac{ad'c'' - ac'd'' + ca'd'' - da'c'' + dc'a'' - cd'a''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}.$$

Eliminating x and y, we have:

$$z = \frac{ab'd'' - ad'b'' + da'b'' - ba'd'' + bd'a'' - db'a''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}.$$

524. Considering the results in articles 512, 521, and 523, we see:

1. (512) That for an equation of the first degree, involving one unknown, the number of terms in the numerator and in the denominator may be reduced to 1.

2. (521) That for two simultaneous equations, involving two unknowns, the number may be reduced to 2 or 1×2.

3. (523) That for three simultaneous equations, involving three unknowns, the number may be reduced to 6 or $1 \times 2 \times 3$.

These numbers would be 24 or $1 \times 2 \times 3 \times 4$ for four simultaneous equations, involving four unknowns; 120 or $1 \times 2 \times 3 \times 4 \times 5$ for five simultaneous equations, involving five unknowns, and so on.

525. The use of the primes in the notation of the coefficients gave rise to a rule for the formation of the numerators and denominators of the values of the unknowns. *Considering the two equations with the two unknowns* (521):

1st. *To obtain the common denominator of the two values of the unknowns,* form with the letters a and b, which are the coefficients of the letters x and y in the first equation $ax + by = c$, the two permutations ab and ba; separate these permutations by the sign $-$, which gives $ab - ba$, and place a prime over the last letter in each term, which gives the common denominator:

$$ab' - ba'.$$

2d. *To obtain the numerator relative to each of the unknowns,* replace, in the denominator, the letters which represent the coefficients of the unknown, by the letters which represent the

known quantities, leaving the primes as they were. Thus, for the unknowns x and y, the denominator $ab' - ba'$ gives respectively the numerators $cb' - bc'$ and $ac' - ca'$.

Considering the case of three equations and three unknowns:

1st. *To obtain the common denominator*, introduce the letter c successively at the right, in the middle and at the left of each of the permutations ab and ba; this gives six new permutations, which are separated alternatively by the signs + and −, thus:

$$abc - acb + cab - bac + bca - cba.$$

Placing in each of the six terms of this polynomial one prime on the second letter and a double prime on the third letter, the common denominator is obtained:

$$ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''.$$

2d. *To obtain the numerator of each of the values of the unknowns*, substitute, in the denominator, the constant quantity for the coefficient of the unknown, leaving the primes as before. Thus, for example, to obtain the numerator of the value of x, substitute d for a, which gives:

$$db'c'' - dc'b'' + cd'b'' - bd'c'' + bc'd'' - cb'd''.$$

NEGATIVE, IMPOSSIBLE, AND INDETERMINATE ROOTS OF EQUATIONS

526. *Examples of some singular roots which may be obtained in the solution of a problem.*

1st. *Negative Roots.* a being the age of a father and b that of his son, in how long a time, x, will the father be three times the age of the son? Writing the problem in the form of an equation,

$$a + x = 3\,(b + x), \text{ and } x = \frac{a - 3\,b}{2}.$$

Inspecting this formula, it is seen that the value of x is positive or negative according as a is greater or less than $3\,b$, which can be stated: according as $a > 3\,b$ or $a < 3\,b$ should the time x be reckoned in the future or the past.

For $a = 45$ and $b = 11$, we have $x = \frac{45 - 33}{2} = 6$ yrs.; that is, in six years the father will be three times as old as his son.

For $a = 55$ and $b = 23$, we have $x = \frac{55 - 69}{2} = -7$ yrs.;

that is, seven years ago the father was three times as old as his son.

2d. *Impossible Roots.*

One-half plus one-third of a certain number plus 5 equals $\frac{5}{6}$ of the same number plus 7; what is the number?

From inspection it is seen that this problem is impossible, since $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$ we cannot have:

$$\frac{x}{2}+\frac{x}{3}+5=\frac{5}{6}x+7.$$

Solving this equation, we have:

$3\,x+2\,x+30=5\,x+42$ or $(3+2-5)\,x=42-30$, and $x=\dfrac{12}{5-5}$, that is,

$$0\times x=12 \quad\text{or}\quad x=\frac{12}{0}=\infty.$$

This formula indicates the impossibility of assigning to x a value which will fulfill the conditions of the problem. The sign ∞ is that of *infinity.*

In general the symbol of impossibility is:

$$\frac{a}{0}=\infty \quad\text{or}\quad \frac{a}{\infty}=0.$$

3d. *Indeterminate Roots.*

One-half plus one-third of a certain number plus 7 equals $\frac{5}{6}$ of the same number plus 7; what is the number?

Writing the problem in the form of an equation:

$$\frac{x}{2}+\frac{x}{3}+7=\frac{5}{6}x+7.$$

Since $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$, this equation is an identity for any value given to x, and is therefore indeterminate. Solving the equation,

$3\,x+2\,x+42=5\,x+42$, or $(5-5)\,x=42-42$, and $x=\dfrac{42-42}{5-5}$, that is,

$$0\times x=0 \text{ or } x=\frac{0}{0},$$

which is the symbol of indetermination.

Remark. However, the symbol $\frac{0}{0}$ does not always indicate

that the equation is indeterminate; as, for example, when the numerator and denominator contain a common factor which becomes zero for certain values of the letters (509, 4th). In this case the common factor must be canceled in order to obtain the value of x. Suppose the following to be the solutions of several equations:

$$x = \frac{a^3 - b^3}{a^2 - b^2}, \qquad x = \frac{2(a-b)^2}{3(a^2 - b^2)}, \qquad x = \frac{2(a^2 - b^2)}{3(a-b)^2},$$

which take the form $\frac{0}{0}$ when $a = b$. The factor $a - b$, which becomes zero when $a = b$, being common to both terms, may be canceled, which gives,

$$x = \frac{a^2 + ab + b^2}{a + b}, \qquad x = \frac{2(a-b)}{3(a+b)}, \qquad x = \frac{2(a+b)}{3(a-b)}.$$

Supposing $a = b$, we have

$$x = \frac{3a}{2}, \qquad x = \frac{0}{6a} = 0, \qquad x = \frac{4a}{0} = \infty,$$

which are respectively *finite, zero,* and *infinite* (509, 4th).

INEQUALITIES

527. Two algebraic expressions separated by the sign $>$ or $<$ form an *inequality*. These two expressions are the *members* of the inequality.

It is understood that in a general way a quantity, A, is greater than a quantity, B, when the difference, $A - B$, is positive; and that $A < B$ when the difference is negative. From this it follows that all positive quantities are greater than zero, and that all negative quantities are less than zero, being as much less as their absolute value is greater. Thus we have,

$$\frac{1}{2} > 0, \qquad 0 > -6, \qquad 3 > -4, \qquad -3 > -7,$$

because

$$\frac{1}{2} - 0 = \frac{1}{2}, \quad 0 - (-6) = 6, \quad 3 - (-4) = 7, \quad -3 - (-7) = 4.$$

528. With a few modifications, the principles given in (509) for equations apply as well to inequalities.

1st. An inequality is not reversed when the same quantity is added to or subtracted from both its members. Thus:

$$5 > 3, \quad \text{we have} \quad 5 - 7 > 3 - 7 \quad \text{or} \quad -2 > -4.$$

It follows also that a term may be transposed from one member to the other by changing its sign.

But if the signs of all the terms are changed, the inequality is reversed. Thus:

$$5 > 3, \quad \text{and} \quad -5 < -3.$$

2d. An inequality is not reversed when both members are multiplied or divided by the same positive number, but is reversed when the number is negative. Thus:

$12 > 4$ gives: $12 \times 2 > 4 \times 2$, $\frac{12}{2} > \frac{4}{2}$, $12 \times -2 < 4 \times -2$;

having $12 - 4 = 8$, we have,

$12 \times 2 - 4 \times 2 = 8 \times 2$, $\frac{12}{2} - \frac{4}{2} = \frac{8}{2}$, $12 \times -2 - (4 \times -2) = 8 \times -2$.

3d. The sum of the members of several inequalities in the same sense gives an inequality also in that sense.

4th. According as the two members of an inequality are positive or negative, their squares form an inequality in the same sense as the first or reversed:

$$a > b \quad \text{gives} \quad a^2 > b^2 \quad \text{and} \quad -a > -b \quad \text{gives} \quad a^2 < b^2.$$

529. By aid of these principles an inequality may be solved, following the same steps as in solving an equation (**511**). The x which should satisfy the condition

$$\frac{3x}{2} - 7 > x + \frac{2}{3},$$

we have successively:

$$9x - 42 > 6x + 4, \quad 9x - 6x > 42 + 4, \quad 3x > 46, \quad x > \frac{46}{3}.$$

Any quantity greater than $\frac{46}{3}$ fulfills the conditions of the given inequality.

BOOK III

POWERS AND ROOTS OF ALGEBRAIC QUANTITIES

SQUARE ROOTS

530. The powers and roots in Algebra have the same signification as in Arithmetic (**85**, **236**, **430**, **444**).

531. *The square of a product* is equal to the product of the squares of the factors:

$$(3\,a^2b^3c)^2 = 9\,a^4b^6c^2. \qquad (\mathbf{299},\ \mathbf{465})$$

532. A *fraction is squared* by squaring its terms:

$$\left(\frac{3\,a}{b^2}\right)^2 = \frac{9\,a^2}{b^4}. \qquad (\mathbf{300})$$

533. *The square of a binomial* is equal to the square of the first term plus twice the product of the first term and the second, plus the square of the second. The double product is positive or negative according as the terms have like or unlike signs (**479**, **480**). (See Art. **485** for *square of any polynomial.*)

534. Since in forming the square of the square root of a quantity the quantity is obtained, it follows from (**465**) that in order to *extract the square root of a monomial,* extract the square root of its coefficient and divide its exponents by 2:

$$\sqrt{36\,a^8b^2c^6} = 6\,a^4bc^3.$$

From this rule it follows that a monomial is not a perfect square, and that its square root cannot be extracted when its coefficient is not a perfect square (**248**), and its exponents even numbers.

When the square root of an imperfect square is to be extracted, simply indicate the operation by putting the quantity under a radical. Thus, having to extract the square root of $35\,a^4b$, write simply

$$\sqrt{35\,a^4b}.$$

Such quantities are called *irrational monomials* (**447**), or *surds.*

535. *The square root of the product of two or any number of factors is equal to the product of the square roots of these factors* (**301**, **531**).

$$\sqrt{36\,a^2b^3c^5} = \sqrt{36} \times \sqrt{a^2} \times \sqrt{b^3} \times \sqrt{c^5}.$$

536. From this it follows that in order *to simplify an irrational monomial* (534), separate it into factors and extract the root of the perfect squares, leaving the surds under the radical. Thus,

$$\sqrt{36\,a^2b^3c^5} = 6\,a\,\sqrt{b^3c^5}, \text{ and } \sqrt{8\,ab^4c^5}\ \ 2\,b^2\,\sqrt{2\,ac^5}.$$

In the above expressions $6\,a$ and $2\,b^2$ are the *coefficients of the surd*, and the second member is called a *mixed surd.*

537. The square of a positive or negative quantity being always positive (465), it follows that a *positive monomial has two equal square roots opposite in sign.* Thus,

$$\sqrt{4\,a^6b^2} = \pm\,2\,a^3b.$$

538. The square of any quantity being positive (465), it follows that the extraction of the square root of a negative quantity is impossible. Thus,

$$\sqrt{-16}=4\,\sqrt{-1},\ \ \sqrt{-4\,a^6b^2}=2\,a^3b\,\sqrt{-1},\ \sqrt{-3\,ab^2} = b\,\sqrt{3\,a}\,\sqrt{-1}$$

are algebraic symbols which represent impossible operations. They are called *imaginary expressions.* Problems in the second degree often conduct to these results.

The general form of an imaginary quantity is $a\sqrt{-1}$, in which a is real.

Any imaginary root in an equation of the second degree may be put in the form $a \pm b\,\sqrt{-1}$, in which a and b are real quantities (572).

539. *The square root of a fraction is obtained by extracting the square root of each of its terms:*

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}. \qquad (302, 532)$$

540. Two radicals are *similar* when they differ only in their coefficients (536). Such are:

$$3\,\sqrt{ab^3},\ \ (c+d)\,\sqrt{ab^3},\ \ 2\,(c+2\,d)\,\sqrt{ab^3}.$$

541. *The combination of similar radicals by addition or subtraction.* Perform the operations upon the coefficients and use the result as coefficient of the radical. Thus,

$$3\,\sqrt{ab^3} + (c+d)\,\sqrt{ab^3} = (3+c+d)\,\sqrt{ab^3},$$
$$3\,\sqrt{ab^3} - (c+d)\,\sqrt{ab^3} = (3-c-d)\,\sqrt{ab^3}.$$

If the radicals were not similar, the operations would simply be indicated. Thus, adding $\sqrt{a}$ and $3\sqrt{b}$, we have:

$$\sqrt{a} + 3\sqrt{b}.$$

and subtracting we have:

$$\sqrt{a} - 3\sqrt{b}.$$

542. *To multiply a radical of the second degree by another*, multiply the quantities under the radicals together, and for coefficient of the product take the product of the coefficients of the given radicals. Thus,

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab},\ \ 3\sqrt{5\,a^2b} \times -5\sqrt{ab} = -15\sqrt{5\,a^3b^2},$$
$$2\sqrt{3\,a + b^2} \times 5\,c\sqrt{3\,a + b^2} = 10\,c\sqrt{(3\,a + b^2)^2} = 10\,c\,(3\,a + b^2).$$

It is evident, that if the radicals are similar, as in this last case, the product is obtained by neglecting the $\sqrt{\ }$ sign and multiplying the quantity under it by the product of the coefficients of the given radicals.

543. *To divide a radical of the second degree by another.* Divide the quantities under the radical separately, taking the quotient of the coefficients for the coefficient of the result. Thus,

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}},\ \ \frac{5\,a\sqrt{b}}{2\,b\sqrt{c}} = \frac{5\,a}{2\,b}\sqrt{\frac{b}{c}},\ \ \frac{12\,ac\sqrt{6\,bc}}{4\,c\ \sqrt{2\,b}} = 3\,a\sqrt{3\,c}.$$

544. *To remove factors which are perfect squares from under the radical*, write their square root outside of the radical as factors of the coefficient (536). Thus,

$$\sqrt{3\,a^2b^4c} = ab^2\sqrt{3\,c},\ \ 8\,d\sqrt{a^3b^2c^4} = 8\,bc^2d\sqrt{a^3}.$$

To place a factor of the coefficient under the radical, square it and write it under the sign $\sqrt{\ }$ as a factor of the radical. Thus,

$$3\sqrt{a} = \sqrt{9\,a},\ \ a\sqrt{b} = \sqrt{a^2b},$$
$$4\,a\sqrt{b + c} = 4\sqrt{a^2(b + c)} = \sqrt{16\,a^2(b + c)}.$$

545. A calculation involving irrational expressions may often

be simplified by eliminating the radicals from the denominators. Examples:

$$\frac{7}{2\sqrt{5}} = \frac{7\sqrt{5}}{10}, \quad \frac{m}{\sqrt{a}+\sqrt{b}} = \frac{m(\sqrt{a}-\sqrt{b})}{a-b}, \quad \frac{\sqrt{m}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{ma}+\sqrt{mb}}{a-b},$$

$$\frac{3\sqrt{11}}{4\sqrt{2}+2\sqrt{3}} = \frac{3\sqrt{11}(4\sqrt{2}-2\sqrt{3})}{16\times 2 - 4\times 3} = \frac{12\sqrt{22}-6\sqrt{33}}{20}$$
$$= \frac{6\sqrt{22}-3\sqrt{33}}{10}.$$

The two terms of the fractions were multiplied respectively by $\sqrt{5}$, $\sqrt{a}-\sqrt{b}$, $\sqrt{a}+\sqrt{b}$, $4\sqrt{2}-2\sqrt{3}$, so as to *rationalize* the denominators (**484**). In the following example, the two terms of the given fraction are first multiplied by $(\sqrt{a}+\sqrt{b})+\sqrt{c}$; then the terms of the fraction thus obtained by $(a+b-c) - 2\sqrt{ab}$:

$$\frac{\sqrt{m}}{\sqrt{a}+\sqrt{b}-\sqrt{c}}$$

or

$$\frac{\sqrt{m}}{(\sqrt{a}+\sqrt{b})-\sqrt{c}} = \frac{\sqrt{ma}+\sqrt{mb}+\sqrt{mc}}{(\sqrt{a}+\sqrt{b})^2-c} = \frac{\sqrt{ma}+\sqrt{mb}+\sqrt{mc}}{a+b-c+2\sqrt{ab}}$$

or

$$\frac{\sqrt{ma}+\sqrt{mb}+\sqrt{mc}}{(a+b-)+2\sqrt{ab}} = \frac{(\sqrt{ma}+\sqrt{mb}+\sqrt{mc})(a+b-c-2\sqrt{ab})}{(a+b-c)^2-4ab}.$$

546. From what was said in Art. **485** concerning the square of any polynomial, it follows that in order *to extract the square root of a polynomial*, the expression must be arranged according to the powers of some letter (see example below); extract the square root of the first term at the left, $4a^6$, which gives the first term, $2a^3$, of the root; neglect the first term of the polynomial and divide the first term, $28a^5$, of the remainder by twice the first term of the root, $4a^3$, which gives the second term, $7a^2$, of the root; subtract from the first remainder the double product, $28a^5$, of the first term of the root and the second, and the square, $49a^4$, of the second; divide the first term, $12a^3$, of the second remainder, by twice the first term of the root, which gives the third term of the root; subtract from the second remainder the double products, $12a^3$ and $42a^2$, of the first and second term of the root by the

third, and the square, 9, of the third term; divide the first term of the third remainder by twice the first term of the root, which gives the fourth term of the root, and so on. Given, for example, the polynomial $49\,a^4 + 12\,a^3 + 9 + 4\,a^6 + 42\,a^2 + 28\,a^5$, to extract the square root, which is done as follows:

$$
\begin{array}{l r|l|r}
\text{Square} & 4a^6+28a^5+49a^4+12a^3+42a^2+9 & \multicolumn{2}{l}{2a^3+\ 7a^2+3 \text{ root.}} \\
\cline{3-4}
 & -4a^6\qquad\qquad\qquad\qquad\qquad\qquad & 4a^3+\ 7a^2 & 4a^3+14a^2+3 \\
\cline{2-2}
\text{1st remainder} & 28a^5+49a^4+12a^3+42a^2+9 & 7a^2 & 3 \\
\cline{3-4}
 & -28a^5-49a^4\qquad\qquad\qquad & 28a^5+49a^4 & 12a^3+42a^2+9 \\
\cline{2-2}
\text{2d remainder} & 12a^3+42a^2+9 & & \\
 & -12a^3-42a^2-9 & & \\
\cline{2-2}
\text{3d remainder} & 0 & & \\
\end{array}
$$

The root may have either the sign + or − (537).

POWERS AND ROOTS OF ALGEBRAIC QUANTITIES OF ANY DEGREE

547. *To raise a monomial to the mth power*, raise its coefficient to the mth power and multiply the exponent of each letter by m (465). If m is an even number, the mth power has always the sign $+$; but if m is odd, the mth power has the sign of the given monomial (463):

$$(3\,a^2b^3c)^m = 3^m a^{2m} b^{3m} c^m, \quad (-\,3\,a^2b^3c)^m = (-\,3)^m a^{2m} b^{3m} c^m.$$

REMARK. These examples show that the mth power of a product is equal to the product of the mth powers of the factors (531).

548. *The mth power of a fraction* is obtained by raising each of the terms to the mth power (532):

$$\left(\frac{3\,a}{b^2}\right)^m = \frac{3^m a^m}{b^{2m}}.$$

549. In general, designating the absolute value of $\sqrt[m]{a}$ by a' (450), we have:

when m is even,	$\sqrt[m]{a} = \pm\, a'$,		(547)
when m is even,	$\sqrt[m]{-\,a} = a'\sqrt{-\,1}$	imaginary;	(538)
when m is odd,	$\sqrt[m]{a} = a'$,		
when m is odd,	$\sqrt[m]{-\,a} = -\,a'$.		

Thus: $\sqrt{4} = \pm\, 2,\quad \sqrt[4]{16} = \pm\, 2,\quad \sqrt[4]{-16} = 2\sqrt{-\,1},$

$\sqrt[3]{27} = 3,\quad \sqrt[3]{-\,27} = -\,3.$

550. *To extract the mth root of a monomial,* extract the mth root of the coefficient and divide the exponent of each letter by m (**537, 547**). Thus,

$$\sqrt[3]{64\,a^3b^6} = 4\,ab^2, \quad \sqrt[5]{32\,a^{10}b^5} = 2\,a^2b.$$

REMARK. These examples show that the mth root of a product is equal to the product of the mth roots of the factors (**547**).

551. *The mth root of a fraction* is obtained by extracting the mth root of each of its terms (**539, 548**):

$$\sqrt[m]{\frac{3\,a}{b^2}} = \frac{\sqrt[m]{3\,a}}{\sqrt[m]{b^2}}.$$

552. The rule given in (**550**), applied in its most general sense conducts to the *notation of positive and negative fractional exponents,* invented by Descartes (**306**):

$$\sqrt[3]{a^2} = a^{\frac{2}{3}}, \quad \sqrt[5]{32\,a^4b^6c} = 2\,a^{\frac{4}{5}}b^{\frac{6}{5}}c^{\frac{1}{5}}, \quad \sqrt[n]{a^m} = a^{\frac{m}{n}}.$$

553. To divide a^m by a^n subtract the exponent of the divisor from that of the dividend (**487, 482**). Thus:

$$\frac{a^m}{a^n} = a^{m-n}.$$

When $m = n$, we have:

$$\frac{a^m}{a^m} = a^{m-m} = a^0 = 1,$$

which shows that any quantity raised to the 0 power gives 1.

When $m < n$, this division gives a negative exponent. Thus:

$$\frac{a^m}{a^{m+p}} = a^{-p},$$

or

$$\frac{a^m}{a^{m+p}} = \frac{1}{a^p} \text{ and } \frac{1}{a^p} = a^{-p}.$$

The expression a^{-p} is therefore the symbol of a division which could not be performed, and its true value is 1 divided by a^p. Thus,

$$a^{-3} = \frac{1}{a^3} \text{ and } a^{-5} = \frac{1}{a^5}.$$

554. *Negative fractional exponents.*

Since

$$\frac{1}{a^m} = a^{-m} \text{ we have:}$$

$$\sqrt[n]{\frac{1}{a^m}} = \sqrt[n]{a^{-m}} = a^{-\frac{m}{n}}. \qquad \textbf{(550)}$$

Thus, in summing up the preceding (552, 553, 554), we have:

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}, \frac{1}{a^p} = a^{-p}, \quad \sqrt[n]{\frac{1}{a^m}} = a^{-\frac{m}{n}}.$$

555. *Positive and negative fractional exponents are operated upon in the same manner as whole exponents,* and as the exponent 2, for example. The following examples show the manner of operating in the different cases:

1st.

$$\sqrt[5]{a^3} \times \sqrt[3]{a^2} = a^{\frac{3}{5}} \times a^{\frac{2}{3}} = a^{\frac{3}{5}+\frac{2}{3}} = a^{\frac{19}{15}},$$

$$\sqrt[4]{\frac{1}{a^3}} \times \sqrt[6]{a^5} = a^{-\frac{3}{4}} \times a^{\frac{5}{6}} = a^{-\frac{3}{4}+\frac{5}{6}} = a^{\frac{1}{12}},$$

$$a^{\frac{3}{4}}b^{-\frac{1}{2}}c^{-1} \times a^2 b^{\frac{2}{3}} c^{\frac{3}{5}} = a^{\frac{11}{4}} b^{\frac{1}{6}} c^{-\frac{2}{5}};$$

2d.

$$a^{\frac{2}{3}} : a^{-\frac{3}{4}} = a^{\frac{2}{3}-\left(-\frac{3}{4}\right)} = a^{\frac{2}{3}+\frac{3}{4}} = a^{\frac{17}{12}},$$

$$a^{\frac{2}{5}} b^{\frac{3}{4}} : a^{-\frac{1}{2}} b^{\frac{7}{8}} = a^{\frac{9}{10}} b^{-\frac{1}{8}}.$$

3d. *To raise a monomial having any exponent to any power,* multiply the exponent of each letter by the exponent of the power. Thus,

$$(a^2)^3 = a^6, \quad (a^2b^5)^7 = a^{14}b^{35},$$

$$\left(a^{\frac{3}{4}}\right)^5 = a^{\frac{3}{4}\times 5} = a^{\frac{15}{4}}, \quad \left(a^{-\frac{5}{6}}\right)^{12} = a^{-10},$$

$$\left(2\,a^{-\frac{1}{2}}b^{\frac{3}{4}}\right)^6 = 64\,a^{-3}b^{\frac{9}{2}},$$

$$\left(a^{\frac{m}{n}}\right)^{-\frac{r}{s}} = a^{\frac{m}{n}\times -\frac{r}{s}} = a^{-\frac{mr}{ns}}.$$

4th. *To extract any root of a monomial,* divide the exponent of each letter by the index of the root. Thus,

$$\sqrt[3]{a^3} = a, \qquad \sqrt[2]{a^2b^6} = ab^3,$$

$$\sqrt[3]{a^{\frac{2}{3}}} = a^{\frac{2}{9}}, \quad \sqrt[2]{a^{-\frac{3}{4}}} = a^{-\frac{3}{8}}, \quad \sqrt[3]{a^{\frac{3}{5}}b^{-2}} = a^{\frac{1}{5}}b^{-\frac{2}{3}},$$

$$\sqrt[s]{\sqrt[n]{\frac{1}{a^{mr}}}} = \sqrt[s]{\sqrt[n]{a^{-mr}}} = \sqrt[s]{a^{-\frac{mr}{n}}} = a^{-\frac{mr}{ns}}.$$

THE USE OF LOGARITHMS IN ALGEBRAIC CALCULATIONS

556. What was said in Arithmetic in regard to logarithms may be repeated here (396). The following examples sum up the uses which may be made of logarithms in shortening the arithmetical calculations which may arise in algebraic operations:

1st. $\operatorname{Log}(abc) = \log a + \log b + \log c$;

2d. $\operatorname{Log}\left(\frac{ab}{cd}\right) = \log a + \log b - \log c - \log d$;

3d. $\operatorname{Log}(a^m b^n c^p) = m \log a + n \log b + p \log c$;

4th. $\operatorname{Log}\left(\frac{ab^m}{c^n}\right) = \log a + m \log b - n \log c$;

5th. $\operatorname{Log}(a^2 - b^2) = \log[(a+b)(a-b)] = \log(a+b) + \log(a-b)$;

6th. $\operatorname{Log}\sqrt{(a^2 - b^2)} = \frac{1}{2}\log(a+b) + \frac{1}{2}\log(a-b)$; **(484)**

7th. $\operatorname{Log}(a^3\sqrt[4]{a^3}) = \log a^3 + \log\sqrt[4]{a^3} = 3\log a + \frac{3}{4}\log a = \frac{15}{4}\log a$;

8th. $\operatorname{Log}\sqrt[n]{(a^3 - b^3)^m} = \frac{m}{n}\log[(a-b)(a^2 + ab + b^2)]$

$$= \frac{m}{n}\log(a-b) - \frac{m}{n}\log(a^2 + ab + b^2);$$

9th. $\operatorname{Log}\frac{\sqrt{(a^2 - b^2)}}{(a+b)^2} = \frac{1}{2}\log(a+b) + \frac{1}{2}\log(a-b) - 2\log(a+b)$

$$= \frac{1}{2}\log(a-b) - \frac{3}{2}\log(a+b).$$

ARRANGEMENTS, PERMUTATIONS, COMBINATIONS

557. Having m distinct objects, m letters for example:

1st. An *arrangement* of these m letters, in groups containing n letters, is made by taking n of them in as many different ways as possible and placing them in a horizontal line. Any two arrangements differ by their letters or only by the order which they occupy.

The three letters, a, b, c, taken in groups of 2, give six arrangements:

$$ab,\ ac,\ ba,\ bc,\ ca,\ cb.$$

2d. The different groups which may be formed with these m letters, placing one by the other on the same line, are called *permutations*. Each permutation contains all the letters, and therefore any two permutations can differ only in the order of the letters.

The three letters, a, b, c, give six permutations:

$$abc,\ acb,\ cab,\ bac,\ bca,\ cba.$$

3d. All the possible different groups of n letters, which can

be made with these m letters, in such a manner that each group differs from the others by at least one letter, are called *combinations*. No attention is paid to the order of the letters, so that if the letters represent different quantities, the combinations represent all the different products which may be obtained by taking n of the m quantities in all possible manners as factors. The letters, a, b, c, taken in twos, give three combinations,

$$ab, ac, bc.$$

558. The following series of m letters are arrangements in groups of 1, of m letters:

$$a, b, c, d, \ldots\ldots, k,$$

and their number, $A_m^1 = m$.

The arrangements of m letters in groups of 2 are obtained by writing at the right of the letter a of the preceding series successively each of the $m - 1$ other letters; then at the right of the letter b each of the $m - 1$ other letters, and so on. The arrangements thus obtained are given in the table below:

$$\begin{array}{l} ab,\ ac,\ ad,\ \ldots,\ ak, \\ ba,\ bc,\ bd,\ \ldots,\ bk, \\ ca,\ cb,\ cd,\ \ldots,\ ck, \\ \ldots\ldots\ldots\ldots\ldots \\ ka,\ kb,\ kc,\ \ldots,\ kh, \end{array}$$

and their number, $A_m^2 = m(m - 1)$.

The arrangements of m letters in groups of 3 are obtained in the same manner, by writing at the right of each arrangement in the preceding table successively each of the $m - 2$ other letters which do not appear in that particular arrangement; which gives:

$$\begin{array}{l} abc,\ abd,\ abe,\ \ldots,\ abk, \\ acb,\ acd,\ ace,\ \ldots,\ ack, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ bac,\ bad,\ bae,\ \ldots,\ bak, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

The number of these arrangements is $A_m^3 = m(m - 1)(m - 2)$. Therefore the number of arrangements of m letters n in a group is:

$$A_m^n = m(m - 1)(m - 2)\ldots(m - n + 1).$$

Example. How many different numbers may be formed with 4 significative figures? $m = 9$ and $n = 4$:

$$A_9^4 = 9 \times 8 \times 7 \times 6 = 3024.$$

559. The permutations of m letters are simply the arrangements of these m letters in groups containing all the letters. The number of permutations is:

$$P_m = A_m^m = m(m-1)(m-2)\ldots 3 \cdot 2 \cdot 1 = 1 \cdot 2 \cdot 3 \cdot 4 \cdots m.$$

With 1 letter we have $P_m = 1$.
With 2 letters we have $P_m = 1 \cdot 2$.

ab *ba*

To form the permutations of 3 letters, introduce the letter c at the right, in the middle, and at the left of the preceding permutations of 2 letters, which gives:

abc, acb, cab,
bac, bca, cba,

and

$$P_m = 1 . 2 . 3.$$

Thus it is seen that in general the permutations of any number of letters is formed as here below:

$$P_m = 1 . 2 . 3 . 4 \ldots m.$$

Example. In how many ways may 5 soldiers be lined up? From the preceding formula:

$$P_5 = 1 . 2 . 3 . 4 . 5 = 120.$$

560. Suppose that all the combinations of m letters n in a group have been made, if permutations are made of the letters in each combination, the arrangements of m letters n in a group will be formed, and the number of arrangements will be equal to the number of combinations of m letters n in a group multiplied by the number of permutations of n letters. Thus we have:

$$A_m^n = C_m^n \times P_n, \text{ and } C_m^n = \frac{A_m^n}{P_n}.$$

Replacing A_m^n and P_n by their values (558, 559), we have:

$$C_m^n = \frac{m(m-1)(m-2)\cdots(m-n+1)}{1 \cdot 2 \cdot 3 \cdots n}.$$

For $n = m$ this formula gives $c_m^m = 1$.

For $m = 7$ and $n = 3$, we have:

$$c_7^3 = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 35.$$

It is seen that the successive numbers from 1 to n are found in the denominator, and that the numerator contains the same number of successive numbers, starting at n and descending.

561. The number of combinations of m objects in groups of n is equal to the number of combinations of m objects, $m - n$, in a group:

$$c_m^n = c_m^{m-n},$$

which is easily proved by aid of the formula in the preceding article.

562. The number of combinations of m objects in groups of n is equal to the number of combinations of $m - 1$ objects n in a group plus the number of combinations of $m - 1$ $n - 1$ in a group:

$$c_m^n = c_{m-1}^n + c_{m-1}^{n-1}.$$

NEWTON'S BINOMIAL THEOREM

563. From the rule for obtaining the product of any number of polynomials (468, 469), it follows that this product is the sum of the products obtained by taking in all possible ways one term in each of the polynomial factors. Find the product

$$(x + a)(x + b)(x + c) \ldots (x + h)(x + k),$$

of m binomials which have the same first term x, arranged according to the descending powers of x.

Taking the first term x in each of the binomial factors, we have the first term x^m of the product.

Taking successively the second term a in the first binomial with the first term x in all the others, the second term b of the second binomial with the first term x in all the others, and so on, the partial products ax^{m-1}, bx^{-1}, $\ldots$ kx^{m-1}, are obtained, and their sum

$$(a + b + c + \cdots + k)\, x^{m-1} \text{ or } S_1 x^{m-1}$$

is the second term of the product.

Taking successively the second terms in any two binomial

factors and the first x in the $m - 2$ others, the partial products abx^{m-2}, acx^{m-2}, bcx^{m-2}, are obtained, and their sum

$$(ab + ac + \cdots)\, x^{m-2} \text{ or } S_2x^{m-2}$$

is the third term of the product.

The fourth term is:

$$(abc + abd + \cdots)\, x^{m-3}, \text{ or } S_3x^{m-3}.$$

Any term of the $m - n$ degree is obtained by taking successively the second terms in any n factors and the first x in the others and summing these partial products.

The result may be written in the following manner:

$$S_nx^{m-n}.$$

The next to the last term is:

$$S_{m-1}x.$$

Finally the last term is simply the product

$$abc \ldots k, \text{ or } S_m,$$

of all the second terms of the binomial factors.

Therefore the desired product is:

$$x^m + S_1x^{m-1} + S^2x^{m-2} + \cdots + S_nx^{m-n} + \cdots + S_{m-1}x + S_m.$$

It may be noted that

$$S_1,\ S_2,\ S_3,\ \cdots,\ S_n,\ \cdots,\ S_{m-1},\ S_m$$

are nothing but the sums of the combinations obtained by taking the m second terms of the binomial factors respectively, 1, 2, 3, $\ldots n, \ldots m - 1, m$ in a group (557).

564. *To raise a binomial* $(x + a)$ *to the mth power.* This is done by supposing each of the second terms, $a, b, c, \ldots k$ of the binomial factors, to be equal to a; then we have (**560**):

$$S_1 = a + a + a + \cdots = C^1_m a = ma,$$

$$S_2 = a^2 + a^2 + a^2 + \cdots = C^2_m a^2 = \frac{m(m-1)}{1 \cdot 2} a^2,$$

$$S_3 = a^3 + a^3 + a^3 + \cdots = C^3_m a^3 = \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} a^3,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$S_m = a^m = C\ a^m = a^m,$$

therefore,

$$(x+a)^m = x^m + max^{m-1} + \frac{m(m-1)}{1 \cdot 2} a^2 x^{m-2}$$
$$+ \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} a^3 x^{m-3} + \cdots + ma^{m-1}x + a^m.$$

This formula is known as *Newton's binomial theorem*, and has the following properties:

1st. $(x + a)^m$ is composed of $m + 1$ terms, of which the first is x^m and the last a^m.

2d. The exponent of x decreases by 1 in passing from one term to the next, and therefore becomes 0 for the last term; the exponent of a increases by 1 from one term to the next, starting at the first term as 0, and becoming m for the last term. Thus it is seen that in any term the sum of the exponents of x and a is equal to m.

3d. The coefficient of any term is obtained by multiplying the coefficient of preceding term by the exponent of x in that term, and dividing the product 1 plus the exponent of a in the same term.

4th. The coefficients of two terms equally distant from the extremes are equal, and therefore the coefficients of two terms equally distant from the middle term if m is even, and from the middle if m is odd, are equal. Thus, having calculated at least half of the terms, we may write the coefficients of the remaining terms without further calculation.

Applying these rules to the two following examples, we have:

$$(x + a)^8 = x^8 + 8\,ax^7 + 28\,a^2x^6 + 56\,a^3x^5 + 70\,a^4x^4 + 56\,a^5x^3 + 28\,a^6x^2 + 8a^7x + a^8;$$

$$(x+a)^7 = x^7 + 7\,ax^6 + 21\,a^2x^5 + 35\,a^3x^4 + 35\,a^4x^3 + 21\,a^5x^2 + 7\,a^6x + a^7.$$

The term which we represented by $S_n x^{m-n}$, is:

$$\frac{m(m-1)(m-2)\ldots(m-n+1)}{1 \cdot 2 \cdot 3 \ldots n} a^n x^{m-n}.$$

This term is called a *general term*, and having it any term may be calculated without having the others, by substituting the values of m and n in the above formula.

Thus the $(n + 1)$th = fourth term of the value of $(x + a)^{m=8}$ is:

$$\frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} a^3 x^{8-8} = 56\,a^3 x^5.$$

If in the binomial formula we replace a by $-a$, we have:

$$(x-a)^m = x^m - max^{m-1} + \frac{m(m-1)}{1 \cdot 2} a^2 x^{m-2} - \cdots \pm a^m,$$

which differs from the first in that the signs of the terms are alternately positive and negative.

565. In the following table, known as *Pascal's triangle*, the figures in the horizontal lines are the coefficients of Newton's binomial for different values of m.

The vertical column 1 contains the number of combinations in groups of 1 of 1, 2, 3, . . . objects (560); column 2 contains the number of combinations in groups of 2 of 2, 3, 4, . . . objects; and in general the column n contains the number of combinations in groups of n of n, $n+1$, $n+2$, . . . objects.

		1st	2d	3d	4th	5th	6th	7th	8th	9th	10th
$m = 1$	1	1	.	.	.	.	.	.	.	.	.
$m = 2$	1	2	1	.	.	.	.	.	.	.	.
$m = 3$	1	3	3	1	.	.	.	.	.	.	.
$m = 4$	1	4	6	4	1	.	.	.	.	.	.
$m = 5$	1	5	10	10	5	1	.	.	.	.	.
$m = 6$	1	6	15	20	15	6	1	.	.	.	.
$m = 7$	1	7	21	35	35	21	7	1	.	.	.
$m = 8$	1	8	28	56	70	56	28	8	1	.	.
$m = 9$	1	9	36	84	126	126	84	36	9	1	.
$m = 10$	1	10	45	120	210	252	210	120	45	10	1
......	...	...	...	...	...	...	...	...	...	...	...

A number in the column n and the horizontal row m, expresses the number C^n_m of combinations of m objects in groups of n (500). Thus, 8 objects combined in groups of 5 give:

$$C^n_m = 56.$$

Any number in the arithmetical triangle is equal to the one above it plus the one at the left of that one. Thus, the number 56 in the 8th horizontal row is equal to 35 + 21. This follows from the relation,

$$C^n_m = C^n_{m-1} + C^{n-1}_{m-1}. \qquad \textbf{(562)}$$

From this relation the formation of the arithmetical triangle is easy.

The mth number of any column is equal to the sum of the

m first numbers of the preceding column. Thus, considering the 4th number 35 in the 4th column, we have:

$$35 = 15 + 20, \quad 15 = 5 + 10, \quad 5 = 1 + 4,$$

and

$$35 = 20 + 10 + 4 + 1.$$

In general, the mth number in the nth vertical column is found in the $(m + n - 1)$th row; that is,

$$C^n_{m+n-1} = \frac{(m+n-1)(m+n-2)\cdots m}{1 \cdot 2 \cdot 3 \cdots n} = \frac{m(m+1)\cdots(m+n-1)}{1 \cdot 2 \cdot 3 \cdots n}.$$

566. *The number of balls contained in a pile which has a triangular base.*

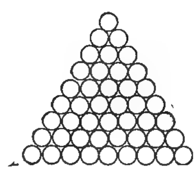


Fig. 2

A triangle of m balls on a side being formed of m rows which contain respectively 1, 2, 3, . . . m balls, corresponds to the whole consecutive numbers contained in the first column of the arithmetical triangle. These numbers are called *figurate numbers of the first order*, and the triangle contains

$$1 + 2 + 3 + \cdots + m = \frac{m(m+1)}{1 \cdot 2} \text{ balls,} \tag{565}$$

a number which is the mth in the second column of the arithmetical triangle (565). For $m = 6$, there are 21 balls in the triangle.

Thus the numbers 1, 3, 6 . . . in the second column of the arithmetical triangle are the *triangular* or *figurate numbers of the second order*; they represent the number of balls contained in the successive layers of a triangular pile; and the sum of the first m layers, that is,

$$\frac{m(m+1)(m+2)}{1 \cdot 2 \cdot 3}, \tag{565}$$

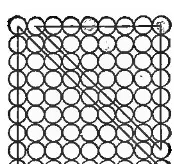


Fig. 3

is the number of balls contained in the pyramid, and is represented by the mth number in the third column of the arithmetical triangle.

For $m = 6$ there are 56 balls in the pyramid. Thus the numbers contained in the third column are the *pyramidal numbers*.

567. *A pyramid with a square base* having m balls on a side may be considered as being formed of two tri-

angular pyramids, the edges of which contain m and $m-1$ balls; the total number of balls which it contains is (566):

$$\frac{m(m+1)(m+2)}{1\cdot 2\cdot 3}+\frac{(m-1)m(m+1)}{1\cdot 2\cdot 3}=\frac{m(m+1)(2m+1)}{6},$$

and this number is the sum,

$$1^2+2^2+3^2+\cdots+m^2,$$

of the squares of the first m whole successive numbers, since these squares express the number of balls contained in the successive layers of the quadrangular pyramid.

For $m=48$, we have:

$$1^2+2^2+3^2+\cdots+48^2=\frac{48\times 49\times 97}{6}=38{,}024 \text{ balls.}$$

568. Considering a *pile with a rectangular base,* one of the sides of the base containing m and the other $n<m$ balls, as being made up of a pile with a square base n balls on a side, and a prism having a height equal to $m-n$ balls and a triangular base of n balls on a side, the number of balls which it contains is (566, 567):

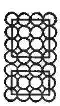


Fig. 4

$$\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)(m-n)}{2}=\frac{n(n+1)(3m-n+1)}{6}$$

For $m=25$ and $n=10$, the pile contains

$$\frac{10\times 11\times 66}{6}=1210 \text{ balls.}$$

569. *The sum S_m of the mth powers of n numbers, a, b, $c \ldots j$, k in an arithmetical progression whose common difference is r* (357).

Having $b=a+r$, $c=b+r, \cdots, k=j+r$, we have (564):

$$b^{m+1}=a^{m+1}+\frac{m+1}{1}ra^m+\frac{(m+1)m}{1.2}r^2a^{m-1}+\cdots+\frac{m+1}{1}r^ma+r^{m+1},$$

$$c^{m+1}=b^{m+1}+\frac{m+1}{1}rb^m+\frac{(m+1)m}{1.2}r^2b^{m-1}+\cdots+\frac{m+1}{1}r^mb+r^{m+1},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$k^{m+1}=j^{m+1}+\frac{m+1}{1}rj^m+\frac{(m+1)m}{1.2}r^2j^{m-1}+\cdots+\frac{m+1}{1}r^mj+r^{m+1},$$

$$(k+r)^{m+1}=k^{m+1}+\frac{m+1}{1}rk^m+\frac{(m+1)m}{1.2}r^2k^{m-1}+\cdots+\frac{m+1}{1}r^mk+r^{m+1}.$$

Adding these equalities, and cancelling the terms b^{m+1}, $c^{m+1}, \ldots k^{m+1}$, which are common to both members of the resulting equation, and making $a^m + b^m + \cdots k^m = S_m$, $a^{m-1} + b^{m-1} + \cdots + k^{m-1} = S_{m-1} \cdots$, $a + b + \cdots + k = S$, and $n = S_0$, we have:

$$(k + r)^{m+1} = a^{m+1} + \frac{m+1}{1} rS_m + \frac{(m+1)m}{1.2} r^2 S_{m-1} + \cdots + \frac{m+1}{1} r^m S_1 + r^{m+1} S_0;$$

from which

$$S^m = \frac{(k + r)^{m+1} - a^{m+1}}{(m+1)r} - \frac{m}{2} rS_{m-1} - \cdots - r^{m-1} S_1 - \frac{r^m}{m+1} S_0.$$

By means of this formula, commencing with $S_0 = n$, S_1, S_2, S_3, ... may be successively calculated.

For $a = 1$, $r = 1$, and $m = 1$, from which $k = S_0 = n$, S_m becomes $S_1 = 1 + 2 + 3 + \ldots + n$, and the preceding formula gives:

$$S_1 = \frac{(n + 1)^2 - 1}{2} - \frac{n}{2} = \frac{n(n + 1)}{2}. \tag{361}$$

For $a = 1$, $r = 1$, and $m = 2$, from which $k = S_0 = n$, S_m becomes $S_2 = 1^2 + 2^2 + 3^2 + \ldots n^2$, and the formula gives:

$$S_2 = \frac{(n + 1)^3 - 1}{3} - S_1 - \frac{1}{3} S_0 = \frac{(n + 1)^3 - 1}{3} - \frac{n(n + 1)}{2} - \frac{n}{3}$$
$$= \frac{n(n + 1)(2n + 1)}{6}.$$

These formulas for S_1 and S_2 are identical to those found in articles (566 and 567), except that m is replaced by n.

For $a = 1$, $r = 1$, and $m = 3$, from which $k = S_0 = n$, S_m becomes $S_3 = 1^3 + 2^3 + 3^3 + \ldots n^3$, and the formula gives:

$$S_3 = \frac{n^2 (n + 1)^2}{4}.$$

For $a = 1$, $r = 2$, and $m = 2$, from which $k = 2n - 1$, $S_0 = n$, $S_1 = 1 + 3 + 5 + \ldots + 2n - 1 = n^2$, S_m becomes $S_2 = 1^2 + 3^2 + 5^2 + \ldots (2n - 1)^2$, and the formula gives,

$$S_2 = \frac{(2n + 1)^3 - 1}{6} - 2n^2 - \frac{4}{3} n = \frac{n(4n^2 - 1)}{3}.$$

The two preceding formulas for the values of S_2 are used in the calculation of the lengths of rods used in suspension bridges.

BOOK IV

EQUATIONS OF THE SECOND DEGREE QUADRATICS

EQUATIONS OF THE SECOND DEGREE INVOLVING ONE UNKNOWN

570. There are two kinds of quadratic equations involving one unknown:

1st. *Pure quadratic equations,* which have only terms containing the square of the unknown and known terms. Such are

$$3\,x^2 = 5,\quad 4\,x^2 - 7 = 2\,x^2 + 9,\quad \frac{1}{3}x^2 - 3 + \frac{5}{12}x^2 = \frac{7}{24}.$$

Operating as in article (511), these become:

$$3\,x^2 = 5, \qquad 2\,x^2 = 16, \qquad 18\,x^2 = 79,$$

which shows that a pure quadratic may always be reduced to the general form:

$$ax^2 = b.$$

This is why they are called *two-term equations.*

2d. The *complete quadratic equations,* which contain both the square and the first power of the unknown. Such are:

$$5\,x^2 - 7\,x = 34, \qquad 4\,x^2 + \frac{1}{2}x + 3 = 8 + \frac{1}{3}x.$$

Operating as in article (511), these become:

$$x^2 - \frac{7}{5}x = \frac{34}{5}, \qquad x^2 + \frac{1}{24}x = \frac{5}{4},$$

which shows that all complete quadratic equations may be reduced to the general form:

$$x^2 + px = q.$$

This is why they are called *three-term equations.*

571. *To solve a pure quadratic equation*, reduce it to the form:

$$ax^2 = b, \text{ extract the root } x^2 = \frac{b}{a}, \quad x = \pm\sqrt{\frac{b}{a}}. \tag{537}$$

Thus the unknown x has two equal values opposite in sign, which are obtained by extracting the square root of the known quantity. This is why the solution of an equation of the second or any degree involving one unknown, is called a *root* (**505**).

572. *To solve a complete quadratic equation*, reduce it to the form (**576**):

$$x^2 + px = q. \tag{570}$$

Noting that $x^2 + px$ are the first two terms of the square $x^2 + px + \frac{p^2}{4}$ of $x + \frac{p}{2}$ (**479**), add $\frac{p^2}{4}$ to both members of the equation, obtaining:

$$x^2 + px + \frac{p^2}{4} \text{ or } \left(x + \frac{p}{2}\right)^2 = \frac{p^2}{4} + q.$$

Extracting the square root of both members:

$$x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} + q},$$

and

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}. \tag{1}$$

The sign $\pm$ which precedes the radical shows that the unknown x has two values.

The roots of the equation equal half the coefficient of x with reversed sign, plus or minus the square root of the sum of the square of this half and the known term. Letting the roots be represented by x' and x'', we have:

$$x' = -\frac{p}{2} + \sqrt{\frac{p^2}{4} + q}, \quad x'' = -\frac{p}{2} - \sqrt{\frac{p^2}{4} + q}. \tag{576}$$

The formula (1) may be written:

$$x = \frac{-p \pm \sqrt{p^2 + 4q}}{2}.$$

When the quantity placed under the radical is positive, the square root is real.

When the quantity under the radical is 0, both roots are equal to $-\frac{p}{2}$.

If the quantity under the radical is negative, its square root is imaginary, as are also the roots of the equation (**538**).

573. Adding the roots of the equation, we have:

$$x' + x'' = -\frac{p}{2} + \sqrt{\frac{p^2}{4} + q} - \frac{p}{2} - \sqrt{\frac{p^2}{4} + q} = -p.$$

Thus the sum of the roots is equal to the coefficient p of the term x taken with its sign reversed (**460**).

Further, having (**484**),

$$x'x'' = \left(-\frac{p}{2} + \sqrt{\frac{p^2}{4} + q}\right)\left(-\frac{p}{2} - \sqrt{\frac{p^2}{4} + q}\right) = -q;$$

the product of the roots is therefore equal to the known quantity taken with its sign reversed.

These values of the sum and product of the roots of an equation of the second degree furnish two very simple methods for determining the exactness of these roots (**575**).

574. *An equation of the second degree may be formed having its roots given,* $x = 5$ and $x'' = -2$, for example. From the preceding article we have:

$$-p = x' + x'' = 5 - 2 = 3 \text{ and } -q = x'x'' = 5 \times -2 = -10,$$

and, therefore, $x^2 - 3x = 10$.

575. *Equations of the second degree to be solved.*

EXAMPLE 1. $\frac{5}{6}x^2 - \frac{1}{2}x + \frac{3}{4} = 8 - \frac{2}{3}x - x^2 + \frac{273}{12}$.

This equation becomes (**570**, 2d):

$$x^2 + \frac{2}{22}x = \frac{360}{22}, \text{ and } (572)\begin{cases} x' = -\frac{1}{22} + \sqrt{\left(\frac{1}{22}\right)^2 + \frac{360}{22}} = 4 \\ x'' = -\frac{1}{22} - \sqrt{\left(\frac{1}{22}\right)^2 + \frac{360}{22}} = -\frac{45}{11}. \end{cases}$$

Having $4 - \frac{45}{11} = -\frac{2}{12}$, and $4 \times -\frac{45}{11} = -\frac{360}{22}$, the roots are exact (**573**). They also fulfill the conditions of the equation.

EXAMPLE 2. $6x^2 - 37x = -57$.

This equation becomes:

$$x^2 - \frac{37}{6}x = -\frac{57}{6}, \text{ and } \begin{cases} x' = \dfrac{37}{12} + \sqrt{\left(\dfrac{37}{12}\right)^2 - \dfrac{57}{6}} = \dfrac{19}{6} \\ x'' = \dfrac{37}{12} - \sqrt{\left(\dfrac{37}{12}\right)^2 - \dfrac{57}{6}} = 3. \end{cases}$$

The roots are correct, since $\frac{19}{6} + 3 = \frac{37}{6}$, and $\frac{19}{6} \times 3 = \frac{57}{6}$.

EXAMPLE 3. $4a^2 - 2x^2 + 2ax = 18ab - 18b^2$.

Transposing and solving:

$$x^2 - ax = 2a^2 - 9ab + 9b^2$$

and

$$\begin{cases} x' = \dfrac{a}{2} + \sqrt{\dfrac{a^2}{4} + 2a^2 - 9ab + 9b^2} = 2a - 3b \\ x'' = \dfrac{a}{2} - \sqrt{\dfrac{a^2}{4} + 2a^2 - 9ab + 9b^2} = -a + 3b. \end{cases}$$

In obtaining these values of x, it may be noted that the quantity under the radical $\frac{9}{4}a^2 - 9ab + 9b^2$ is the square of $\frac{3}{2}a - 3b$ (479). The roots are correct, since $2a - 3b - a + 3b = a$, and $(2a - 3b)(-a + 3b) = -2a^2 + 9ab - 9b^2$ (468).

EXAMPLE 4. $ax^2 + bx = 0$.

This equation, in which the known quantity is 0, dividing by a gives:

$$x^2 + \frac{b}{a}x = 0, \text{ from which } \begin{cases} x' = -\dfrac{b}{2a} + \sqrt{\dfrac{b^2}{4a^2}} = 0 \\ x'' = -\dfrac{b}{2a} - \sqrt{\dfrac{b^2}{4a^2}} = -\dfrac{b}{a}. \end{cases}$$

576. The roots of the complete quadratic $ax^2 + bx = c$ may be obtained without reducing the equation to the form $x^2 + px = q$, that is, without making $\frac{b}{a} = p$ and $\frac{c}{a} = q$ (572).

Substituting $p = \frac{b}{a}$ and $q = \frac{c}{a}$ in the following:

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q},$$

we have:

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} + \frac{c}{a}} = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}. \tag{1}$$

This formula is more generally used than the former because the calculations are simpler. In this case we have:

$$x' + x'' = -\frac{b}{a}, \quad \text{and} \quad x'x'' = -\frac{c}{a}.$$

When the coefficient b of x is even, we can write $b = 2b'$ in the formula, which gives:

$$x = \frac{-2b' \pm \sqrt{4b'^2 + 4ac}}{2a} = \frac{-b' \pm \sqrt{b'^2 + ac}}{a}.$$

In this form the arithmetical calculations are still simpler.

From the equation $3x^2 - 25x + 49 = 0$

$$3x^2 - 28x = -49,$$

we have:

$$x = \frac{14 \pm \sqrt{14^2 - 13 \times 49}}{3} = \frac{14 \pm 7}{3};$$

that is, $\quad x' = 7 \text{ and } x = \frac{7}{3}.$

When $a = 1$, the formula (1) becomes:

$$x = \frac{-b \pm \sqrt{b^2 + 4c}}{2},$$

which is the same as the general formula in article (572).

577. *To resolve a trinomial of the second degree $x^2 + px + q = 0$ into two factors of the first degree.*

1st. Since this trinomial comes from the equation $x^2 + px = -q$, we have (572, 573):

$$x' + x'' = -p \text{ or } -(x' + x'') = p \text{ and } x'x'' = q.$$

Substituting these values for p and q in the trinomial, we have:

$$x^2 - (x' + x'')x + x'x'' = 0,$$

or

$$(x - x')(x - x'') = 0,$$

and in general,

$$x^2 + px + q = (x - x')(x - x''). \tag{1}$$

For example, the equation $x^2 + 4x - 12 = 0$, giving $x' = -6$ and $x'' = 2$, we have:

$$x^2 + 4x - 12 = (x + 6)(x - 2).$$

The trinomial $ax^2 + bx + c = 0$ in the same manner gives:

$$ax^2 + bx + c = a(x - x')(x - x'').$$

Thus the roots of the equation $3x^2 - 7x + 2 = 0$, being $x' = 2$ and $x'' = \frac{1}{3}$, we have:

$$3x^2 - 7x + 2 = 3(x - 2)\left(x - \frac{1}{3}\right).$$

2d. Having the trinomial,

$$x^2 + px + q = P. \qquad (2)$$

Adding and subtracting $\frac{p^2}{4}$ in the first member, we have:

$$x^2 + px + \frac{p^2}{4} + q - \frac{p^2}{4} = P,$$

or

$$\left(x + \frac{p}{2}\right)^2 - \left(\frac{p^2}{2} - q\right) = P,$$

or

$$\left(x + \frac{p}{2}\right)^2 - \left(\sqrt{\frac{p^2}{4} - q}\right)^2 = P.$$

Designating the two roots of the trinomial (2) by x' and x'', when the trinomial is made equal to zero, the difference of these two squares may be written:

$$\left(x + \frac{p}{2} + \sqrt{\frac{p^2}{4} - q}\right)\left(x + \frac{p}{2} - \sqrt{\frac{p^2}{4} - q}\right) = P,$$

and

$$(x - x')(x - x'') = P.$$

In the same manner, having:

$$ax^2 + bx + c = P.$$

Designating the roots of this trinomial by x' and x'' and making it equal to zero, we may write:

$$P = a(x - x')(x - x'') = ax^2 + bx + c, \qquad (3)$$

in which expression x' and x'' have certain fixed values and x has absolutely any value. In giving x a positive or negative value, calculate the corresponding value P of the trinomial $ax^2 + bx + c = P$.

Example. Given the trinomial,

$$P = 3x^2 - 6x - 45,$$

to be resolved into factors of the first degree. Find the roots of the equation:

$$3x^2 - 6x - 45 = 0,$$

or

$$x^2 - 2x - 15 = 0,$$

$$x = +1 \pm \sqrt{1 + 15},$$

$$x' = 5, \quad x'' = -3.$$

Therefore, the given trinomial may be written in the form:

$$P = 3(x - 5)(x + 3).$$

In this form we can study the values of P corresponding to different values of x. Some of these values are given below.

For	$x = 0$	$P = -45$
	1	-48
	2	-45
	3	-36
	4	-21
	5	0
	-1	-36
	-2	-21
	-3	0
	-4	$+27$

EQUATIONS OF THE SECOND DEGREE INVOLVING SEVERAL UNKNOWNS

578. *The solution of a system of two simultaneous equations, involving two unknowns, one or both of which are of the second degree.*

1st. *If one of them is of the first degree,* express one of the unknowns in terms of the other and substitute in the other equation, which will give a second degree equation involving only one unknown; this may be solved and the value obtained substituted in the first equation, which in turn will give the value of the other unknown.

Thus, having

$$ax + by = 2s \text{ and } xy = t,$$

from the first equation (**511**):

$$y = \frac{2s - ax}{b}.$$

Substituting this value of y in the second,

$$x\left(\frac{2s - ax}{b}\right) = t \quad \text{or} \quad -\frac{a}{b}x^2 + \frac{2s}{b}x = t;$$

eliminating the denominators and changing the signs,

$$ax^2 - 2sx + bt = 0,$$

and therefore (576),

$$x = \frac{s \pm \sqrt{s^2 - abt}}{a}.$$

Substituting this value of x in the first of the given equations, and solving:

$$y = \frac{s \pm \sqrt{s^2 - abt}}{b}.$$

The system of equations has two direct solutions, because evidently $s > \sqrt{s^2 - abt}$; but in order that they be real, s^2 must be greater than, or equal to, abt.

These two solutions when separated are:

$$x = \frac{s + \sqrt{s^2 - abt}}{a}, \quad y = \frac{s - \sqrt{s^2 - abt}}{b};$$

and $$x = \frac{s - \sqrt{s^2 - abt}}{a}, \quad y = \frac{s + \sqrt{s^2 - abt}}{b}.$$

For $a = b = 1$, the given equations become $x + y = 2s$, $xy = t$, and the values of x and y are reduced to:

$$x = s \pm \sqrt{s^2 - t} \text{ and } y = s \mp \sqrt{s^2 - t},$$

which shows that the two values of y are equal to those of x taken in an inverse order, that is, if $s + \sqrt{s^2 - t}$ is the value of x, $s - \sqrt{s^2 - t}$ is the corresponding value of y, and conversely.

Special Method. Noting that the solution of the system

$$x + y = 2s \text{ and } xy = t$$

amounts to finding two numbers x and y, the sum and product of which are known, it is seen that they are the roots of the equation (573, 574):

$$z^2 - 2sz + t = 0,$$

which gives directly (542):

$$z' = s + \sqrt{s^2 - t} \text{ and } z'' = s - \sqrt{s^2 - t}.$$

The solutions of the equation are therefore, putting successively $x = z'$ and $y = z''$,

$$x = s + \sqrt{s^2 - t}, \quad y = 2s - x = s - \sqrt{s^2 - t}$$

and
$$x = s - \sqrt{s^2 - t}, \quad y = 2s - x = s + \sqrt{s^2 - t},$$

values found by the general method.

This special method may be applied to the system:

$$x - y = 2, \quad xy = 15.$$

Putting $y = -y_1$,

$$x + y_1 = 2, \quad xy_1 = -15$$

x and y_1, being the roots of the equation

$$z^2 - 2z = 15,$$

which gives,

$$z' = 5 \text{ and } z'' = -3;$$

$$x = 5, \quad y_1 = 2 - 5 = -3;$$
$$x = -3, \quad y_1 = 2 + 3 = 5.$$

Therefore the solutions of the given system are:

$$x = 5, \quad y = 3;$$
$$x = -3, \quad y = -5.$$

This special method may also be applied to the system:

$$x + y = 8, \quad x^2 + y^2 = 34.$$

If the first equation is squared,

$$x^2 + 2xy + y^2 = 64,$$

and the second one subtracted from it, we have:

$$2xy = 30 \text{ or } xy = 15;$$

and we have again,

$$x + y = 8 \text{ and } xy = 15,$$

x and y being the roots of the equation

$$z^2 - 8z = -15,$$

which gives

$$z' = 5 \text{ and } z'' = 3,$$

and the solutions of the system are:

$$x = 5, \quad y = 8 - 5 = 3;$$
$$x = 3, \quad y = 8 - 3 = 5.$$

2d. *When one of the equations is of the first degree with reference to one of its letters only,* solve for the value of this unknown and

substituting in the other equation an equation of the third degree is obtained. Thus, having

$$ax^2 + by = 2\,s \text{ and } xy = t,$$

from the first equation:

$$y = \frac{2\,s}{b} - \frac{ax^2}{b}.$$

Substituting in the second equation,

$$\frac{2\,s}{b}x - \frac{a}{b}x^3 = t\,;$$

eliminating the denominators and changing the signs,

$$ax^3 - 2\,sx + bt = 0.$$

3d. *A system of two simultaneous equations of the second degree involving two unknowns.*

$$x^2 + y^2 = 25, \quad xy = 12.$$

The second equation gives $y = \dfrac{12}{x}$, and substituting this in the first, we have:

$$x^2 + \frac{144}{x^2} = 25, \text{ or } x^4 - 25\,x^2 + 144 = 0.$$

Thus we have an equation of the fourth degree; but this equation, being a quadratic, is easily solved (579).

Thus the system may be solved by multiplying the second equation by 2 and adding it to the first:

$$x^2 + 2\,xy + y^2 \text{ or } (x + y)^2 = 49, \text{ from which } x + y = \pm 7. \quad (1)$$

Subtracting the second multiplied by 2 from the first, we have:

$$x^2 - 2\,xy + y^2 \text{ or } (x - y)^2 = 1, \text{ from which } x - 1 = \pm 1. \quad (2)$$

The equations (1) and (2) giving the sum and difference of the quantities x and y, the quantities themselves may be easily found. These equations added and subtracted give:

$$x = \frac{\pm 7 \pm 1}{2}, \quad \text{and} \quad y = \frac{\pm 7 \mp 1}{2}.$$

The roots of the given system are:

$$\begin{aligned} x &= 4, & y &= 3; \\ x &= 3, & y &= 4; \\ x &= -4, & y &= -3; \\ x &= -3, & y &= -4. \end{aligned}$$

These roots satisfy the system.

The elimination of one of the unknowns in two complete quadratic equations involving two unknowns gives an equation of the fourth degree.

Considering the following:

$$ay^2 + bxy + cx^2 + dy + fx + g = 0,$$
$$a'y^2 + b'xy + c'x^2 + d'y + f'x + g' = 0,$$

arranged according to x,

$$cx^2 + (by + f)\,x + ay^2 + dy + g = 0,$$
$$c'x^2 + (b'y + f')\,x + a'y^2 + d'y + g' = 0.$$

If the coefficients of x^2 were the same in the two equations, by subtracting them an equation of the first degree of x, which could be substituted in one of the given equations would be obtained; from this equation the value of x in the terms of y may be found, and substituting this value in one of the given equations, an equation is obtained which contains only one unknown y (520, 3d).

Or if each term of the first equation is multiplied by c', and those of the second by c, we have:

$$cc'x^2 + (by + f)\,c'x + (ay^2 + dy + g)\,c' = 0,$$
$$cc'x^2 + (b'y + f')\,cx + (a'y^2 + d'y + g')\,c = 0.$$

Subtracting one from the other, we obtain:

$$[(bc' - cb')y + fc' - cf']x + (ac' - ca')y^2 + (dc' - cd')y + gc' - cg' = 0,$$

which gives:

$$x = \frac{(ca' - ac')\,y^2 + (cd' - dc')\,y + cg' - gc'}{(bc' - cb')\,y + fc' - cf'}.$$

This value of x substituted in one of the given equations will give the final equation for y. Without making this substitution, which would be somewhat complicated, it is easily seen that the equation in y would be of the fourth degree.

TRINOMIAL EQUATIONS

579. *The trinomial equations are of a degree greater than the second, and their solution may be brought to that of an equation of the second degree involving one unknown.* The general form is:

$$ax^{2m} + bx^m = c.$$

They are called trinomial equations because they involve three kinds of terms: the terms in x^{2m}, the terms in x^m, and the known terms.

Putting $x^m = y$, the equation is of the second degree:

$$ay^2 - by = c.$$

Having calculated the values of y from this equation, those of x are given by the formula:

$$x = \sqrt{y}.$$

If *m is an even number*, all positive real values of y give two equal real values of opposite sign for x; while the negative values of y give imaginary values of x (**514**). If *m is odd*, all real values of y give but one value of x, which is real and of the same sign as y.

Given the trinomial equation,

$$x^4 - 25\,x^2 + 144 = 0.$$

Putting $x^2 = y$, we have,

$$y^2 - 25\,y + 144 = 0,$$

and

$$y = \frac{25 \pm \sqrt{25^2 - 4 \times 144}}{2}. \qquad \textbf{(572, 576)}$$

But $x^2 = y$ and $x = \pm\sqrt{y}$;

$$x = \pm\sqrt{\frac{25 \pm \sqrt{25^2 - 4 \times 144}}{2}},$$

which shows that the equation has 4 roots, equal in pairs and opposite in sign.

Effecting the calculations, we find first:

$$y = 16 \quad \text{and} \quad y = 9.$$

Then $\qquad x = \pm 4 \quad \text{and} \quad x = \pm 3.$

Which values satisfy the given equation.

EQUATIONS OF ANY DEGREE

580. *A graphical method of obtaining an approximate solution of an equation of any degree.*

Given, the equation,

$$x^5 + 5\,x^4 + x^3 - 16\,x^2 - 20\,x - 16 = 0$$

with all its terms in the first member.

Draw two axes Ox and Oy perpendicular to one another. The different values given to x in the equation are laid off to a convenient scale on the axis Ox or Ox' according as they are positive or negative. Perpendiculars are raised at the points thus obtained on xx', and on these the values y of the first member for different values of x are laid off to a convenient scale, which need not be the same as the first. Having obtained a sufficient number of points, a smooth curve is drawn through them, and the distances from O to the points where this curve crosses xx' are the roots of the equation.

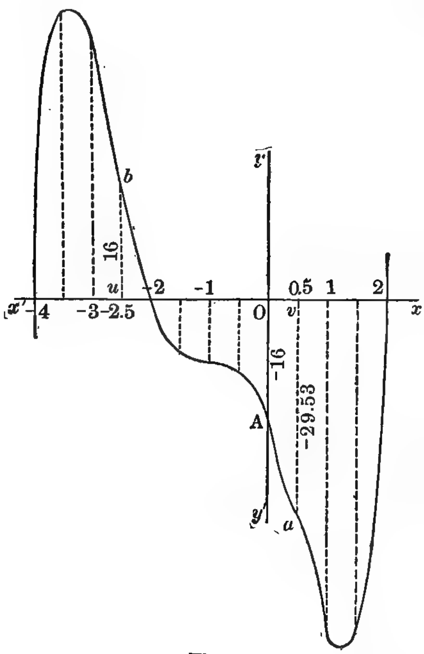


Fig. 5

For $x = 0$, the value y of the first member of the equation is -16, which gives $OA = -16$.

For $$x = Ov = 0.5,$$
$$y = va = 0.5^5 + 5 \times 0.5^4 + 0.5^3 - 16 \times 0.5^2 - 20 \times 0.5 - 16 = -29.53.$$
For $$x = Ou = -2.5,$$
$$y = ub = 2.5^5 + 5 \times 2.5^4 - 2.5^3 - 16 \times 2.5^2 + 20 \times 2.5 - 16 = 16.$$

According as x is positive or negative, the different terms which enter in the value of y will have the signs of the first or the second of these last two inequalities, which makes it necessary to find the signs but once for each sign of x.

Constructing a table of these values, we have:

$x=$	0	0.5	1	1.5	2	-0.5	-1	-1.5	-2	-2.5	-3	-3.5	-4,
$y=$	-16	-29.53	-45	-45.72	0	-9.84	-9	-7.65	0	16	35	40	0.

Having obtained $y = 0$ for the values 2, -2 and -4 of x, these are the real roots of the equation. If the curve is plotted, it will cut the axis xx' at the points for which $x = 2$, $x = -2$, and $x = -4$. An examination of the equation shows that for

values of x greater than 2, the values of y are all greater than 0 and positive; and furthermore, since the curve does not cut the axis xx' between $x = 2$ and $x = 0$, 2 is the only positive real root of the equation. In the same manner it is shown that -2 and -4 are the only real negative roots.

When, as in the preceding example, the roots are whole, they may be obtained rapidly enough without tracing the curve. Having obtained a value of y which approaches 0, upon augmenting or diminishing x, $y = 0$ is quickly found, and the corresponding value of x is the required root.

In engineering practice the positive root is the one which is most often used. In this case the negative values of x are not used, and no curve is plotted on the negative end of the xx' axis. Furthermore, the nature of the problem generally permits of a fair guess as to the value of x, and the curve need be drawn only near this point.

The graphical method is most useful when the roots are not whole or when they contain a great number of figures.

Given, to solve the equation,

$$x^3 - 3x^2 + 7x - 40 = 0.$$

$$\begin{array}{lll}
\text{For } x=0, & \text{we have } y=AO & =-40; \\
\quad x=1=Ov, & y=va & =1-3+7-40=-35; \\
\quad x=2=Ov', & y=v'a' & =8-12+14-40=-30; \\
\quad x=3=Ov'', & y=v''a'' & =27-27+21-40=-19; \\
\quad x=4=Ov''', & y=v'''a''' & =64-48+28-40=4.
\end{array}$$

y having become positive indicates that the equation has a positive root between 3 and 4. Further, the equation shows that for values of x greater than 4, y would always be positive and greater than 0. Thus there is only one positive real root, and this is shown by the curve. The point c where the curve intersects Ox gives, with the exactitude furnished by a plotted curve, $v''c = 0.9$ of $v''v'''$, or of 1 in practice, and we have 3.9 for the root of the equation.

If it is desired to prove the correctness of this root or to determine it more accurately, the following method is employed:

For $x = 39$, the equation gives $y = 0.99$. This indicates that 3.9 is too great.

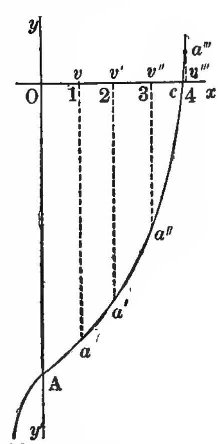


Fig. 6

For $x = 3.8$, we have, $y = -1.85$.

Therefore, x lies between 3.8 and 3.9.

The value of x augmenting from 3.8 to $3.9 = 0.1$, for an augmentation of $1.85 + 0.99 = 2.84$ of y, supposing that the increments remain proportional, which amounts to supposing the curve to be a straight line between those points, for the augmentation 1.85 of y, x would augment $0.1 \frac{1.85}{2.84} = 0.065$. Therefore, the required root is 3.865; and substituting in the equation, we have $y = 0.0234$, which is more than accurate enough for ordinary practice.

$$x = 3.866$$

gives $$y = +0.005.$$

If the negative roots are desired, they may be obtained in the same manner.

581. *Solution of an equation of any degree by successive approximations.*

Given, the equation,

$$x^5 + 200\,x = 5000, \text{ or } x^5 + 200\,x - 5000 = 0.$$

Suppose $x = 0$ in all the terms which contain x except one; ordinarily the term which contains x with the largest exponent is excepted, because the value of x is more rapidly approached when the coefficients of the other terms of an elevated degree are not very great. Making $x = 0$ in the second term of the given equation,

$$x^5 = 5000, \text{ or } 5 \log x = \log 5000, \text{ and } x = 5.4928$$

Substitute this value for x in the terms which were first made equal to zero.

$$x^5 + 200 \times 5.4928 = 5000; \text{ and } x = 5.2269.$$

Substituting this new value in the equation

$$x^5 + 200 \times 5.2269 = 5000; \text{ and } x = 5.2411.$$

This value when substituted gives a fourth $x = 5.2403$, which gives a fifth $x = 5.2403 \ldots$.

The value $x = 5.240$ may be taken as the root; and substituting, we have:

$$y = -1.45.$$
$$x = 5.241.$$
$$y = 2.51.$$

Instead of starting with $x = 0$, it is possible to start with any value which the nature of the problem may indicate as being near the true value.

MAXIMA AND MINIMA

582. When an expression takes different successive values, it is said to have reached a *maximum* or *minimum* when its value is less or greater than the values which immediately precede or follow it.

A *maximum* or a *minimum* is said to be *absolute* when the expression has no value which is larger than this maximum and none which is smaller than the minimum. In other cases it is a *relative maximum or minimum.*

At this point, only problems which may be solved by means of second degree equations will be treated, leaving the general treatment of maxima and minima for a later chapter.

583. The maximum of the product $xy = z$ of two variable factors x, y, whose sum $x + y = a$ is constant, occurs when these two factors are equal, that is, when $x = y = \frac{a}{2}$.

1st. Having (**481**)

$$(x + y)^2 - (x - y)^2 = 4\,xy$$

$(x + y)^2$ being a positive constant quantity, the product $4\,xy$, and therefore, xy will increase in proportion as $x - y$ decreases in absolute value, and will be a maximum when

$$x - y = 0, \text{ that is, } x = y = \frac{a}{2}.$$

2d. Having $x + y = a$, and $xy = z$, it follows (**574**) that x and y are the roots of the equation $u^2 - au + z = 0$, which gives (**572**):

$$x = \frac{a}{2} + \sqrt{\frac{a^2}{4} - z}, \quad y = \frac{a}{2} - \sqrt{\frac{a^2}{4} - z}.$$

If x and y are to have real values, $z = xy$ should not be greater than $\frac{a^2}{4}$, which is the maximum value. But when $z = xy = \frac{a^2}{4}$, the two roots x and y of the equation are equal, and we have as in 1st,

$$x = y = \frac{a}{2}.$$

3d. On a straight line AB, take successively the lengths AC and CB, representing to some chosen scale the numbers x and y, the sum of which $x + y = a = AB$ is constant; on AB as diameter describe a semicircle, and at C erect a perpendicular to AB. Representing z by CD, we have, no matter what the position of C may be, that is, what the values of x and y may be,

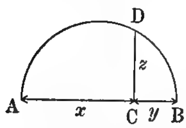


Fig. 7

$$z^2 = xy.$$

The maximum of xy corresponds, therefore, to that of z^2 or z; but z is a maximum when z is at the center of the semicircle, and we have:

$$z = x = y = \frac{a}{2}.$$

584. From the preceding article (583), it follows:

1st. *That of all rectangles of the same perimeter, the square has the maximum area.*

2d. *That of all the right triangles the sum of whose legs is constant, the isosceles has the maximum area.*

3d. *That of all triangles of the same base a, and the same perimeter 2 p, the isosceles has the maximum area.*

The expression for the area s of a triangle being (see Trigonometry)

$$s = \sqrt{p(p-a)(p-b)(p-c)},$$

the factors p and $p - a$ being constants, s will be a maximum when the product $(p - b)(p - c)$ is a maximum; and since the sum $2p - b - c = a$ is a constant, this will be when $p - b = p - c$ or $b = c$.

585. *The product of any number of n positive factors, the sum a of which is constant, is a maximum when all the factors are equal.* Because if only two factors are unequal, replacing each by their arithmetical mean (337), the product of the factors is increased, but the sum remains unchanged.

From this it follows:

1st. *That the arithmetical mean of n positive numbers which are not equal is greater than their geometrical mean.* Thus, having $abc \cdots < \left(\frac{a+b+c+\cdots}{n}\right)^n$, we have $\frac{a+b+c+\cdots}{n} > \sqrt[n]{abc\cdots}$

2d. *That of all triangles having the same perimeter* $2p$, *the equilateral triangle has the maximum area.* Thus, having (584)

$$s = \sqrt{p(p-a)(p-b)(p-c)},$$

since p is positive, each of the three factors should be positive; because if one or all of them were negative, s would have an imaginary value; and if two were negative, $p - b$ and $p - c$, for example, we would have $2p < b + c$, which is impossible. p being constant, s will be a maximum when the product of the three other factors is a maximum, that is, when

$$p - a = p - b = p - c, \text{ or } a = b = c.$$

586. *The product* $abc \ldots$ *of any number* n *of positive factors, the sum* $a^m + b^m + c^m + \cdots$ *of the mth powers of which is constant, is a maximum when the factors are equal.*

Let it be given to *find the rectangle of maximum area which may be inscribed in a given circle.*

S being the area of the inscribed rectangle, x and y the dimensions, and d the diameter of the given circle or the diagonal of the rectangle, we have:

$$xy = s, \text{ or } x^2y^2 = s^2, \text{ and } x^2 + y^2 = d^2.$$

The sum d^2 of the factors $x^2 + y^2$ being constant, in order that s^2, and therefore the area of the rectangle, be a maximum, x^2 must equal y^2 and $x = y$. Thus the *square is the largest rectangle which may be inscribed in a circle.*

587. *The sum* $x + y = a$, *of two positive numbers* x *and* y, *being given, find the maximum of the product* x^my^n, *wherein* m *and* n *are whole positive numbers.*

We have:

$$x^my^n = m^mn^n\frac{x^m}{m^m} = \frac{y^n}{n^n}.$$

m^mn^n being a constant, the product x^my^n will be a maximum when $\frac{x^m}{m^n} \times \frac{y^n}{n^n}$ is a maximum. But this last product is composed of m factors $\frac{x}{m}$ and n factors $\frac{y}{n}$, the sum $m\frac{x}{m} + n\frac{y}{n} = x + y$ of which is constant; therefore, it is a maximum when all these factors are equal, that is, when

$$\frac{x}{m} = \frac{y}{n}.$$

Thus the product $x^m y^n$ is a maximum when x and y are proportional to their exponents m and n.

This applies, no matter how many factors there may be.

From the two equations

$$x + y = a \text{ and } \frac{x}{m} = \frac{y}{n},$$

we deduce (520):

$$x = \frac{ma}{m+n} \text{ and } y = \frac{na}{m+n}.$$

EXAMPLE 1. *Inscribe an isosceles triangle ABC of a maximum area in a circle of a given radius r.*

Let $2x$ be the base of the triangle, y its height, s its area, and CD the diameter perpendicular to the base AB. Then we have

$$xy = s \text{ and } x^2 = y(2r - y).$$

The second equation expresses that x is a mean proportional between the two segments of the diameter. (See Geometry.)

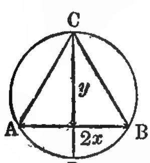


Fig. 8

s will be a maximum when xy or $x^2y^2 = y^3(2r - y)$ is a maximum. But in this last product, which is obtained by multiplying the value of x^2 by y^2, the sum $y + (2r - y) = 2r$ is constant. Therefore, 3 being the exponent of the first factor y^3, and 1 that of $(2r - y)$, we have for a maximum:

$$\frac{y}{2r - y} = \frac{3}{1}, \text{ and } y = \frac{3}{2}r.$$

This value of y indicates that the maximum triangle is an equilateral.

EXAMPLE 2. *Construct a box having a maximum capacity, with a square $ABCD$ of cardboard.*

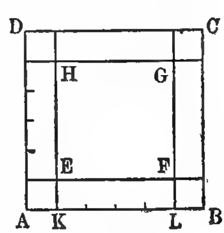


Fig. 9

To construct such a box, draw parallel lines at equal distances from the sides; remove the four squares at the corners and fold the four rectangles, such as $EFLK$, so as to form the sides of the box. The base of the box is the square $EFGH$.

Designating the constant AB by $2l$ and the variable AK by x, the capacity c of the box i

$$c = (2l - 2x)^2 x = 4(l - x)^2 x,$$

and the sum $(l - x)^2 + x$ being constant, the maximum of c corresponds to

$$\frac{l-x}{x} = \frac{2}{1}, \text{ and } x = \frac{l}{3} = \frac{2\,l}{6}.$$

Thus, to obtain the largest box divide AB and AD into six equal parts and draw parallels through the first points of division.

EXAMPLE 3. In an analogous manner *find the largest cylinder which can be inscribed in a sphere.*

Let r be the radius of the sphere, x the radius of the base of the cylinder, and $2\,y$ the height, then

$$y = \frac{r}{\sqrt{3}} \text{ or } 2\,y = r\frac{r}{\sqrt{3}}, \text{ and } x = r\sqrt{\frac{2}{3}}.$$

EXAMPLE 4. *Circumscribe a given cylinder by a cone of minimum volume.*

Let h be the height of the cylinder, r the radius of its base, y the height of the cone, and x the radius of its base, then we find that for a minimum volume,

$$y = 3\,h \text{ and } x = \frac{3}{2}r.$$

588. *Resolve a given number into two factors x and y, the sum z of which should be a minimum.* Having

$$x + y = z, \text{ and } xy = a,$$

x and y are, for any value of z, the roots of the equation $u^2 - zu = -a$, which gives (572, 573):

$$x = \frac{z}{2} + \sqrt{\frac{z^2}{4} - a}, \quad y = \frac{z}{2} - \sqrt{\frac{z^2}{4} - a}.$$

If x and y should have real values, $\frac{z^2}{4}$ should at least be equal to a, or $z = 2\sqrt{a}$; at this lower limit, the two roots are equal, and we have:

$$x = y = \frac{z}{2} = \sqrt{a}.$$

Thus *the minimum of the sum $x + y$ of two variable positive factors, the product $xy = a$ of which is a constant, occurs when each of these factors is equal to the square root of the given product* (583).

From this it follows:

1st. *That of all rectangles, which have the same area, the square has the shortest perimeter.*

2d. *That of all the right triangles, which have the same area, the sum of the legs of the isosceles is the least.*

589. *The minimum of the sum of any number n of variable positive factors, of which the product a is constant, occurs when all the factors are equal, that is, when each of them is equal to* $\sqrt[n]{a}$. Because if only two of the factors were unequal, replacing each by their geometrical mean, their sum would be diminished, as would also the total sum, without changing the product of the factors (585).

590. *The sum $x^2 + y^2 = z$ of the squares of two variable quantities x and y, the sum $x + y = a$ of which is constant, when the two quantities are equal,* and therefore, *each equal to* $\frac{a}{2}$.

Squaring both members of the equation,

$$x + y = a, \text{ we have } x^2 + y^2 = a^2 - 2\,xy,$$

and it is seen that $x^2 + y^2$ will be a minimum when xy is a maximum, that is, when (583) $x = y = \frac{a}{2}$.

From this it follows that:

1st. *Of all right triangles, of which the sum of the legs is constant, the isosceles has the shortest hypotenuse.*

2d. *Of all the rectangles having the same perimeter, the square has the shortest diagonal.*

3d. *Of all the squares inscribed in a given square, the one whose corners bisect the sides of the given square is the smallest.*

591. The preceding comes under the general head of *finding the maximum and minimum of a trinomial*

$$ax^2 + bx + c.$$

Designating the variable value of the trinomial by y, we have:

$$ax^2 + bx + c = y \text{ or } ax^2 + bx + c - y = 0,$$

from which (576):

$$x = \frac{-b \pm \sqrt{4\,ay - (4\,ac - b^2)}}{2\,a}.$$

Thus, in order to obtain a real value of x, the following condition must be fulfilled:

$$4\,ay \gtreqless 4\,ac - b^2; \tag{1}$$

and there are two cases, according as the coefficient of x^2 is positive or negative.

Case 1. For $a > 0$, the relation (1) gives:

$$y \geqq \frac{4\,ac - b^2}{4\,a}.$$

It is seen that in this case for real values of x *the smallest value of y is* $\dfrac{4\,ac - b^2}{4\,a}$, and since for this minimum value the radical becomes 0, we have $x = -\dfrac{b}{2\,a}$.

Thus the trinomials

$$3\,x^2 - 7\,x + 2 \text{ and } x^2 + x + 1,$$

in which the coefficient of x^2 is positive, have respectively for their *absolute minimum values*,

$\dfrac{4 \times 3 \times 2 - 7 \times 7}{4 \times 3} = -\dfrac{25}{12}$, which corresponds to $x = -\dfrac{-7}{2 \times 3} = \dfrac{7}{6}$;

$\dfrac{4 \times 1 \times 1 - 1 \times 1}{4 \times 1} = \dfrac{3}{4}$, which corresponds to $x = -\dfrac{1}{2}$.

Case 2. For $a < 0$, the relation (1) gives:

$$y \leqq \frac{4\,ac - b^2}{4\,a} \quad \text{(since } 4\,a \text{ is negative).}$$

It is seen that the *greatest value of y* is $\dfrac{4\,ac - b^2}{4\,a}$, and this maximum corresponds to $x = -\dfrac{b}{2\,a}$.

Thus the trinomial $-9\,x^2 + 6\,x - 1$, in which the coefficient of x^2 is negative, has for an *absolute maximum value*,

$$\frac{4 \times -9 \times -1 - 6 \times 6}{4 \times -9} = \frac{36 - 36}{-36} = 0,$$

which corresponds to $\quad x = -\dfrac{6}{2 \times -9} = \dfrac{1}{3}$.

PROPERTIES OF TRINOMIALS OF THE SECOND DEGREE

The properties of the trinomials of the second degree written in the form

$$y = ax^2 + bx + c$$

may be summed up as follows:

First property. (Unequal roots.) If in making a trinomial of the second degree equal to zero, two real unequal roots are obtained, any quantity lying between these two roots, substi-

tuted for x in the trinomial, will give signs which are the opposite of that of the coefficient a of the first term of the second degree; and any quantity lying outside of the roots, that is greater or less than the roots, substituted for x in the trinomial, gives to this trinomial the same sign as that of the coefficient a of its first term.

To demonstrate this, assume that a is positive, and let x' and x'' be the roots of the trinomial; then from the transformation in article (**543**) we may write:

$$y = a(x - x')(x - x'').$$

Replacing x by a number α, which lies between the roots, that is,

$$x' > \alpha > x''$$

and

$$\alpha - x' < 0,$$
$$\alpha - x'' < 0,$$

we have the product

$$a(\alpha - x')(\alpha - x'') = y,$$

with the opposite sign to that of the coefficient a of its first term.

From the above relations:

$$\begin{matrix} \alpha - x' > 0 \\ \alpha - x'' > 0 \end{matrix} \quad \text{or} \quad \begin{matrix} \alpha - x' < 0, \\ \alpha - x'' < 0, \end{matrix}$$

we have the product

$$a(\alpha - x')(\alpha - x'') = y,$$

with the same sign as that of a, since the two factors $(\alpha - x')$ and $(\alpha - x'')$ are of the same sign, and the value of y approaches infinity as the value of α increases.

Second property. (Equal roots.) If the roots of the trinomial are equal, any number α substituted for x in the trinomial will give the same sign as that of the coefficient a of the first term.

The trinomial may be written in the form

$$y = a(x - x')^2,$$

and will always have the same sign as a for any value positive or negative given to x, and will approach infinity for increasing values of $\alpha = x$.

Third property. (Imaginary roots.) In case the roots are imaginary, any value substituted for x in the trinomial will give the same sign as that of the coefficient a of the first term.

Solving the equation,

$$ax^2 + bx + c = 0, \tag{1}$$

we obtain,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};$$

since the roots are imaginary, we have:

$$4ac > b^2,$$

and

$$\frac{c}{a} > \frac{b^2}{4a^2}.$$

The quantity $\frac{c}{a}$ being greater than a positive quantity, we may write:

$$\frac{c}{a} = \frac{b^2}{4a^2} + k^2. \tag{2}$$

The relation (1) may be written:

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0.$$

Substituting the value of $\frac{c}{a}$ (2)

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + k^2\right) = 0,$$

$$a\left[\left(x + \frac{b}{2a}\right)^2 + k^2\right] = 0.$$

In this form it is seen that by replacing x by any value, a result y of the same sign as a would be obtained; therefore, in the case of imaginary roots, the trinomial

$$ax^2 + bx + c = y$$

always retains the same sign as the coefficient a of its first term; when x is replaced by any value, positive or negative, and the value of the trinomial approaches infinity, $a = x$ is increased. For $x = -\frac{b}{2a}$ the trinomial has a minimum value.

Example 1. It is desired to study the following fraction; find its maximum and its minimum when x is varied.

Write:

$$\frac{x^2 - 2x + 21}{6x - 14} = y,$$

then

$$x^2 - 2x + 21 = 6xy - 14y$$

or

$$x^2 - 2x(1 + 3y) + 14y + 21 = 0,$$

$$x = 1 + 3y \pm \sqrt{9y^2 - 8y - 20}.$$

If x is to be real, the trinomial $9y^2 - 8y - 20$ must be positive; the roots of this trinomial are:

$$9y^2 - 8y - 20 = 0,$$

$$y = \frac{4 \pm \sqrt{16 + 9 \times 20}}{9}$$

$$y' = 2; \quad y'' + -\frac{10}{9}$$

Thus two unequal roots are obtained, and the first property of trinomials of the second degree is applicable, and gives, for all values of y between y' and y'', a negative trinomial and imaginary x; and for all values not between y' and y'', a positive trinomial and a real y; therefore, y may be varied from 2 to + infinity and from $-\frac{10}{9}$ to $-$ infinity, and 2 is the minimum and $-\frac{10}{9}$ the maximum value of the given fraction.

It remains to determine the corresponding values of x. The maximum and minimum of y were deduced from the relation

$$9y^2 - 8y - 20 = 0,$$

which does away with the radical and gives for x:

$$x = 1 + 3y.$$

Substituting successively

$$y' = 2 \text{ and } y'' = -\frac{10}{9} \text{ for } y,$$

we obtain:

$$x' = 7 \text{ for } y' = 2 \text{ (minimum)}$$

and

$$x'' = -\frac{7}{3} \text{ for } y'' = -\frac{10}{9} \text{ (maximum)}$$

EXAMPLE 2. Study the variation of the expression,

$$y = x \pm \sqrt{2x^2 - x};$$

that is, determine the maximum and minimum of y when the quantity x varies in all possible manners.

Find the roots of the polynomial

$$2x^2 - x = 0,$$

which may be written,

$$x(2x - 1) = 0,$$

$$x' = 0 \text{ and } x'' = \frac{1}{2}.$$

Thus two unequal roots are obtained, and the first property must be applied in order to study the variation of the quantity $2x^2 - x$; any quantity between 0 and $\frac{1}{2}$ substituted for x would make the quantity $2x^2 - x$ negative, and thus give an imaginary value to y, while any quantity not lying between those values would make the quantity $2x^2 - x$ positive; from this it follows that the quantity x can vary from $\frac{1}{2}$ to $+\infty$ and from 0 to $-\infty$ for all real values of y, and that $x'' = \frac{1}{2}$ is a minimum, and $x = 0$ a maximum; therefore, the corresponding values of y may be calculated, which give:

$$y' = x' = 0, \text{ corresponding to the maximum of } x,$$

$$y'' = x'' = \frac{1}{2}, \text{ corresponding to the minimum of } x.$$

As to the maxima or minima of y, it is seen that the relation

$$y = +x \pm \sqrt{2x^2 - x} = +x \pm \sqrt{x(2x - 1)}$$

gives greater absolute values of y for greater absolute values of x, therefore, y varies from 0 to $+\infty$ and from $\frac{1}{2}$ to $-\infty$.

Example 3. Study the variation,

$$y = x^2 + 6x + 9.$$

The roots of the trinomial are:

$$x = -3 \pm \sqrt{9 - 9} = -3.$$

These roots being equal, the above trinomial may be written,

$$y = (x + 3)(x + 3) = (x + 3)^2.$$

In this form it is seen that any value positive or negative would give a positive value to y; but for $x = -3$ the quantity

y equals 0; therefore, y varies from 0 to $+\infty$, and x varies from $+\infty$ to $-\infty$.

EXAMPLE 4. Study the variation,

$$y = x^2 - 4x + 15.$$

Putting the trinomial equal to 0 and solving for x,

$$x^2 - 4x + 15 = 0,$$
$$x = 2 \pm \sqrt{4 - 15} = 2 \pm \sqrt{-11}.$$

The values of x being imaginary, the third property of trinomials must be applied in order to study the variation of the trinomial, that is, that any value substituted for x will give the trinomial the same sign as that of the coefficient of x^2. The above trinomial may be written:

$$x^2 - 4x + 15 = (x - 2)^2 - 4 + 15,$$
$$y = (x - 2)^2 + 11.$$

In this form it is seen that y is positive for all values of x, positive or negative, and that the value of y increases with that of x; but for $x = 2$, the quantity y is a minimum and is equal to;

$$y = 11.$$

From this minimum, y varies to $+\infty$.

EXAMPLE 5. Study the variation,

$$y = 3x - 1 \pm \sqrt{x^2 - 4x + 15},$$
$$x^2 - 4x + 15 = 0,$$
$$x = 2 \pm \sqrt{-11}.$$

Referring to Example 4, we may write,

$$y = 3x - 1 \pm \sqrt{(x - 2)^2 + 11}.$$

In this form the radical is positive for any value, positive or negative, given to x; and x may vary from $-\infty$ through 0 to $+\infty$.

As to y, its maximum and minimum are obtained by making the radical as small as possible, that is, taking $x = 2$, which gives for y:

$$y = 3 \times 2 - 1 \pm \sqrt{11},$$
$$y' = 5 + \sqrt{11} \text{ (minimum)},$$
$$y'' = 5 - \sqrt{11} \text{ (maximum)}.$$

These values are the limits; therefore, y varies from y' to $+\infty$, and from y'' to $-\infty$, but there is no value of x which can make $y = 0$.

EQUATION OF THE THIRD DEGREE

592. Transformations which permit the solution of an equation of the third degree.

The most general form of an equation of the third degree is:

$$ax^3 + bx^2 + cx + d = 0. \tag{1}$$

All the terms may be divided by a, which will give

$$x^3 + Bx^2 + Cx + D = 0 \tag{2}$$

The term x^2 may be eliminated by proceeding as in the following special case.

Given:

$$x^3 - 4x^2 + 5x - 2 = 0. \tag{3}$$

Let $x = y + h$; h being indeterminate, and y a new unknown.

Then substituting this value of x in equation (3),

$$y^3 + 3y^2h + 3yh^2 + h^3 - 4y^2 - 8yh - 4h^2 + 5y + 5h - 2 = 0,$$

or

$$y^3 + y^2(3h - 4) + y(3h^2 - 8h + 5) + h^3 - 4h^2 + 5h - 2 = 0.$$

This relation is true for all values of h; therefore, we can put

$$3h - 4 = 0$$

$$h = \frac{4}{3}.$$

Then substituting this value for h in all the terms of the last equation, we have an equation of the form:

$$y^3 + py + q = 0, \tag{4}$$

wherein p is the numerical coefficient of the term y, and q the sum of all the known terms. It is in this form (4) that an equation of the third degree is most often solved, or, which is the same thing, in the form:

$$x^3 + px + q = 0.$$

The solution of third degree equations.

$$x^3 + px + q = 0. \tag{a}$$

Let x be replaced by the sum of two unknowns.

$$x = y + z. \tag{b}$$

Substituting in (a),

$$y^3 + 3\,y^2z + 3\,yz^2 + z^3 + p\,(y + z) + q = 0,$$

or

$$y^3 + z^3 + (y + z)\,(3\,yz + p) + q = 0. \tag{c}$$

The unknowns y and z should satisfy only the relation (b), therefore the following condition may be imposed:

$$3\,yz + p = 0. \tag{d}$$

Then reducing (c),

$$y^3 + z^3 + q = 0. \tag{e}$$

From equation (d),

$$y^3z^3 = -\frac{p^3}{27},$$

and from equation (e),

$$y^3 + z^3 = -q.$$

From these it follows that the quantities y^3 and z^3 are the roots of the following equation,

$$t^2 + qt - \frac{p^3}{27} = 0,$$

and

$$y^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \quad z^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}};$$

substituting $x = y + z$,

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}. \tag{A}$$

When the square root is positive, the calculation may be effected without difficulty and the roots of the equation determined. The other roots are imaginary, and are calculated from the following formulas:

Let A and B be the values of the two cubic radicals, then the three roots of the equation of the third degree are:

$$x_1 = A + B,$$
$$x_2 = A\alpha + B\alpha^2,$$
$$x_3 = A\alpha^2 + B\alpha,$$

wherein α represents one of the two imaginary cubic roots of

unity, or one of the roots of the equation $x^3 = 1$, which gives besides $\alpha = 1$, the two roots:

$$\alpha = \frac{-1 + \sqrt{3}\sqrt{-1}}{2} \quad \text{and} \quad \alpha = \frac{-1 - \sqrt{3}\sqrt{-1}}{2}.$$

NOTE. — See examples at end of Trigonometry (1072).

REMARK. When the quantity $\frac{q^2}{4} + \frac{p^3}{27}$ is negative, the square roots are imaginary, and consequently so are the cube roots, and it appears that the roots should be imaginary. But here is a peculiarity of the third degree equation, because the three roots are real. It is called the irreducible case of the third degree equation, and trigonometric transformations must be used to express the roots. (See end of Trigonometry.)

In many cases numerical equations of the third degree may be solved without recourse to the general formula (A), by a process similar to that in (580).

Thus, having given:

$$3x^3 - 4x^2 + 5x - 18 = 0,$$

write

$$y = 3x^3 - 4x^2 + 5x - 18,$$

then make

$$x = 0, 1, 2, 3, -1, -2, -3, \text{etc.},$$

and calculate the corresponding values of y, and plot the graph of the equation (546). The points where the graph cuts the x-axis will determine the roots of the equation with a sufficient degree of accuracy.

593. *The solution of an equation in annuities by the graphic method.*

Calculate the rate of an annuity, $a = \$11,986$, corresponding to a loan of $c = \$200,000$ for 50 years.

Referring to article (410), it is seen that the solution of this problem is expressed by the formula (3). Therefore, the relation

$$r = \frac{a}{c} - \frac{a}{c\,(1-r)^n} \tag{1}$$

wherein r = rate (unknown), $c = \$200,000$, $a = \$11,986$, $n = 50$ years, is to be solved.

It is noted that the second term of the second member of the equation is smaller than the first $\frac{a}{c}$; if the second term is neglected, the value of r will be too large.

$$r = \frac{11{,}986}{200{,}000} = 0.05993.$$

Substituting this value or 0.06 in equation (1),

$$r = 0.05993 - \frac{11{,}986}{200{,}000\ (1.06)^{50}};$$

then with logarithms,

$$r = 0.05669.$$

To find if this value is too large or too small, write (1) in the form

$$y = \frac{a}{c} - \frac{a}{c\ (1+r)^n} - r. \tag{2}$$

Substituting 0.05669 for r,

$$y = -r = -0.05669.$$

Now it is seen that this value is too large; try $r = 0.056$, the equation (2) gives:

$$y = -0.0000007.$$

This very small value indicates that the value of r is very nearly correct. If r is taken as 0.055, we find $y = +0.0008089$, which shows that the value of r lies between 0.056 and 0.055.

Below are the various values obtained in the trials:

Values of r	Values of y
0.05993	$-$ 0.05669
0.056	$-$ 0.0000007
0.0555	$+$ 0.00041

Thus the method of trial and error consists in giving values to r which give opposite signs to y, and in the given example, it is found that the value of r lies between 0.056 and 0.055. Trying $r = 0.0558$, we still get a positive value for y, which shows that r lies between 0.056 and 0.0558, and so on. The same is found to be true for $r = 0.0559$; thus the value $r = 0.056$ is correct to less than one thousandth.

PART III

GEOMETRY

DEFINITIONS

594. The *volume* of a body is that portion of space occupied by the body.

The limit of a body or its volume is the *surface* of the body or the volume.

The limit of a portion of the surface is a *line.*

The extremities of a portion of a line are called *points.*

Remark. A volume has three dimensions: *Length, breadth, and thickness*; a surface has two, *length* and *breadth;* a line has only one, *length;* a point has none.

595. Volumes, surfaces, and lines come under the common head of *geometrical figures.*

Geometrical figures are represented to the eye by material objects; but geometry has nothing to do with the material, it is simply the shape and size which are studied.

596. Two *figures coincide* when they have the same shape and size and are superposed one upon the other.

Two *equal figures* have the same shape and size, and coincide throughout their extent when superposed one upon the other.

Two *equivalent figures* have the same size.

Remark. Two equal figures are always equivalent, but two equivalent figures are not necessarily equal.

597. A straight or right line may be thought of as a thread tightly stretched between two points.

A straight line is the shortest distance between two points A and B. Only one straight line can be drawn between two points A and B; two straight lines which have two points in common coincide throughout their length, and two points are sufficient to determine a straight line.

Fig. 10

598. *The direction of any straight line AB* is the line itself prolonged indefinitely from its extremities A and B.

599. *Directions of a line.* Every straight line may be considered as having two directions: thus in Fig. 10 we have the directions AB and BA, which are distinguished by the order of the letters.

600. A *broken line ACDB* is composed of a series of different successive straight lines.

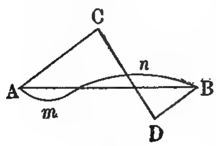


Fig. 11

601. A *curved line AmnB* is a line no part of which is straight. It is the limit which a broken line approaches when the number of its elements is indefinitely increased (136).

602. A plane is an indefinite surface, such that a straight line joining any two points in that surface will lie wholly in the surface.

603. A *plane may be constructed to contain: First,* any three points not in a straight line; *Second,* any two intersecting straight lines; *Third,* any line and a point which lies outside of the line; but only one such plane can be constructed, because all planes containing three points, two intersecting lines or a point and a line, coincide and are one.

604. The intersection of a plane and a line is a point.

The intersection of two planes is a straight line, which contains all the points common to both.

605. A figure is a *plane figure* when it has all its points in the same plane.

606. The *contour or perimeter* of a surface is the line which bounds the surface on all sides.

607. A *broken surface* is a surface composed of several plane surfaces not situated in the same plane (600).

608. A *curved surface* is a surface no part of which is plane. It is the limit approached by a broken surface when the number of its elements is indefinitely increased (601).

609. A figure which contains all the points that fulfill a certain set of conditions is called a *geometrical locus* (585).

610. Geometry is the science which treats of position, form, and magnitude.

Plane geometry treats of plane figures.

Solid geometry treats of solids and space.

PLANE GEOMETRY

BOOK I

STRAIGHT LINES

611. Two straight lines AB and AC drawn from the same point A and in different directions form a geometrical figure called an *angle;* the lines AB and AC, which may be prolonged indefinitely, are the *sides of the angle;* and the common point A is the *vertex of the angle.*

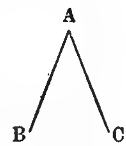


Fig. 12

The *magnitude of an angle* is independent of that of the sides. A very clear idea of an angle and its magnitude may be obtained by supposing the lines to coincide first, and then that they be spread apart like a compass; the angle, at first 0, increases in value as the legs of the compass are separated.

A single angle is designated by the letter at its vertex; thus, one would say *the angle A*. But when there are several angles which have the same vertex, each is designated by the three letters BAC or CAB, with the letter which represents the vertex in the middle.

The angle A is the angle between the two straight lines AB and AC (Fig. 12); and, in general, the angle between the two straight lines AB and CD (Fig. 13), which may or may not be situated in the same plane, is the angle $BC'D'$ formed by one of the lines AB and a line $C'D'$ parallel to CD and intersecting AB in any point C'.

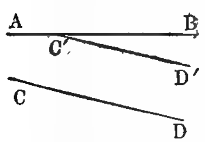


Fig. 13

It is seen that an angle between two straight lines is determined by the direction of the lines; thus, for the direction AB and CD the angle would be $AC'D'$.

612. *Two angles BAC and CAD* are adjacent when they have the same vertex A, and one side common, and are exterior to one another (Fig. 14).

613. Two angles are *vertical angles* when they have the same

vertex and the sides of one are prolongations of the sides of the other. Such are angles

AOC and *BOD*, *AOD* and *BOC* (Fig. 15).

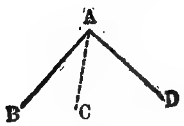


Fig. 14

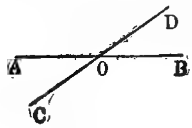


Fig. 15

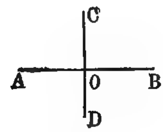


Fig. 16

Vertical angles are equal.

614. A *straight line is perpendicular to another* when by the intersection of one with the other equal adjacent angles are formed. Thus (Fig. 16), supposing $AOC = BOC$, CD is perpendicular to AB; and therefore, AB is also perpendicular to CD.

When one line is perpendicular to another, the latter is also perpendicular to the former.

Lines which intersect and are not perpendicular are *oblique lines*. Such are AB and CD in (Fig. 15).

615. A *vertical line* is one if prolonged would pass through the center of the earth.

All straight lines perpendicular to a vertical are *horizontal* (766).

616. The angles formed by the intersection of two lines perpendicular to one another are called *right angles*. Such are AOC and BOC in (Fig. 16).

All right angles are equal.

All angles BOD (Fig. 15), less than a right angle, are *acute* angles; and all angles AOD (Fig. 15), greater than a right angle, are *obtuse angles*.

617. *Two angles are complementary or complements* when their sum is equal to a right angle; such are the angles BAC and CAD (Fig. 14), supposing their sum BAD to be a right angle.

Two angles are supplementary or supplements when their sum is equal to two right angles or a *straight angle*. Such are the two angles AOD, BOD (Fig. 15).

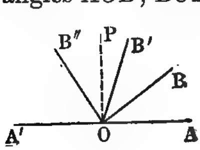


Fig. 17

618. The sum of all the consecutive adjacent angles AOB, BOB', $B'OB''$, $B''OA'$, about a point A on one side of a straight line $A'A$, is equal to a straight angle or two right angles. The perpendicular PO erected at the point

O on AA' determines two right angles AOP and POA' which are equal to the sum of AOB, BOB', $B'OB''$, $B''OA'$.

If two angles AOB, BOA', are supplementary (617), the exterior sides OA, OA', form a straight line.

The sum of all the consecutive adjacent angles AOB, BOB', $B'OB''$ . . ., formed about a point O by any number of straight lines radiating from the point, is equal to four right angles.

619. From any point a perpendicular may be drawn to a given line, but only one can be drawn from that given point.

To erect a perpendicular OC upon a straight line AB (Fig. 16), is to draw a perpendicular through the line at a point O taken on the line.

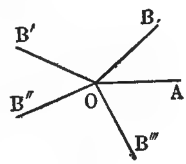


Fig. 18

To drop a perpendicular CO upon a straight line AB (Fig. 16), is to draw a perpendicular to the line passing through a given point C outside of the line.

620. From a point A outside of a given straight line BC, drop a perpendicular AD and several obliques AE, AF, and AG; then: First, the perpendicular is shorter than any oblique; second, the two obliques AE, AF, which cut off equal distances at the foot of the perpendicular, are equal; third, of the two obliques AE, AG, the one AE, which cuts off the shorter distance from the base of the perpendicular, is the shorter line. The converse holds for all these statements.

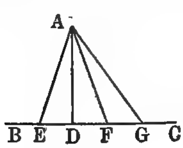


Fig. 19

The perpendicular AD, being the shortest distance from the point A to the straight line, is *the distance from the point to the line.*

621. A *perpendicular CD erected at the middle of a line AB is the geometrical locus of all points equidistant from the extremities of the line* (609). That is, that any point C, taken on CD, gives $AC = BC$, and any point E not on the line CD, we have $AE > BE$ or $AE < BE$, according as E is on the right or left of CD.

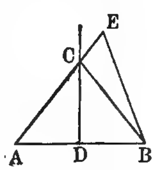


Fig. 20

622. The *bisector of an angle* is a straight line which divides the angle into two equal parts.

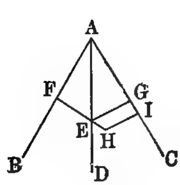


Fig. 21

The bisector AD of an angle BAC is the geometrical locus of all the points within the angle and equidistant from the sides (609). That is, if from any point E taken on AD the perpendiculars

EG and *EF* are drawn to the sides, these perpendiculars are equal; if a point *H* is taken outside of *AD*, the perpendicular *HF* will be greater than *HI*.

The bisectors of two vertical angles form a straight line (613).

The bisectors of two supplementary adjacent angles are perpendicular to one another and form a right angle (612, 614, 617).

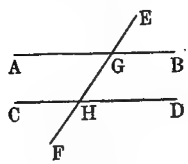


Fig. 22

623. Two straight lines *AB* and *CD* (Fig. 22) are *parallel* when being in the same plane they may be indefinitely prolonged without meeting (598).

Through a point *A* (Fig. 22) exterior to a given line *CD*, one and only one parallel to this line can be drawn.

624. When any two straight lines *AB*, *CD*, situated in the same plane, are cut by a third straight line *EF*, called a *transversal*, we have the following angles formed:

1st. *Interior angles*, each of the four angles formed between the two given lines. Such are *AGH*, *BGH*, *CHG*, *DHG*.

2d. *Exterior angles*, each of the four angles formed outside of the two given lines. Such are *AGE*, *BGE*, *CHF*, *DHF*.

3d. The *alternate-interior angles* are the two angles formed on opposite sides of the transversal, interior and not adjacent. Such are *AGH* and *DHG*, *BGH* and *CHG*.

4th. The *interior-exterior angles* are two angles, one exterior and one interior, both on the same side of the transversal and not adjacent. Such are *AGH* and *CHF*, *BGH* and *DHF*, *CHG* and *AGE*, *DGH* and *BGE*.

5th. The *alternate-exterior angles* are the two angles formed on opposite sides of the transversal, exterior and not adjacent. Such are *AGE* and *DHF*, *BGE* and *CHF*.

625. *When the two lines AB and CD are parallel* (Fig. 22):

1st. The sum of the two interior angles on the same side of the transversal is equal to two right angles; and conversely, if the sum of two interior angles situated on the same side of a transversal is equal to two right angles the lines are parallel.

2d. The sum of the two exterior angles on the same side of the transversal is equal to two right angles, and conversely.

3d. Any two angles of the same name, alternate-interior or alternate-exterior, are equal, and conversely.

626. Two straight lines AB and $A'B'$, perpendicular to a third straight line CD, are parallel to one another (614 and 623).

627. *Any straight line CD perpendicular to one of two parallels is perpendicular to the other.*

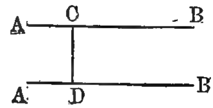


Fig. 23

The part intercepted by the two parallels on the perpendicular CD is a constant, that is, *the parallels are everywhere equidistant from one another.*

628. The two straight lines AB and $A'B'$ being parallel to one another (Fig. 24), any straight line EF, which is parallel to one, is also parallel to the other.

629. *Two angles whose sides are perpendicular are either equal or supplementary* (617).

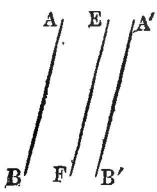


Fig. 24

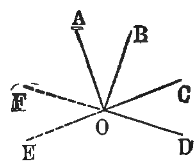


Fig. 25

OA being perpendicular to OC, and OB to OD, we have $AOB = COD$ or EOF, and AOB is the supplement of DOE or COF.

Remark. The same holds where the angles have not the same vertex.

630. *Two angles whose sides are parallel each to each, are either equal or supplementary.*

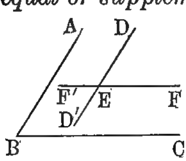


Fig. 26

AB being parallel to DE, and BC to EF, we have $ABC = DEF$ or $D'EF'$, and ABC is supplementary to DEF' or $D'EF$.

The two angles are *equal* when their sides extend in the same direction or in opposite directions from their vertices, and supplementary when two of the parallel sides extend in one direction and two in the other.

BOOK II

POLYGONS

631. A *polygon* is a plane figure bounded on all sides by a broken line (600, 605). Such is the figure $ABCDE$.

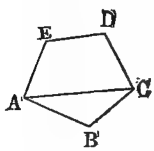


Fig. 27

Each of the straight lines AB, BC, . . ., which form the perimeter of the polygon, is a *side of the polygon.*

Each of the angles EAB, ABC . . ., formed by two adjacent sides of the polygon, is an *angle of the polygon.*

Any line AC joining two vertices not adjacent is a *diagonal of the polygon.*

632. A polygon of three sides is called a *triangle;* one of four sides, a *quadrilateral;* one of five, a *pentagon;* one of six, a *hexagon;* one of seven, a *heptagon;* one of eight, an *octagon;* one of nine, an *enneagon;* one of ten, a *decagon;* one of eleven, *endecagon;* one of twelve, a *dodecagon;* one of fifteen, a *pentadecagon;* one of twenty, an *icosagon.*

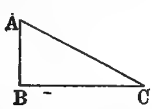


Fig. 28

633. A *triangle ABC* is a *right triangle* when one of its angles is a right angle (616).

The hypotenuse of a right triangle is the side AC opposite the right angle ABC.

634. A triangle is an *obtuse triangle* when one of its angles is obtuse (616).

A triangle is an *acute triangle* when all of its angles are acute.

635. A triangle ABC is an *isosceles triangle* when two of its sides AB and AC are equal.

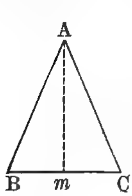


Fig. 29

REMARK. In an isosceles triangle, the angles B and C opposite the equal sides are equal; and conversely, if in a triangle two angles B and C are equal, the sides opposite these angles are equal and the triangle is isosceles. In an isosceles triangle the *altitude Am* bisects the angle A and the base BC (639).

636. A triangle is *equilateral* when its three sides are equal.

Remark. In an equilateral triangle the angles are all equal; and, conversely, if all the angles are equal, the triangle is equilateral.

A triangle is a *scalene triangle* when none of its sides nor angles are equal.

637. In any triangle ABC, any side AC is smaller than the sum $AB + BC$ of the other two sides and greater than their difference $AB - BC$.

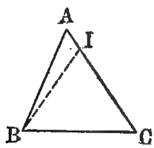


Fig. 30

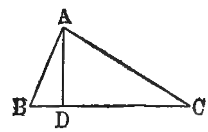


Fig. 31

638. In a triangle ABC (Fig. 30), of two unequal sides AB and AC, the smaller side is opposite the smaller angle; and, conversely, the side AB being smaller than the side AC, the angle C is smaller than the angle B.

639. The *base of a triangle* may be any side.

In the isosceles triangle (Fig. 29), the side BC which is not equal to the others is taken as the base.

The *vertex of a triangle* is the vertex of the angle opposite the base.

The *altitude of a triangle* is the perpendicular distance from the base to the vertex.

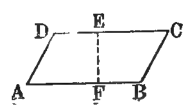


Fig. 32

Thus, having BC as base (Fig. 31), the vertex is A, the altitude is AD.

640. A *parallelogram* is a quadrilateral whose opposite sides are parallel. Such is $ABCD$.

In a parallelogram the opposite sides and angles are equal.

In order that a quadrilateral be a parallelogram, two opposite sides must be equal and parallel. It is also a parallelogram when the opposite sides are equal each to each, or when the opposite angles are equal each to each.

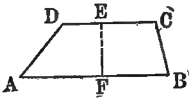


Fig. 33

641. Any side may be taken as the *base of a parallelogram.*

The altitude of a parallelogram is the distance from the base to the opposite side.

Thus, having taken AB for the base (Fig. 32), the altitude is the perpendicular EF intercepted by the base and the side DC (627).

642. A *trapezoid* is a quadrilateral which has two sides and only two sides parallel. Such is $ABCD$ (Fig. 33).

The *bases of a trapezoid* are the two parallel sides AB and DC.

The *altitude of a trapezoid* is the distance EF between the two bases (627).

A trapezoid is *rectangular* when one of the non-parallel sides is perpendicular to the base.

Fig. 34

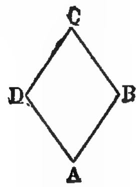


Fig. 35

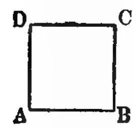


Fig. 36

A trapezoid is *isosceles* or *symmetrical* when its non-parallel sides or *legs* are equal.

643. A *rectangle* is a parallelogram $ABCD$ whose angles are right angles (Fig. 34).

644. The *base of a rectangle* may be any side.

The *altitude of a rectangle* is the length of either side adjacent to the base.

645. A *rhombus* is a parallelogram $ABCD$ whose sides are all equal (Fig. 35).

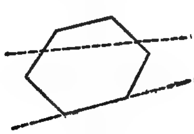


Fig. 37

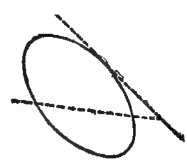


Fig. 38

Any side may be taken as the *base of the rhombus*. (641)

The *altitude of the rhombus* is the distance from the base to the opposite side (627).

646. A *square* is a rectangle $ABCD$ with equal sides (Fig. 36).

The *base* is any one of the sides, and the *altitude* the adjacent side.

647. A polygon is *equiangular* when all its angles are equal. Such are the equilateral triangle and the rectangle (636, 643).

A polygon is *equilateral* when all its sides are equal (600, 609).

REMARK. A polygon can be equiangular and equilateral at the same time. Such are the equilateral triangle and the square.

648. A broken line or a curved line (600, 601) is said to be convex when it lies entirely on one side of any one of its straight line elements, finite in (Fig. 37) and infinitely small in (Fig. 38).

A straight line can not cut a convex line in more than two points.

A polygon is *convex* when bounded by a convex line.

649. A certain convex line $AEFGB$ is greater than any other convex line $ACDB$ which is included by the first when the two have their extremities at the same points A and B.

Since $DB < DIB$, $CI < CHGI$, and $AH < AEFH$, we have $ACDB < ACIB < AHGB < AEFGB$. The exterior line may be formed by two sides of a triangle, and the interior line by two lines joining a point within the triangle to the extremities of the base.

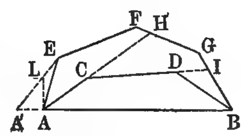


Fig. 39

When the exterior convex line $A'EFGB$ meets the line AB prolonged in A' so that the perpendicular $AL < A'L$, we still have $ACDB < A'EFGB$.

A closed convex line is greater than any convex line totally included by it.

REMARK. All which has been said applies to convex lines which are wholly or only partly composed of curves as well as to broken lines.

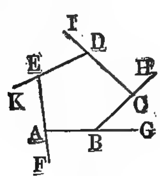


Fig. 40

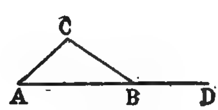


Fig. 41

650. Angles formed by one side of a polygon and the prolongation of an adjacent side are called *exterior angles* of the polygon. Such is the angle DCH, formed by the side CD and the prolongation CH of the adjacent side CB. EDI, AEK, etc., are exterior angles (653).

651. The two angles of a triangle not adjacent to the exterior angle are called *opposite interior angles*. Such are A and C with reference to the exterior angle CBD (653).

652. *The sum of the interior angles of a polygon is equal to two right angles taken as many times less two as the figure has sides.*

Thus, s being the sum of the angles, and n the number of sides of a polygon, we have:

$$s = 2\,(n - 2) = (2\,n - 4)\ rt\ \angle s\ \text{(right angles)}.$$

For the triangle	$n = 3,$	$s = 2\,(3 - 2) = 2\ rt\ \angle s.$
For the quadrilateral	$n = 4,$	$s = 2\,(4 - 2) = 4\ rt\ \angle s.$
For the pentagon	$n = 5,$	$s = 2\,(5 - 2) = 6\ rt\ \angle s.$
For the hexagon	$n = 6,$	$s = 2\,(6 - 2) = 8\ rt\ \angle s.$

and so on for any number of sides.

Remark. The sum of the angles of a triangle being equal to two right angles, it follows that if one of the three angles is right or obtuse, the two others are acute.

The two acute angles of a right triangle are complementary (617, 633).

653. The exterior angle CBD (Fig. 41) of a triangle is equal to the sum of the two opposite interior angles A and C, and consequently greater than either of them.

When the successive sides of a polygon are prolonged as in (Fig. 40), the sum $CBG + DCH + EDI + \ldots$ of the exterior angles is always equal to four right angles.

654. *Any two triangles ABC, $A'B'C'$, are equal:*

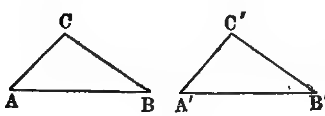


Fig. 42

1st. When two sides and the included angle of one are equal to two sides and the included angle of the other: $\angle A = \angle A'$, $AB = A'B'$, $AC = A'C'$.

2d. When one side and the adjacent angles of one are equal to one side and the adjacent angles of the other: $AB = A'B'$, $\angle A = \angle A'$, $\angle B = \angle B'$.

3d. When they have three sides equal each to each (663).

655. *Two right triangles ABC, $A'B'C'$, are equal:*

1st. When the hypotenuse and an acute angle of one are equal to the hypotenuse and an acute angle of the other: $BC = B'C'$, $\angle B = \angle B'$.

2d. When the hypotenuse and one leg of one is equal to the hypotenuse and one leg of the other: $B'C' = BC$, $A'B' = AB$.

656. *Two parallelograms are equal* when two adjacent sides

and the included angle of one are equal to two adjacent sides and the included angle of the other (640).

Two rectangles are equal when two adjacent sides of one are equal to two adjacent sides of the other (643).

Two rhombuses are equal when one side and one angle of one are equal to one side and one angle of the other (645).

Two squares are equal when one side of one is equal to one side of the other (646).

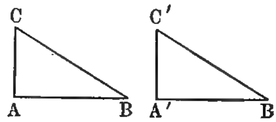


Fig. 43

657. *Two polygons of n sides are equal* when they have $n - 2$ angles or sides equal each to each, and situated in the same order, and respectively $n - 1$ sides or angles equal each to each, and situated in the same order.

The number of conditions necessary for the equality of two polygons of n sides is, therefore, $(n - 2) + (n - 1) = 2n - 3$, and these conditions suffice when they are properly chosen.

658. When two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.

Conversely, when two sides of a triangle are equal respectively to two sides of another, but the third side of the first is greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

659. In an isosceles triangle (Fig. 29), the line Am drawn from the vertex to the middle of the base is perpendicular to the base and bisects the angle at the vertex.

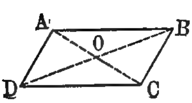


Fig. 44

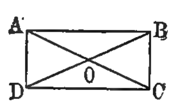


Fig. 45

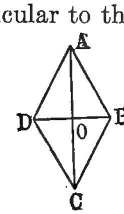


Fig. 46

660. The diagonals of a parallelogram bisect each other; conversely, if the diagonals of a quadrilateral bisect each other, the figure is a parallelogram (Fig. 44).

Besides these properties of a parallelogram:

1st. The diagonals of a rectangle are equal (Fig. 45); from this it follows that in a right triangle BCD, the middle point O of the hypotenuse is equidistant from the three vertices, B, C, D.

2d. The diagonals of a rhombus are perpendicular to one another (Fig. 46).

3d. The diagonals of a square are equal and perpendicular to each other.

The converse statements of the above are true.

661. The diagonal of a parallelogram divides the figure into two equal triangles (Fig. 44). The diagonals of a rhombus and a square divide the figure into four equal right triangles (Fig. 46).

The point O of intersection of the two diagonals of any parallelogram is the center of the figure (Figs. 44–46), that is, the point O lies in the middle of any transversal which contains it and terminates in the perimeter of the parallelogram. Drawing two such transversals and connecting their extremities by straight lines, we have a parallelogram. All transversals which pass through the point O divide the parallelogram into two equal polygons.

662. In any trapezoid: First, the straight line MN, which joins the middles of the opposite non-parallel sides, or legs, is parallel to the bases and equal to half their sum, $MN = \frac{AB+DC}{2}$; second, the straight line EF, which joins the middles of the diagonals, coincides with MN and is equal to half the difference of the bases, $EF = \frac{AB-DC}{2}$.

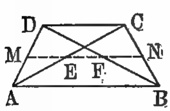


Fig. 47

In any trapezoid the middles of the bases, the point of intersection of the diagonals, and the vertex of the angle formed by producing the legs, lie in the same straight line.

663. *A triangle may be constructed:*

1st. When two sides and the included angle are given.

2d. When one side and two angles are given.

3d. When the three sides are given.

4th. When two sides and an angle opposite one of the sides are given (**654**). (See problems in Geometry.)

664. *A parallelogram may be constructed* when two adjacent sides and the included angle are given; a rectangle, when two adjacent sides are given; a rhombus, when one side and one angle are given; a square, when one side is given (**656**).

BOOK III

THE CIRCLE

665. The *circle* is a plane surface bounded by a curved line called the *circumference,* all points of which are equally distant from a point O within, called the *center.* Any straight line drawn from the center to the circumference is called a *radius.*

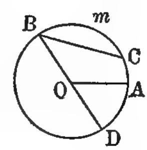


Fig. 48

Thus the circumference is the geometrical locus of all points situated at a distance equal to the radius from the center (609).

Two circles of the same radius are equal, and their circumferences are equal.

666. An *arc of a circle* is a portion BmC of the circumference.

The *chord* of an arc is a straight line BC joining the extremities of the arc.

Any chord BD which passes through the center, is called a diameter, and divides the circle and its circumference into two equal parts.

The diameter is equal to two radii; and since the radii of the same circle are all equal, so are the diameters.

Any chord BC, which does not pass through the center, is less than the diameter.

The diameter divides the circle and circumference into two equal parts; and any chord, other than a diameter, divides them into two unequal parts.

667. Any angle AOD, whose vertex is at the center, is called *an angle at the center.*

An *arc is intercepted by an angle at the center* when the radii which form the sides of the angle are drawn to the extremities of the arc.

668. That part of a circle BmC, bounded by an arc and its chord, is called a *segment of the circle.* The chord is the *base of the segment.* That part AOD of a circle bounded by an arc and two radii is called a *sector of a circle.* The arc is *the base of the sector;* the center of the circle is the *vertex.*

669. The longest chord which can be drawn through a point m, which lies within the circle, is the diameter DD' which passes through the point; the shortest chord is the chord AB perpendicular to the diameter DD'.

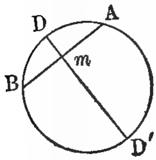


Fig. 49

670. The shortest and the longest line which can be drawn from a point to the circumference of a circle have the same direction as a line drawn from the given point to the center of the circle, whether the point be within or without the circle. Thus (Fig. 49), the longest line from the point m to the circumference is mD', and the shortest is mD.

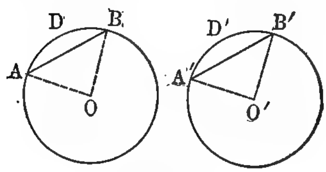


Fig. 50

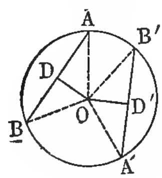


Fig. 51

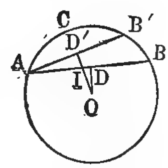


Fig. 52

671. Any diameter DD' (Fig. 49), perpendicular to a chord AB, divides the chord and each of its subtended arcs into two equal parts, $mA = mB$, $DA = DB$, $D'A = D'B$.

A perpendicular erected at the middle of a chord passes through the center of the circle (621).

672. In the same circle or two equal circles:

1st. Two equal arcs ADB, $A'D'B'$ (Fig. 50), not greater than a semicircumference, are subtended by equal chords AB, $A'B'$, and conversely.

2d. Of two arcs the greater is subtended by the greater chord, and conversely.

3d. Two equal chords AB, $A'B'$, are equally distant from the center, $OD = OD'$ (Fig. 51), and conversely.

4th. Of the two chords AB, AB' (Fig. 52), the longer is nearer the center, $OD < OD'$, and conversely.

5th. Equal arcs ADB, $A'D'B'$, are subtended by equal angles at the center (Fig. 50), and conversely.

6th. A greater arc is subtended by a greater angle at the center, and conversely.

7th. The two equal chords AB $A'B'$ (Fig. 50), are the bases of equal segments, and conversely.

8th. Two equal arcs ADB, $A'D'B'$ (Fig. 50), are the bases of equal sectors, and conversely.

673. *A straight line BC is inscribed in a circle* (Fig. 48) when it has its extremities in the circumference of that circle.

The angle CBD formed by two chords which meet at the circumference is called an inscribed angle (Fig. 48).

An angle is inscribed in a segment when its vertex lies in the circumference and its sides pass through the ends of the base of the segment.

All angles inscribed in the same segment are equal (684).

A polygon is inscribed in a circle when its sides are inscribed in the circle (Fig. 62). *The polygon is circumscribed* by the circle.

674. A straight line can not cut a circumference in more than two points, and all lines which cut the circumference in two points are called *secants*.

675. *A straight line AB is tangent to a circle* when they have but one point m in common. A tangent may be thought of as being the limit of a secant where the two points of intersection approach each other and finally coincide.

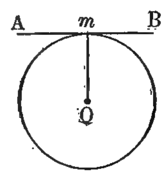


Fig. 53

The perpendicular AB erected at the extremity of a radius Om is tangent to the circle.

The perpendicular Om erected at the point of contact of the tangent AB is *normal to the circumference* at the point m.

All normals to the circumference pass through the center, and all radii are normal to the circumference. The shortest and longest distance from a point to the circumference of a circle are the normals to the circumference which pass through the point (670).

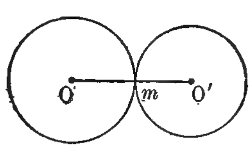


Fig. 54

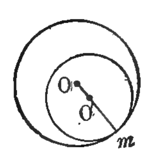


Fig. 55

Two circles O and O′ are tangent to each other when they have one point m in common. They are *externally* or *internally* tangent according as one lies wholly without or within the other.

Two circles tangent to the same line at the same point are tangent to each other. The point common to the tangent and

the circumference (Fig. 53), or to the two tangent circumferences (Figs. 54 and 55), is called the *point of contact* or *point of tangency.*

676. *Two parallels intercept equal arcs upon the circumference;* this is true when they are two tangents EF, GH, or chords AB, CD, or a chord and a tangent AB, EF.

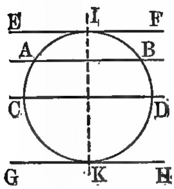


Fig. 56

Conversely, two chords, two tangents, or a chord and a tangent which intercept equal arcs, are parallel.

677. *A polygon is circumscribed about a circle* when each of its sides is tangent to the circle at a point between the extremities (Fig. 63). The circle is *inscribed* in the polygon.

678. *A straight line is normal or oblique to a circumference or to an arc which it meets in a point,* according as it is perpendicular or oblique to the tangent drawn to the circumference or arc at that point (675).

679. *Two circles are concentric* when they have the same center.

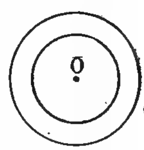


Fig. 57

When two non-concentric circles are in the same plane, a line passing through their centers is called the *line of centers.*

680. A point can always be found which is equidistant from three others not in a straight line, and in the same plane with them; but only one can be found, and this is the center of a circle, whose circumference passes through the three points. (See Problems.)

A circle, and only one, can be drawn through three points which are not in the same straight line (688).

Two circles can not intersect in more than two points.

The center is the only point from which more than two equal lines can be drawn to the circumference.

681. When two circles are tangent externally (Fig. 54), the distance between centers is equal to the sum of the radii. If the two circles are tangent internally (Fig. 55), the distance between centers is equal to the difference of the radii.

The line of centers passes through the point of contact.

682. When two circles have no point in common (Figs. 58 and 59), the distance between centers is either greater than the sum of the radii or less than the difference, according as one circle lies wholly without or within the other.

683. When two circles intersect (Fig. 60), the line of centers is the perpendicular bisector of the chord mp which joins the points common to both, and the distance between centers is less than the sum of the radii and greater than their difference; we have $OO' < Om + O'm$ and $OO' > Om - O'm$ (637).

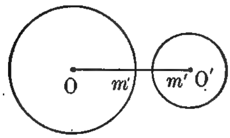


Fig. 58

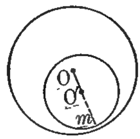


Fig. 59

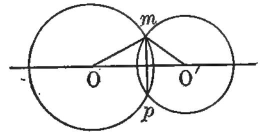


Fig. 60

Conversely, when the distance between centers is less than the sum of the radii and greater than their difference, the circles intersect each other.

When two circles have a common point m outside of the line of centers, they cut each other in a second point p, situated on the other side of the line of centers on a perpendicular to the line of centers and the same distance from it as the other point.

684. Any inscribed angle BCD is equal to half the angle at the center BOD, which intercepts the same arc BD (673).

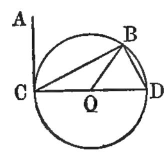


Fig. 61

All angles inscribed in a semicircle are right angles.

A circle drawn upon a given line as diameter is the locus of the vertices of all the right triangles which have the given line for hypotenuse (609). Any angle ACB formed by a tangent AC and a chord CB is equal to half the angle at the center COB, which is subtended by the same arc CB; and therefore, it is equal to any angle inscribed in the segment CDB which has the chord CB for a base.

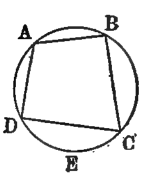


Fig. 62

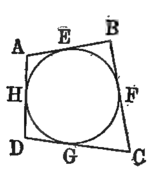


Fig. 63

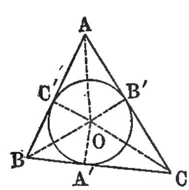


Fig. 64

685. The opposite angles of any quadrilateral inscribed in a circle are supplementary, $A + C = B + D = 2$ right angles, and conversely.

686. The sum $AB + DC$ of the opposite sides of a quadrilateral circumscribed about a circle (677) is equal to the sum $AD + BC$ of the other two sides, and conversely.

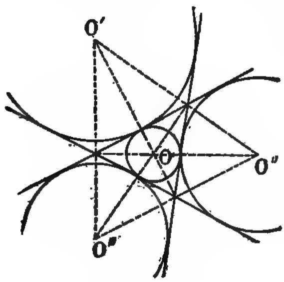


Fig. 65

687. The three bisectors of the angles of a triangle intersect in the same point O (Fig. 64), which is the center of a circle inscribed in the triangle.

The three bisectors of the exterior angles of a triangle (Fig. 65) meet in pairs on each of the bisectors of the interior angles produced, and these points of intersection O', O'', O''', are centers of circles each tangent to one of the sides of the triangle and the other two sides prolonged. These circles are called *escribed circles.*

688. The perpendicular bisectors of the sides of a triangle intersect in a point O (Fig. 66), which is the center of a circumscribed circle (680).

689. The three *medians,* that is, the three lines joining the vertices and the middles of the opposite sides, meet in the same point, which is the center of gravity of the triangle.

690. The *radical axis* of two circles (Fig. 67) is a geometrical locus XX', such that if tangents MT and MT' to the circles be drawn from any point M on the line they will be equal, XX' being perpendicular to the line of centers OO'. Drawing a common exterior tangent KK' to the two circles and bisecting it, we can construct the locus by drawing a perpendicular to the line of centers through the middle point of the common tangent.

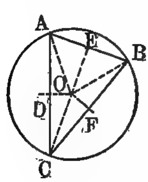


Fig. 66

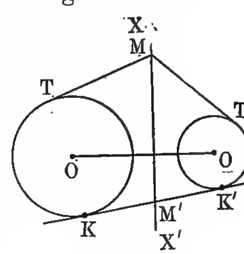


Fig. 67

If the two circles are internally or externally tangent, the radical axis is the common tangent drawn through the point of contact; and if the two circles intersect each other, the radical axis is the common chord indefinitely produced in both directions.

BOOK IV

SIMILAR POLYGONS AND THE MEASUREMENT OF ANGLES

691. *Two lengths are proportional to two other lengths* when their ratio is equal to that of the others (326).

Lengths being measured in certain fixed units, these units may be substituted in the ratios and the arithmetical operations performed.

692. *To divide a line in extreme and mean ratio,* is to divide it into two parts such that the larger part is the mean proportional between the whole line and the other part (330, 344, and Problems).

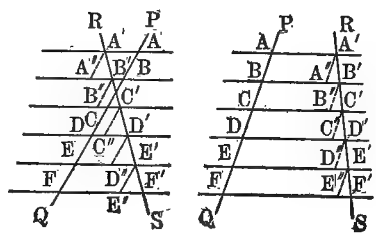


Fig. 68

693. The parallels AA', BB', $CC' \ldots$, intercept proportional segments on the transversals PQ, RS. Thus:

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} \cdots$$

These ratios are also equal to that $\frac{AE}{A'E'}$ of any segments such as AE and $A'E'$.

If the segments or intercepts on one transversal are equal, $AB = BC = CD \ldots$, those on another transversal are also equal, $A'B' = B'C' = C'D' = \ldots$

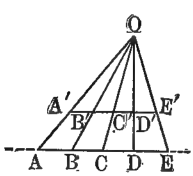


Fig. 69

694. All lines OA, OB, $OC \ldots$, meeting in a common point O, intercept proportional segments on two parallels AE, $A'E'$. Thus:

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} \cdots$$

and these ratios are also equal to $\frac{AD}{A'D'}$ the ratio of any two corresponding segments AD and $A'D'$.

695. *Two polygons ABCDE, A′B′C′D′E′*, are *similar* when the angles of one are equal to the angles of the other and in the same order ($A = A'$, $B = B'$, $C = C' \ldots$), and homologous sides are proportional.

$$\left(\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{DC}{D'C'} \cdots\right).$$

In two similar polygons: *First*, when the angles $A, B \ldots$ of one polygon are respectively equal to the angles $A', B' \ldots$ of another, they are said to be *homologous angles; Second*, the adjacent sides AB and $A'B'$, BC and $B'C'$ of homologous angles are *homologous sides; Third*, the vertices of homologous angles are *homologous vertices; Fourth*, diagonals AC and $A'C' \ldots$ which join homologous vertices are *homologous diagonals; Fifth*, triangles ABC and $A'B'C'$, ACD and $A'C'D'$, which have homologous vertices, are *homologous triangles.*

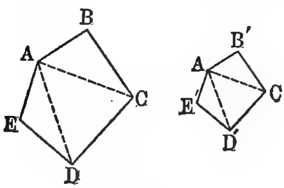


Fig. 70

The ratio of the homologous sides of two similar polygons is the ratio of the *symmetry* of the two figures.

696. The straight lines Aa, Bb, . . ., which join the vertices of two similar polygons, meet in a point O when prolonged; this point is called the *center of symmetry*. We have:

$$\frac{OA}{Oa} = \frac{OB}{Ob} \cdots = \frac{AB}{Ab},$$

ratio of symmetry.

If the figures have equal angles and proportional sides, but placed in an inverse order, they still have a center of symmetry O; and we have:

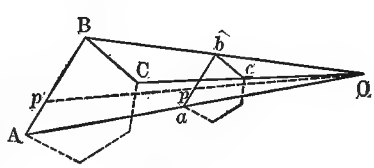


Fig 71.

$$\frac{OA}{Oa} = \frac{OB}{Ob} \cdots \frac{AB}{Ab}.$$

Two points p and p' in two similar figures (Fig. 71), such that a line joining them passes through the center of symmetry when prolonged, are said to be *homologous points.* The same is true in (Fig. 72).

Two circles have two centers of symmetry, one between them O', and one external to them O, which are located at the intersections of their common tangents.

697. All transversals which cut the three sides of a triangle

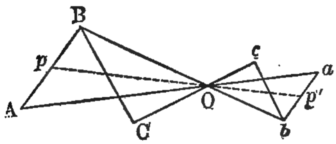


Fig. 72

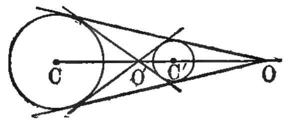


Fig. 73

ABC, determine six segments such that the product of any three which are not consecutive, equals the product of the other three. Thus, the consecutive segments being *BD, DA, AE, EC, CF, FB*, we have:

$$BD \times AE \times CF = DA \times EC \times FB.$$

The six segments are said to be in *involution*. The transversal may cut the sides of the triangle prolonged.

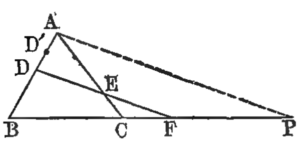


Fig. 74

Conversely, if three points taken on the sides of a triangle determine six segments in involution, these three points are in a straight line.

698. If three unequal but similar figures have their homologous dimensions parallel (Fig. 75), the three centers of symmetry O, O', O'', are in a straight line, and this line is called the *axis of symmetry*.

If one of the figures has its dimensions situated in the inverse

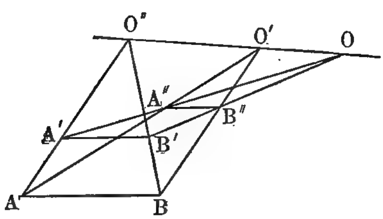


Fig. 75

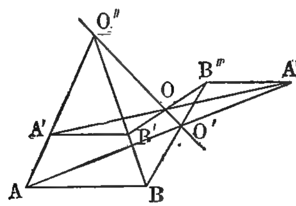


Fig. 76

order of the others (Fig. 76), the centers of symmetry still fall in one straight line.

Three circles have in general, six centers of symmetry, situated in threes, on four axes of symmetry (Fig. 77).

699. In any triangle *ABC* (Fig. 78) a straight line *DE* drawn

parallel to the base, *First*, divides the sides proportionally, $\frac{AD}{AE} = \frac{DB}{EC} = \frac{AB}{AC}$, and conversely; *Second*, forms, together with the adjacent sides of the triangle, a triangle ADE which is similar to the first ABC (693, 695).

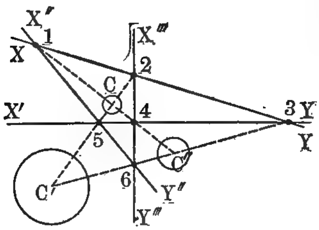


Fig. 77

700. *Two triangles ABC and A′B′C′ are similar:*

1st. When the angles are equal each to each: $A = A'$, $B = B'$, $C = C'$. When two angles are equal, the third must be, and, therefore, two triangles are similar when two angles are equal each to each.

2d. When their sides are proportional:

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}.$$

3d. When they have equal angles between adjacent proportional sides:

$$\angle A = \angle A', \frac{AB}{A'B'} = \frac{AC}{A'C'}.$$

4th. When they have sides parallel (Fig. 79) or perpendicular (Fig. 80) each to each.

5th. When they are right triangles and have the hypotenuse and one leg proportional each to each.

REMARK 1. In two similar triangles the homologous sides are opposite equal angles.

REMARK 2. In two triangles which have their sides parallel

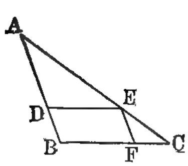


Fig. 78

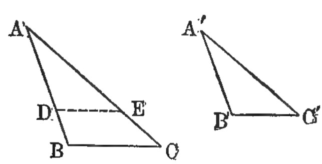


Fig. 79

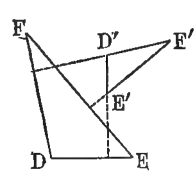


Fig. 80

or perpendicular each to each (4th), the homologous sides are parallel or perpendicular each to each.

701. *Two parallelograms are similar* when they have equal angles between adjacent proportional sides (695).

702. *Two polygons are similar* (Fig. 70) when they can be divided into the same number of similar triangles situated in the same order, and conversely. *Two polygons similar to a third are similar to each other.*

703. In two similar polygons the perimeters and homologous diameters are proportional to the homologous sides; thus we have (Fig. 70):

$$\frac{AB + BC + CD + DE + EA}{A'B' + B'C' + C'D' + D'E' + E'A'} = \frac{AC}{A'C'} = \frac{AB}{A'B'}. \quad (695)$$

704. The bisector of the vertex angle of a triangle divides the base BC into two segments proportional to the adjacent sides, $\frac{BI}{CI} = \frac{AB}{AC}$, and conversely.

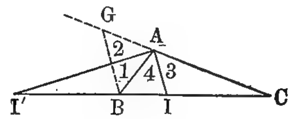


Fig. 81

The bisector of the exterior angle GAB cuts the opposite side produced so as to form segments which are proportional to the adjacent sides, $\frac{BI'}{CI'} = \frac{AB}{AC} = \frac{BI}{CI}$, and conversely.

From the proportion

$$\frac{BI'}{CI'} = \frac{BI}{CI}, \quad (a)$$

we have,

$$CI' \times BI = BI' \times CI,$$

which shows that the product of the whole line CI' and the middle segment BI is equal to the product of the two extreme segments BI' and CI.

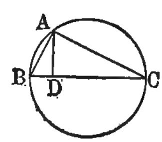


Fig. 82

The proportion (a) is said to be a *harmonical proportion;* the points I', B, I, C, form a *harmonical system;* the points I, I', are called *conjugate harmonics;* the line BC is harmonically divided by the two points I, I'.

Since, for the same line BC, the position of the points I and I' depends upon the ratio $\frac{AB}{AC}$, it is seen that the line BC may be harmonically divided in an infinite number of ways; but the problem is determinate when AB and AC or their ratio is given.

When $AB = AC$ the bisector AI bisects the base BC, and AI_1 is parallel to the base and cuts it in infinity.

705. If in a right triangle ABC a perpendicular AD is drawn from the vertex A of the right angle to the hypotenuse BC: *First,* the triangles ABD, ADC, are similar to each other and similar to the original triangle ABC; *Second,* each leg of the right triangle is a mean proportional between the hypotenuse and its adjacent segment (330). Thus we have:

$$BC : AB = AB : BD \text{ and } BC : AC = AC : CD;$$

Third, the perpendicular is a mean proportional between the segments of the hypotenuse:

$$BD : AD = AD : CD.$$

706. When a perpendicular is drawn from any point A in a circumference of a circle to the diameter BC, and chords AB and AC are drawn between this point and the extremities of the diameter (**648**, and Fig. 82): *First,* each chord is a mean proportional between the diameter and the adjacent segment; *Second,* the perpendicular is a mean proportional between the segments of the diameter.

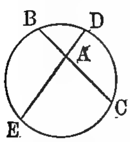


Fig. 83

707. The parts of two chords BC and DE, which intersect, are inversely proportional (326); thus:

$$AB : AD = AE : AC;$$
$$AB \times AC = AD \times AE.$$

From the last equation it is seen that the product of the two parts of all the chords which can be drawn through the same point A are equal. This product is equal to the square of half the chord which is perpendicular to the diameter drawn through the given point.

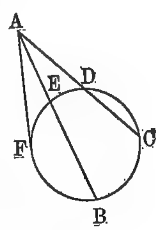


Fig. 84

708. If from a fixed point A without a circle, two secants AB and AC, which terminate in the circumference of the circle, are drawn, they are proportional to their external segments; thus:

$$AB : AC = AD : AE;$$

and

$$AB \times AE = AC \times AD.$$

If from a fixed point A without a circle, a tangent AF and a secant AB, which terminate in the circumference, are drawn, the

tangent is a mean proportional between the secant and its exterior segment:

$$AB : AF = AF : AE \text{ and } AB \times AE = \overline{AF}^2.$$

Thus for a certain point A without the circle, the product of the secant and its external segment is constant and equal to the square of the tangent drawn from that point. This result is analogous to the one obtained when the point was within the circle (707).

709. In the same or equal circles, two angles at the center are to each other as their intercepted arcs (667).

All angles at the center are measured by their intercepted arcs. That is, that the angle contains the unit angle as many times as the arc contains the unit arc. Generally the arc of one degree is taken as the unit arc (222); therefore, the unit angle intercepts an arc of one degree, which is the 360th part of four right angles. The angle of one degree is divided, as is the arc, into 60 equal parts called minutes, and these in turn are subdivided into 60 equal parts called seconds.

It should be noted that when an arc of a certain number of degrees is specified, no length is designated, but simply the number of times this arc contains one 360th part of the circumference which has the same radius as the arc. Thus, arcs of the same number of degrees may be unequal. On the contrary, angles of the same number of degrees are always equal.

710. *An angle inscribed in a circle is measured by one-half its intercepted arc.* The same is true of an angle formed by a tangent and a chord (684, 709).

711. The angle formed by two chords (Fig. 83) intersecting within the circumference is measured by one-half the sum $\frac{EC + BD}{2}$ of the intercepted arcs.

712. An angle formed by two tangents, two secants, or a tangent and a secant, intersecting without the circumference, is measured by one-half the difference of the intercepted arcs.

Thus (Fig. 84), the angle BAC is measured by $\frac{BC - ED}{2}$, and angle FAC is measured by $\frac{FC - FD}{2}$.

BOOK V

THE MENSURATION OF POLYGONS

713. *The length of a line* is the measure of the line, that is, the ratio of the whole line to one of unit length (216, 321).

The area of a surface is the measure of the surface, that is, the ratio of that surface to the unit surface.

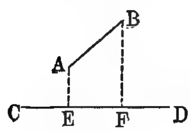


Fig. 85

714. *The product of two lines* is the product of their lengths.

715. *The projection of a point A on a line CD* is the foot E of a perpendicular drawn from that point to the line.

The projection of a line AB on another CD is that part of the latter EF which lies between the projections of the extremities of the first AB on the second CD.

716. *The area of a rectangle is equal to the product of its base and its altitude* (644):

$$S = B \times H.$$

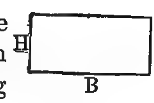


Fig. 86

This expression for the area indicates that the surface contains as many units of surface, which have the unit of length for a side used in expressing B and H, as the product $B \times H$ contains units.

Having $B = 3.5'$ and $H = 2.15'$, we have:

$$S = 3.5 \times 2.15 = 7.525 \text{ square feet.} \qquad (224)$$

717. Two rectangles are to each other as the product of their bases and their altitudes. Thus, having $S = B \times H$ and $S' = B' \times H'$ we have:

$$S : S' = B \times H : B' \times H'.$$

Two rectangles having one equal side are to each other as the other sides. Thus, making $B = B'$ in the preceding proportion we have:

$$S : S' = H : H'.$$

718. *The area of a triangle* is equal to half the product of the base and altitude (639). Let the base, B = 5 feet, and the altitude, H = 3 feet; then:

$$S = \frac{B \times H}{2} = \frac{3 \times 5}{2} = 7.5 \text{ square feet.}$$

719. Two triangles are to each other as the products of their bases and altitudes:

$$S' : S' = B \times H : B' \times H'.$$

Two triangles which have the same bases or the same altitudes are to each other respectively as their altitudes or their bases:

$$S : S' = H : H' \text{ or } S : S' = B : B'.$$

720. Two triangles ABC and ABC' which have the same bases and the same altitudes are equivalent (596, 718). Placing

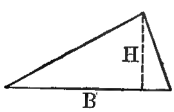


Fig. 87

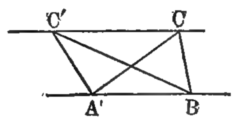


Fig. 88

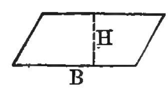


Fig. 89

them so that their bases coincide, their vertices fall on the same line $C'C$ parallel to their common base AB.

721. *The area of a parallelogram* is equal to the product of the base and the altitude (641). Having B = 5 feet and H = 3, we have:

$$S = B \times H = 5 \times 3 = 15 \text{ sq. ft.}$$

It is seen that the area of a parallelogram is double that of a triangle having the same base and altitude (718), and is equal to a rectangle having the same base and altitude (716).

As for rectangles, two parallelograms are to each other as the product of their altitudes and bases, and two parallelograms with the same bases or altitudes are to each other respectively as their altitudes or bases.

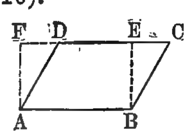


Fig. 90

722. A parallelogram $ABCD$ is equivalent to another parallelogram or rectangle $ABEF$ which has the same base and altitude. Placing them so that their bases coincide, the sides opposite the base will fall on the same line parallel to the base AB.

723. *The area of a trapezoid* is equal to half the sum of the

bases times the altitude (642). $B = 3$ feet and $b = 2$ feet, being the bases, and $H = 1$ foot, the altitude of the trapezoid, the area is:

$$S = \frac{B+b}{2} \times H = \frac{3+2}{2} \times 1 = 2.5 \text{ square feet.}$$

The area of a trapezoid is also equal to the product of the line joining the middle points of the legs and the altitude (662).

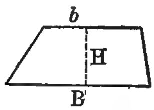


Fig. 91

724. The area of any polygon circumscribed about a circle is equal to half the perimeter times the radius of the circle (677, 718).

Dividing any polygon into triangles by drawing the diagonals, the sum of the areas of these triangles is equal to the area of the polygon (718).

725. Two triangles ABC, $AB'C'$, which have one angle equal, are to each other as the products of the sides which are adjacent to the angle. Thus S and s being the areas of the triangles, we have:

$$S : s = AB \times AC : AB' \times AC'.$$

726. The areas of two similar triangles and, in general, two similar polygons are to each other as the squares of two homolo-

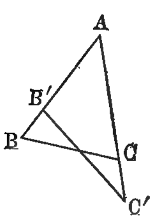


Fig. 92

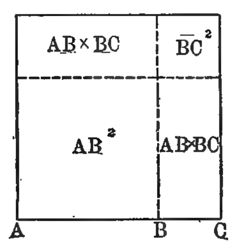


Fig. 93

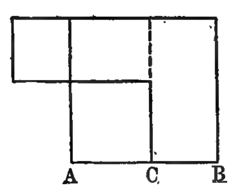


Fig. 94

gous sides or diagonals. The polygons $ABCDE$ and $A'B'C'D'E'$ (Fig. 70) being similar, S and s being their areas, we have:

$$S : s = \overline{AB}^2 : \overline{A'B'}^2 = \overline{AC}^2 : \overline{A'C'}^2. \qquad (703)$$

727. *The square whose side AC is equal to the sum of two lines AB and BC contains* the square of the first line, plus the square of the second, plus twice the rectangle formed by the two lines. Thus we have (479) (Fig. 93):

$$\overline{AC}^2 \text{ or } \overline{(AB + BC)}^2 = \overline{AB}^2 + \overline{BC}^2 + 2\, AB \times BC.$$

728. *The square whose side AC is equal to the difference of two lines AB and BC* is equivalent to the square of the first, plus the square of the second, less twice the rectangle formed by the two lines (**480**) (Fig. 94):

$$\overline{AC}^2 \text{ or } \overline{(AB - BC)}^2 = \overline{AB}^2 + \overline{BC}^2 - 2\,AB \times BC.$$

729. *The rectangle ACED whose sides are respectively equal to the sum and difference of two lines* is equivalent to the difference of the squares of the two lines (**484**):

$$(AB + BC)\,(AB - BC) = \overline{AB}^2 - \overline{BC}^2$$

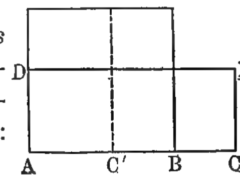


Fig. 95

730. The square constructed on the hypotenuse BC of a right triangle is equal to the sum of the squares on the other two sides. The square of one of the legs is equal to the difference of the squares of the hypotenuse and the other leg. Thus (Fig. 96):

$$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 \text{ and } \overline{AB}^2 = \overline{BC}^2 - \overline{AC}^2$$

$$\text{or } \overline{AC}^2 = \overline{BC}^2 - \overline{AB}^2.$$

731. The square of the diagonal of a rectangle is equal to the sum of the squares of the two adjacent sides (**730**).

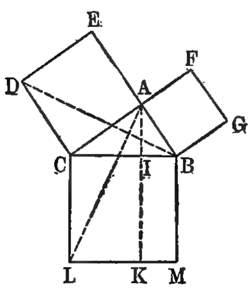


Fig. 96

The square of the diagonal is equal to twice the square of one side; from which it follows that the ratio of diagonal to one side is $\sqrt{2}$.

732. The perpendicular AI, drawn from the vertex of the right angle in a right triangle to the hypotenuse (Fig. 96), divides the hypotenuse into two segments which are to each other as the squares of the sides adjacent to the right angle. We have:

$$BI : IC = \overline{AB}^2 : \overline{AC}^2,$$

and further (**705**),

$$\overline{AI}^2 = BI \times IC, \text{ and } \overline{AC}^2 = CB \times CI,\ \overline{AB}^2 = BC \times BI.$$

733. In any triangle ABC, the products $AB \times AC'$ and $AC \times AB'$ of the two sides AB and AC and their mutual projections upon one another, are equal. Likewise:

$$BC \times BA' = AB \times BC' \text{ and } AC \times CB' = BC \times CA'.$$

734. In any obtuse triangle ABC, the square of the side BC opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the rectangle formed by one of the sides and the projection of the other upon it. Thus:

$$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 + 2\,AC \times AD.$$

In any triangle ABC, the square of a side BC opposite an acute angle A, is equal to the sum of the squares $\overline{AB}^2$ and $\overline{AC}^2$ of the other two sides, less twice the rectangle $AC \times AD$ formed by one side and the projection of the other upon it:

$$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 - 2\,AC \times AD.$$

735. In any triangle:

1st. The sum $\overline{BC}^2 + \overline{BA}^2$, of the square of the two sides adjacent to the vertex is equal to twice the square of the median

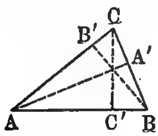


Fig. 97

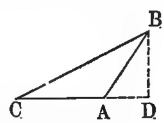


Fig. 98

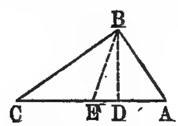


Fig. 99

BE, drawn from the vertex to the middle of the opposite side, plus twice the square of half the base CE:

$$\overline{BC}^2 + \overline{BA}^2 = 2\,\overline{BE}^2 + 2\,\overline{CE}^2;$$

2d. The difference of the squares of these sides is equal to twice the rectangle formed by the base of the triangle and the distance between the foot of the perpendicular to the base drawn from the vertex, and the foot of the median:

$$\overline{BC}^2 - \overline{BA}^2 = 2\,AC \times DE.$$

736. *The product of two sides BC, BA, of a triangle BCA,* is equal to the square of the bisector of the angle which they form, plus the product of the segments formed by this bisector on the third side CA. Thus in (Fig. 99), supposing BE to be the bisector of the angle CBA, we have:

$$BC \times BA = \overline{BE}^2 + CE \times EA.$$

The product of the two sides BC, BA, of a triangle BCA, is also equal to the product of the altitude BD, considering the third

side as base, and the diameter of the circle circumscribed about the triangle (**673**).

737. In any quadrilateral $ABCD$, the sum of the squares of the sides is equal to the sum of the squares of the diagonals, plus four times the square of the line which joins the middle points of the diagonals EF:

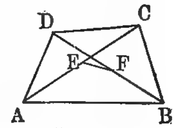


Fig. 100

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2 + 4\,\overline{EF}^2.$$

738. In any trapezoid, the sum of the squares of the legs is equal to the sum of the squares of the diagonals, less twice the product of the bases. Referring to Fig. 47:

$$\overline{AD}^2 + \overline{BC}^2 = \overline{AC}^2 + \overline{BD}^2 - 2\,AB \times DC.$$

739. In all parallelograms, the sum of the squares of the sides is equal to the sum of the squares of the diagonals, and conversely.

BOOK VI

REGULAR POLYGONS AND THE MENSURATION OF THE CIRCLE

740. A regular polygon is a polygon which is equilateral and equiangular (**647**).

The *center* and the *radius* of a regular polygon are the center O and the radius OA of a circle circumscribed about the polygon (**673**).

The *apothem* of a regular polygon is the radius OP of a circle inscribed in the polygon (**677**).

The angle between the radii drawn to the extremities of any side is called the *angle at the center* of the polygon.

The part $OABC$, included between two consecutive radii OA and OC, is called the sector of the polygon.

741. A circumference being divided into three or more equal parts: *First,* the chords which join the consecutive points of division form a regular inscribed polygon; *Second,* the tangents drawn at the points of division form a regular circumscribed polygon.

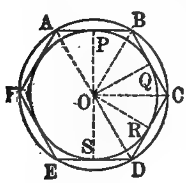


Fig. 101

Conversely: First, the vertices of a regular inscribed polygon divide the circumference into equal parts; *Second,* the points of contact of the sides of a regular circumscribed polygon divide the circumference into equal parts (**673, 677**).

The circle inscribed in and the circle circumscribed about the same regular polygon are concentric (**679**).

When a regular polygon is circumscribed about a circle, each side is divided into two equal parts by the point of contact.

742. One circle, and only one, may be circumscribed about any regular polygon (**741**).

One circle, and only one, may be inscribed in any regular polygon.

743. The area of a regular polygon is equal to one-half the product of its perimeter and its apothem OP (**724, 740**).

744. Two regular polygons having the same number of sides are similar. Their perimeters are to each other as any two homologous linear dimensions; and their surfaces are to each other as the squares of these same dimensions (695, 703, 726, 740).

745. *The side of a square circumscribed about a circle* is equal to the diameter of the circle.

The side c of a square inscribed in a circle of radius R is equal to $\sqrt{2}\,R$ (695).

$$c : R = \sqrt{2} : 1 \quad\text{and}\quad c = R\sqrt{2}.$$

The side of a regular hexagon inscribed in a circle is equal to the radius of the circle.

The side c of an equilateral triangle inscribed in a circle of radius R is equal to $\sqrt{3}\,R$.

$$c : R = \sqrt{3} : 1 \quad\text{and}\quad c = R\sqrt{3}.$$

The side C of an equilateral triangle circumscribed about a circle is equal to double the side of an equilateral triangle inscribed in the same circle.

$$C = 2\,c = \sqrt{3}\,R.$$

The side C′ of a regular hexagon circumscribed about a circle is equal to one-third the side of a circumscribed equilateral triangle about the same circle.

$$C' = \frac{2\sqrt{3}\,R}{3} = \frac{2}{3}R\sqrt{3}.$$

The side of a regular decagon inscribed in a circle is equal to the greater segment of a radius divided in extreme and mean ratio (632, 692).

The side of a regular inscribed pentadecagon is equal to the chord which subtends an arc, which is equal to the difference of the arcs subtended by the sides of a regular inscribed hexagon and decagon.

The difference between the arcs subtended by the sides of a regular inscribed pentagon and hexagon, is subtended by *the side of a regular inscribed polygon of thirty sides.* (See Problems.)

Sides and Apothem of Regular Polygons Inscribed in a Circle of Radius R

	SIDES.	APOTHEMS.
Equilateral triangle	$R\sqrt{3}$	$\frac{1}{2}R$
Square	$R\sqrt{2}$	$\frac{1}{2}R\sqrt{2}$
Pentagon	$\frac{1}{2}R\sqrt{10-2\sqrt{5}}$	$\frac{1}{4}R(1+\sqrt{5})$
Hexagon	R	$\frac{1}{2}R\sqrt{3}$
Octagon.	$R\sqrt{2-\sqrt{2}}$	$\frac{1}{2}R\sqrt{2+\sqrt{2}}$
Decagon	$\frac{1}{2}R(\sqrt{5}-1)$	$\frac{1}{4}R\sqrt{10+2\sqrt{5}}$
Dodecagon	$R\sqrt{2-\sqrt{3}}$	$\frac{1}{2}R\sqrt{2+\sqrt{3}}$
	or $\frac{1}{2}R(\sqrt{6}-\sqrt{2})$	or $\frac{1}{4}R(\sqrt{2}+\sqrt{6})$
Pentadecagon.	side $=\frac{1}{4}R\left[\sqrt{10+2\sqrt{5}}-\sqrt{3}(\sqrt{5}-1)\right]$	

Radii and Apothems of Regular Polygons of the Side c

	RADII.	APOTHEMS.
Equilateral triangle . . .	$\frac{1}{3}c\sqrt{3}$	$\frac{1}{6}c\sqrt{3}$
Square	$\frac{1}{2}c\sqrt{2}$	$\frac{1}{2}c$
Pentagon	$\frac{1}{10}c\sqrt{50+10\sqrt{5}}$	$\frac{1}{10}c\sqrt{25+10\sqrt{5}}$
Hexagon	c	$\frac{1}{2}c\sqrt{3}$
Octagon	$\frac{1}{2}c\sqrt{4+2\sqrt{2}}$	$\frac{1}{2}c(1+\sqrt{2})$
Decagon	$\frac{1}{2}c(1+\sqrt{5})$	$\frac{1}{2}c\sqrt{5+2\sqrt{5}}$
Dodecagon	$c\sqrt{2+\sqrt{3}}$	$\frac{1}{2}c(2+\sqrt{3})$
	or $\frac{1}{2}c(\sqrt{2}+\sqrt{6})$	

Areas of Regular Polygons

	Inscribed in a Circle of Radius R.	Of side c.
Equilateral triangle	$\frac{3}{4} R^2 \sqrt{3}$	$\frac{1}{4} c^2 \sqrt{3}$
Square	$2 R^2$	c^2
Pentagon	$\frac{5}{8} R^2 \sqrt{10 + 2\sqrt{5}}$	$\frac{1}{4} c^2 \sqrt{25 + 10\sqrt{5}}$
Hexagon	$\frac{3}{2} R^2 \sqrt{3}$	$\frac{3}{2} c^2 \sqrt{3}$
Octagon	$2 R^2 \sqrt{2}$	$2 c^2 (1 + \sqrt{2})$
Decagon	$\frac{5}{4} R^2 \sqrt{10 - 2\sqrt{5}}$	$\frac{5}{2} c^2 \sqrt{5 + 2\sqrt{5}}$
Dodecagon	$3 R^2$	$3 c^2 (2 + \sqrt{3})$

$\sqrt{2} = 1.4142135623\ldots$ $\log 2 = 0.3010300$

$\sqrt{3} = 1.7320508075\ldots$ $\log 3 = 0.4771213$

$\sqrt{5} = 2.2360679774\ldots$ $\log 5 = 0.6989700$

Table 1. The values of the radius, the apothems, and the area of a regular polygon, whose side is taken as unity.

Table 2. The values of the side of a regular polygon, according as the radius, the apothem, or the area of the polygon are taken as unity.

Number of Sides of the Polygon.	First. The Side $C=1$.			Second. Value of the Side C for		
	Radius	Apothem	Surface	Radius = 1	Apothem = 1	Surface = 1
3	0.577350	0.288675	0.433013	1.732050	3.464101	1.519671
4	0.707107	0.500000	1.000000	1.414214	2.000000	1.000000
5	0.850651	0.688191	1.720477	1.175570	1.453085	0.762387
6	1.000000	0.866025	2.598076	1.000000	1.154701	0.620403
7	1.152382	1.038261	3.633912	0.867767	0.963149	0.524581
8	1.306563	1.207107	3.828428	0.765367	0.828427	0.455090
9	1.461902	1.373739	6.181823	0.684040	0.727940	0.402200
10	1.618034	1.538842	7.694207	0.618034	0.649839	0.390511
11	1.774732	1.702844	9.365640	0.563465	0.587253	0.326762
12	1.931852	1.866025	11.196150	0.517638	0.535898	0.298858
15	2.404867	2.352315	17.642360	0.415823	0.425113	0.238079
18	2.879385	2.835641	25.520770	0.347296	0.352654	0.197949
20	3.196227	3.156876	31.568760	0.312869	0.316769	0.177980

For the same number of sides, the sides, the apothems, and the radii vary in the same ratio, and the areas vary as the squares of these lengths (744).

Example. Construct a prismatic reservoir which is to contain 36.75 cubic feet, to be 3 feet deep, and its base is to be a regular octagon.

The area of the base $\frac{36.75}{3} = 12.25$ square feet.

Then from the table (2d)

$$c^2 : \overline{0.45509}^2 = 12.25 : 1;$$

and

$$c = 0.45509\ \sqrt{12.25} = 0.45509 \times 3.5 = 1.592815 \text{ feet.}$$

From the table (1st)

$$R : 1.306563 = 1.592815 : 1;$$

and $$R = 1.306563 \times 1.592815 = 2.081 \text{ feet.}$$

Therefore, describe a circle of 2.081 feet radius and lay off the chord 1.592815 feet, eight times, which will give the regular octagon that is to serve as base to the reservoir.

746. Having a regular inscribed polygon, to inscribe a regular polygon of twice the number of sides, join the vertices of the first to the middles of the arcs subtended by the sides of the first.

Having a regular inscribed polygon of an even number of sides, to inscribe a regular polygon of half that number of sides, draw lines connecting every other vertex of the given polygon.

Having a regular circumscribed polygon, to circumscribe a regular polygon of twice the number of sides, draw tangents to the circle at the middle points of the arcs intercepted by the sides of the given polygon.

Having a regular circumscribed polygon of an even number of sides greater than four, to circumscribe a regular polygon of half the number of sides, erase every other side of the given polygon and prolong the remaining sides until they meet.

747. Let p and P be the perimeter of two regular similar polygons, one inscribed in and the other circumscribed about the same circle, designating by p' and P' the perimeters of regular inscribed and circumscribed polygons of double the number of sides, we have:

$$P' = \frac{2\,Pp}{P + p}, \text{ and } p' = \sqrt{P'p} = \sqrt{\frac{2\,Pp^2}{P + p}}.$$

748. The circumference is greater than the perimeter of any inscribed polygon and less than that of any circumscribed poly-

gon. It is the limit which they approach as their sides become smaller and smaller, that is as the number of sides becomes greater (601, 649).

749. Two circles are always similar. Their circumferences C and c are to each other as their radii R and r, or as their diameter D and d, and their areas are to each other as the squares of their linear dimensions:

$$\frac{C}{c}=\frac{R}{r}=\frac{D}{d}, \text{ and } \frac{S}{s}=\frac{R^2}{r^2}=\frac{D^2}{d^2}, \qquad (744)$$

750. In two different circles *arcs, sectors, and segments are said to be similar* when they correspond to the same angles at the center (667).

Similar arcs are to each other as their radii, their diameters, and the chords which subtend them.

Similar sectors and segments are to each other as the squares of their radii, diameters, arcs, and chords (749).

751. *The ratio of a circumference C to its diameter D* is a constant uncommensurable number, which is commonly represented by π.

$$\pi=\frac{C}{D}=3.141\ 592\ 653\ 589\ 793\ 238\ 462\ 643\ldots$$

In practice generally not more than four places are expressed thus:

$$\pi = 3.1416.$$

Tables of the nearest values to the seventh decimal place of the First 9 *multiples of* π, π^2, π^3, $\sqrt{\pi}$, $\sqrt[3]{\pi}$, $\frac{1}{\pi}$, $\frac{1}{\pi^2}$, $\frac{1}{\pi^3}$, $\sqrt{\frac{1}{\pi}}$ and $\sqrt[3]{\frac{1}{\pi}}$, *which are often met with in formulas.*

	π		π^2		π^3		$\sqrt{\pi}$		$\sqrt[3]{\pi}$
1	3.1415927	1	9.8696044	1	31.0062767	1	1.7724539	1	1.4645919
2	6.2831853	2	19.7392088	2	62.0125534	2	3.5449077	2	2.9291838
3	9.4247780	3	29.6088132	3	93.0188300	3	5.3173616	3	4.3937756
4	12.5663706	4	39.4784176	4	124.0251067	4	7.0898154	4	5.8583675
5	15 7079633	5	49.3480220	5	155.0313834	5	8.8622693	5	7.3229594
6	18.8495559	6	59.2176264	6	186.0376601	6	10.6347231	6	8.7875513
7	21.9911486	7	69.0872308	7	217.0439368	7	12.4071770	7	10.2521432
8	25.1327412	8	78.9568352	8	248.0502134	8	14.1796308	8	11.7167351
9	28.2743339	9	88.8264396	9	279.0564901	9	15.9520847	9	13.1813269

	$\frac{1}{\pi}$		$\frac{1}{\pi^2}$		$\frac{1}{\pi^3}$		$\sqrt{\frac{1}{\pi}}$		$\sqrt[3]{\frac{1}{\pi}}$
1	0.3183099	1	0.1013210	1	0.0322515	1	0.5641896	1	0.6827841
2	0.6366198	2	0.2026420	2	0.0645030	2	1.1283792	2	1.3655681
3	0.9549297	3	0.3039631	3	0.0967545	3	1.6925688	3	2.0483522
4	1.2732395	4	0.4052841	4	0.1290060	4	2.2567583	4	2.7311363
5	1.5915494	5	0.5066051	5	0.1612575	5	2.8209479	5	3.4139203
6	1.9098593	6	0.6079261	6	0.1935090	6	3.3851375	6	4.0967044
7	2.2281692	7	0.7092471	7	0.2257605	7	3.9493271	7	4.7794885
8	2.5464791	8	0.8105682	8	0.2580120	8	4.5135167	8	5.4622725
9	2.8647890	9	0.9118892	9	0.2902635	9	5.0777063	9	6.1450566

Log $\pi = 0.4971499$, $\log \pi^2 = 0.9942997$, $\log \pi^3 = 1.4914496$, $\log \sqrt{\pi} = 0.2485749$,

Log $\sqrt[3]{\pi} = 0.1657166$, $\log \frac{1}{\pi} = \bar{1}.5028501$, $\log \frac{1}{\pi^2} = \bar{1}.0057003$, $\log \frac{1}{\pi^3} = \bar{2}.5085504$,

$$\log \sqrt{\frac{1}{\pi}} = \bar{1}.7514251, \quad \log \sqrt[3]{\frac{1}{\pi}} = \bar{1}.8342834.$$

752. The expression of the length C of the circumference as a function of its diameter D or its radius R. Having (751)

$$\pi = \frac{C}{D},$$

then
$$C = \pi D \text{ or } C = 2\,\pi R,$$

and
$$D = \frac{C}{\pi} \text{ and } R = \frac{C}{2\pi}.$$

According as $D = 1$ or $R = 1$, we have:

$$C = \pi \text{ or } C = 2\,\pi.$$

753. The *area S of a circle* is equal to the product of its circumference C and half its radius R, which is equivalent to area of a triangle whose base is equal to the circumference, and whose altitude is equal to the radius (718, 743).

$$S = \pi D \frac{D}{4} = \frac{\pi D^2}{4} \text{ or } S = 2\,\pi R \frac{R}{2} = \pi R^2;$$

then
$$D = 2\sqrt{\frac{S}{\pi}} \text{ and } R = \sqrt{\frac{S}{\pi}}. \qquad (a)$$

According as $D = 1$ or $R = 1$, we have:

$$S = \frac{\pi}{4}, \text{ or } S = \pi.$$

Substituting for R in (a) its value in terms of the circumference C (752), we have:

$$C^2 = 4\,\pi S.$$

754. PROBLEMS.

1st. What is the length of the circumference of a circle whose radius is 13 inches?

From (716) $C = 2\,\pi R = 2 \cdot 3.1416 \cdot 13 = 81.68$ inches.

2d. What is the area of a circle whose radius is 13 inches?

Having calculated the circumference, it is only necessary to multiply it by one-half the radius. Otherwise, according to (753) we have:

$$S = \pi R^2 = 3.1416 \cdot 13 \cdot 13 = 530.9 \text{ square inches.}$$

3d. What is the radius of a circle whose area is equal to 530.9 square inches?

From (751, 753)

$$R = \sqrt{\frac{S}{\pi}} = \sqrt{\frac{1}{\pi}} \times \sqrt{S} = 0.5642\sqrt{530.9} = 13.0 \text{ inches.}$$

755. *The solution of the preceding problems using a table,* which contains, to two decimal figures, the lengths of the circumferences and the areas of circles of whole diameters from 1 to 1000.

1st. *The radius R or the diameter D of a circle being given, to calculate the length of the circumference and the area of the surface.*

Converting the given diameter into units of an order such that the whole part is the greatest possible number less than 1000; if the decimal part of this number is zero, the length of the circumference may be read directly from the table in units of the order given and correct to within one hundredth of these units, and the area may be read directly in units of surface correct to within one hundredth of the chosen units.

EXAMPLE 1. For $D = 2.5$ inches, multiply by 10, which gives 25, then the table gives:

For $\qquad D = 25, \;\; C = 78.5,$ and $S = 490;$

but since the circumferences are to each other as the linear dimensions (749)

$$\frac{C}{C'} = \frac{D}{D'} = \frac{2.5}{25} = \frac{1}{10}. \qquad C = 78.5\frac{1}{10} = 7.85,$$

and the areas are to each other as the squares of any linear dimensions (749)

$$\frac{S}{S'} = \frac{2.5^2}{25^2} = \frac{1}{100}. \qquad S = 490\frac{1}{100} = 4.9 \text{ square inches.}$$

Example 2. For $D = 2520$ feet, divide by 10, and the table gives for $D' = 252$

$$C' = 791.68 \text{ feet and } S' = 49875.92 \text{ square feet,}$$

and since
$$\frac{C}{C'} = \frac{2520}{252} = 10,$$

we have,
$$C = 10,791.68 = 7916.8 \text{ feet}$$

and since
$$\frac{S}{S'} = \frac{(2520)^2}{(252)^2} = 100.$$

$$S = 49,875.92 \cdot 100 = 4,987,592 \text{ square feet.}$$

Example 3. For $d = 0.0252$ inches, multiply by 10,000, then from the table

$$C' = 791.68 \text{ inches} \quad \text{and} \quad S = 49,875.92 \text{ square inches;}$$

but $\dfrac{C}{C'} = \dfrac{0.0252}{252} = \dfrac{1}{10,000}$ or $C = \dfrac{791.68}{10,000} = 0.079168$ inches,

and $\dfrac{S}{S'} = \dfrac{0.0252^2}{252^2} = \dfrac{1}{10,000,000}$ or $S = \dfrac{49,875.92}{10,000,000}$

$$= 0.0004987562 \text{ square inch.}$$

2d. *The circumference C or the area S of a circle being given, to find the diameter D or the radius R.*

Example 1. Let $C = 7.9303$ feet, then it should be expressed in units such that the number be the greatest possible number less than the greatest number in the table. Multiplying by 100 we have $C = 793.03$, and looking in the table we find the nearest circumference is 791.68, which corresponds to a diameter of 252 and may be taken as the required diameter as the error is negligible. Thus:

$$D = \frac{252}{100} = 2.52 \text{ feet.}$$

If greater accuracy is desired, it is better to substitute in the formulas (752), but it is possible to obtain the same result by interpolation in the tables.

Example 2. For $S = 5.0046$ square feet, multiply by 10,000, then we find the nearest surface in the table is 49,875.92, and the corresponding diameter is 252.

$$\frac{D}{D'} = \frac{\sqrt{4.987592}}{\sqrt{49,875.92}} = \frac{1}{\sqrt{10,000}} = \frac{1}{100};$$

$$D = \frac{252}{100} = 2.52 \text{ ft.}$$

756. Circumferences being to each other as any homologous linear dimensions, and areas as the squares of those dimensions (**749**), it follows that having the dimensions of one circle and its area, the corresponding dimensions of another circle may be found if one dimension is known. Thus, let C and S represent the circumference and area of a circle of the diameter D, what are the same dimensions of a circle whose diameter is d?

$$c = C\frac{d}{D} \quad \text{or} \quad s = S\frac{d^2}{D}.$$

Thus, according as

$$d = 2\,D,\ 3\,D,\ 4\,D\ldots$$

or

$$d = \frac{D}{2},\ \frac{D}{3},\ \frac{D}{4}\ldots,$$

we have respectively:

$$c = 2\,C,\ 3\,C,\ 4\,C\ldots \quad \text{or} \quad c = \frac{1}{2}C,\ \frac{1}{3}C,\ \frac{1}{4}C\ldots,$$

and $$s = 4\,S,\ 9\,S,\ 16\,S\ldots \quad \text{or} \quad s = \frac{1}{4}S,\ \frac{1}{9}S,\ \frac{1}{16}S.\ldots$$

757. The surface of a circle being equal to the product of the circumference C and half the radius R or $\frac{1}{4}$ the diameter D, at times the calculations may be shortened when some one of these has already been calculated. Thus:

$$S = C \times \frac{D}{4}; \quad C = \frac{4\,s}{D}.$$

758. *The length of an arc of a circle* is equal to the circumference of the circle multiplied by the ratio of the number of degrees in the arc to 360°. Thus, to find the length of an arc of 25°, 8′ of a circle whose radius is 13 inches.

$C = 81.68$ (**754, 755**), and letting the length of the arc be A we have:

$$A = 81.68\,\frac{25.60 + 8}{36.060} = 5.70 \text{ inches.}$$

The nearest lengths of arcs containing 12 *decimal places* (176) *in circles of unit radius expressed: First, in degrees, minutes, and seconds; Second, in grades.*

Arcs	Lengths	Arcs	Lengths	Arcs	Lengths	Arcs	Lengths
						gr.	
1°	0.017453292520	1′	0.000290888209	1″	0.000004848137	1	0.015707963268
2	0.034906585040	2	0.000581776417	2	0.000009696274	2	0.031415926536
3	0.052359877560	3	0.000872664626	3	0.000014544410	3	0.047123889804
4	0.069813170080	4	0.001163552835	4	0.000019392547	4	0.062831853072
5	0.087266462600	5	0.001454441043	5	0.000024240684	5	0.078539816340
6	0.104719755120	6	0.001745329252	6	0.000029088821	6	0.094247779608
7	0.122173047640	7	0.002036217461	7	0.000033936958	7	0.109955742876
8	0.139626340160	8	0.002327105669	8	0.000038785094	8	0.125663706144
9	0.157079632679	9	0.002617993878	9	0.000043633231	9	0.141371669412

Arc	Length	Arc	Length
1°	0.01745329251994329576 92369	1″	0.00000484813681109535 99359
1′	0.00029088820866572159 61539	1gr.	0.01570796326794896619 23133

Ex. 1. Determine the length of an arc of 126° 45′ 9″, whose radius is 10.4 feet.

Taking the radius of 1, the table gives:

For	100°	1.7453292520
	20°	0.3490658504
	6°	0.1047197551
	40′	0.0116355283
	5′	0.0014544410
	9″	0.0000436332
Total for	126° 45′ 9″	2.2122484600

The length of the arc in feet is

$$10.4 \times 2.2122484600 = 23.007384 \text{ feet.}$$

Ex. 2. Determine the length of an arc of 183.4857 grades whose radius is 600 feet.

Taking the radius as 1, the table gives:

For	100 gr.	1.5707963268
	80	1.2566370614
	3	0.0471238898
	0.4	0.0062831853

	0.08	0.0012566371
	0.005	0.0000785398
	0.0007	0.0000109956
Total for	183.4857 gr.	2.8821866358

The length of the arc in feet is

$$600 \times 2.8821866358 = 1729.311981.$$

759. The table in the preceding article may be used for *reducing angles or arcs expressed in degrees, minutes, and seconds, to grades and vice versa.* To do this, find the length which corresponds to a certain arc in degrees and then find that same length in the other part of the table, which will give the number of grades and vice versa.

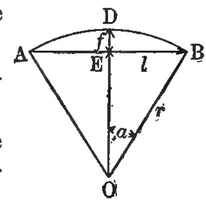


Fig 102

760. *The area of sector* is equal to the product of its base and half its radius (668).

This is equivalent to the area of a triangle which has a base equal to the base of the sector and an altitude equal to the radius (718).

The area of a sector is also equal to the surface of a circle of the same radius multiplied by the ratio of the angle of the sector in degrees, minutes, and seconds to 360°. Thus the radius of a sector being 13 inches and the angle at the center being 25° 8′, the length of the base is calculated to be 5.7 inches (758), and we have:

$$s = 5.7\frac{13}{2} = 37.1 \text{ square inches.}$$

The area of a circle of 13 in. radius being 530.9 (754), we have also:

$$s = 530.9\frac{25 \times 60 + 8}{360 \times 60} = 37.1 \text{ square inches.}$$

761. *The area of a circular segment* is equal to the difference between the areas of the sector and triangle OAB (Fig. 102).

In practice the *span* AB and *rise* DE of an arch are often given and it is required to find the radius OB; the length of the arc ADB; and the area of the segment.

Designating the radius OB by r, half the span BE by l, the rise by f and half the angle at the center by a, the right triangle OBE gives (694):

$$r^2 = l^2 + (r - f)^2, \text{ and } r = \frac{l^2 + f^2}{2f},$$

and also $\sin \alpha = \frac{l}{r}\cdot$ (See Trigonometry.)

Having r, l, and a, we have all that is necessary to calculate the length of the arc ABD, the area, the area of the sector $OADB$, the area of the triangle OAB, and therefore, the area of the segment ADB. The following table contains these various quantities.

Table of the Lengths of Arcs and the Areas of Segments, the Rise Being Taken as Unity

Chords	Arcs	Segments	Chords	Arcs	Segments	Chords	Arcs	Segments
2.00	3.1416	1.5708	4.80	5.337	3.3085	8.50	8.810	5.7289
2.01	3.146	1.5764	4.90	5.427	3.3730	8.60	8.903	5.7947
2.02	3.152	1.5821	5.00	5.517	3.4377	8.70	9.003	5.8606
2.03	3.158	1.5879	5.10	5.608	3.5024	8.80	9.100	5.9266
2.04	3.164	1.5936	5.20	5.698	3.5672	8.90	9.196	5.9927
2.05	3.170	1.5993	5.30	5.789	3.6320	9.00	9.293	6.0587
2.06	3.176	1.6051	5.40	5.881	3.6969	9.10	9.390	6.1248
2.07	3.182	1.6108	5.50	5.973	3.7618	9.20	9.487	6.1909
2.08	3.187	1.6166	5.60	6.065	3.8269	9.30	9.584	6.2570
2.09	3.193	1.6224	5.70	6.157	3.8919	9.40	9.681	6.3230
2.10	3.199	1.6282	5.80	6.249	3.9571	9.50	9.778	6.3890
2.20	3.261	1.6863	5.90	6.342	4.0222	9.60	9.875	6.4551
2.30	3.324	1.7449	6.00	6.435	4.0874	9.70	9.972	6.5212
2.40	3.390	1.8041	6.10	6.528	4.1527	9.80	10.069	6.5873
2.50	3.458	1.8637	6.20	6.621	4.2182	9.90	10.167	6.6533
2.60	3.527	1.9238	6.30	6.715	4.2835	10.00	10.264	6.7194
2.70	3.599	1.9843	6.40	6.809	4.3489	10.10	10.362	6.7854
2.80	3.672	2.0452	6.50	6.903	4.4142	10.20	10.459	6.8515
2.90	3.746	2.1064	6.60	6.997	4.4797	10.30	10.557	6.9176
3.00	3.822	2.1679	6.70	7.091	4.5452	10.40	10.654	6.9837
3.10	3.899	2.2297	6.80	7.185	4.6107	10.50	10.752	7.0498
3.20	3.977	2.2917	6.90	7.280	4.6763	10.60	10.849	7.1160
3.30	4.056	2.3540	7.00	7.375	4.7420	10.70	10.947	7.1822
3.40	4.137	2.4165	7.10	7.470	4.8076	10.80	11.045	7.2484
3.50	4.218	2.4793	7.20	7.565	4.8732	10.90	11.143	7.3146
3.60	4.300	2.5422	7.30	7.660	4.9389	11.00	11.240	7.3809
3.70	4.383	2.6053	7.40	7.755	5.0047	11.10	11.338	7.4471
3.80	4.467	2.6686	7.50	7.850	5.0705	11.20	11.436	7.5133
3.90	4.551	2.7320	7.60	7.946	5.1363	11.30	11.534	7.5795
4.00	4.636	2.7956	7.70	8.042	5.2020	11.40	11.632	7.6457
4.10	4.722	2.8593	7.80	8.137	5.2678	11.50	11.730	7.7119
4.20	4.808	2.9231	7.90	8.233	5.3336	11.60	11.828	7.7781
4.30	4.895	2.9871	8.00	8.329	5.3994	11.70	11.926	7.8454
4.40	4.983	3.0512	8.10	8.425	5.4653	11.80	12.024	7.9117
4.50	5.071	3.1154	8.20	8.521	5.5312	11.90	12.122	7.9770
4.60	5.159	3.1796	8.30	8.617	5.5971	12.00	12.220	8.0433
4.70	5.248	3.2440	8.40	8.714	5.6630	.....	.....	.....

Example. The rise of a circular arch is 2.6 feet, the span 20 feet. What is the length of the arc and the area of the segment formed by the arch?

Taking the rise as 1, the span becomes $\frac{20}{2.6} = 7.692.$

Looking in the table for the nearest chord to 7.692, we find 7.70 and the corresponding length of arc is 8.042 feet, and area is 5.2020 square feet.

For an arch having 2.6 feet rise we have:

$$8.042 \cdot 2.6 = 20.909 \text{ feet}$$

and

$$5.2020 \cdot (2.6)^2 = 35.1655 \text{ square feet.}$$

These results are ordinarily sufficiently accurate, but if a higher degree of approximation is desired, recourse may be had to interpolation (**404**).

In the above example the arc would be:

$$8.042 - (8.042 - 7.946)\,\frac{7.70 - 7.692}{7.70 - 7.60} = 8.034,$$

and the area

$$5.2020 - (5.2020 - 5.1316)\,\frac{7.70 - 7.692}{7.70 - 1.60} = 5.1967,$$

which, when reduced to feet and square feet, become:

$$8.034 \cdot 2.6 = 20.888 \text{ feet}$$

and

$$5.1967 \cdot (2.6)^2 = 35.1297 \text{ square feet.}$$

SOLID GEOMETRY

BOOK I

PLANES (Arts. 602-605)

762. *A line AB is perpendicular to a plane MN,* when any line drawn through the foot of the line AB in the plane MN is perpendicular to the line AB. The line is oblique to the plane when it is not perpendicular to all the lines drawn through its foot and contained in the plane. If AB is perpendicular to two lines CD and EF, which pass through its foot and lie in the plane, it is perpendicular to the plane.

All the perpendiculars CD, EF, . . . drawn through a point B, in a line, lie in the same plane, and that line is perpendicular to the plane.

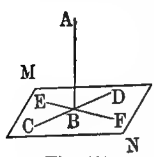


Fig. 103

At a point B in a plane, one, and only one, perpendicular to that plane can be erected.

763. The foot of a perpendicular, drawn from a point A to a plane, is *the projection of the point upon the plane.*

The line formed by the projections of the points of a line upon a plane is *the projection of the line upon the plane* (715).

764. Through a point B taken on a line and a point C taken outside the line, one plane MN, and only one, can be drawn perpendicular to the line.

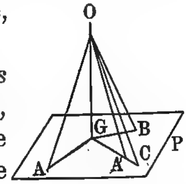


Fig. 104

765. *When a perpendicular and several obliques are drawn from an exterior point to a plane: First,* The perpendicular OG is shorter than any oblique OA; *Second,* Two obliques OA, OB, which are equidistant, $GA = GB$, from the foot of the perpendicular are equal, and conversely; *Third,* Of two obliques OA, OC, which are not equidistant from the foot of the perpendicular, that one OC which is farther is longer, and conversely (620).

The perpendicular OG being the shortest distance from the point O, to the plane, it is *the distance of the point O from the plane.*

The locus of the feet of the equal obliques drawn from the same point O, is a circle whose center is at the foot G of the perpendicular.

From this it follows that in order to draw a perpendicular from a given point to a plane, locate three points in the plane equidistant from the given point, then, drawing a circle through these points, the center of this circle coincides with the foot of the desired perpendicular.

766. *The angle that a line OA makes with a plane* is the smallest angle which is formed by that line and any line drawn through its foot and in the plane. This angle is the one OAG, formed by the line OA and its projection AG on the plane (611, 663).

767. A plane perpendicular to a vertical is *horizontal* (615). The horizontal as well as the vertical varies for each point on the globe.

A plane oblique to the vertical is an *inclined plane.*

768. *The line which has the greatest slope in a plane* is that line in the plane which makes the greatest angle with the horizontal plane and consequently the smallest with the vertical.

Drawing in a plane, first a horizontal then a perpendicular to this horizontal, the perpendicular is the line with the greatest slope of any in the plane.

769. A perpendicular to a circle passing through its center is the geometrical locus of all the points equidistant from the circumference (609, 665).

A plane perpendicular to a line and passing through its middle point is the locus of all points equidistant from the extremities of the line (621).

770. If from the foot B of a perpendicular AB to a plane MN, a straight line is drawn at right angles to any line DE in the plane, the line AC, drawn from its intersection with the line in the plane to any point of the perpendicular, is perpendicular to the line in the plane. (This is called the theorem of the three perpendiculars.)

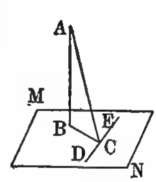


Fig. 105

771. When one straight line AB is perpendicular to a plane, all lines $A'B'$ which are parallel to this line are also perpendicular to the plane.

Conversely, two straight lines perpendicular to the same plane are parallel.

Corollary. Two straight lines parallel to a third straight line are parallel to each other (628).

772. Through any point in space one parallel, and only one, can be drawn to a given straight line (623).

773. A line is parallel to a plane if it can not meet the plane, however far produced (602, 623).

Two planes are parallel if they can not meet, however far they are produced.

774. Every straight line AB, parallel to a certain straight line $A'B'$ in a plane, is parallel to that plane.

Corollary 1. Through a given straight line a plane can be passed parallel to any other given straight line in space.

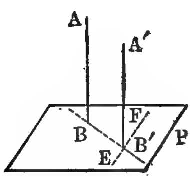


Fig. 106

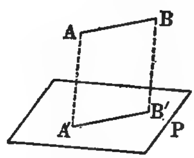
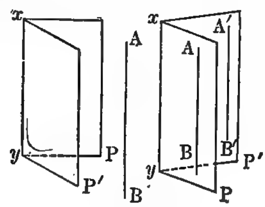


Fig. 107



Figs. 108–9

Corollary 2. Through a given point a plane can be passed parallel to any two given straight lines in space.

775. If a given straight line AB is parallel to a given plane, the intersection $A'B'$ of the given plane with any plane passed through the given line, is parallel to that line.

Corollary 1. If a given straight line AB and a plane are parallel, a parallel $A'B'$ to the given line drawn through any point A' in the plane, lies in the plane.

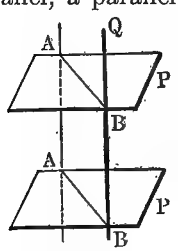


Fig. 110

Corollary 2. Any straight line AB, parallel to two planes P, P' which intersect, is parallel to their intersection xy (Fig. 108).

Corollary 3. The intersection xy of two planes which contain two parallel lines AB and $A'B'$, is parallel to those lines (Fig. 109).

776. Two planes perpendicular to the same straight line are parallel.

777. The intersections AB, $A'B'$, of two parallel planes P, P', by a third plane Q, are parallel lines (Fig. 110).

778. If a straight line AA' is perpendicular to a plane P, it is perpendicular to any plane P' which is parallel to the first.

COROLLARY 1. Two planes parallel to a third plane are parallel to each other.

COROLLARY 2. Through a point taken outside of a given plane one, and only one, plane can be drawn parallel to the given plane.

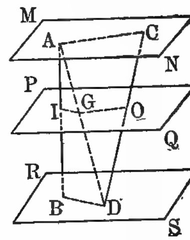


Fig. 111

779. Parallel lines included between parallel planes or between a line and plane which are parallel, are equal.

COROLLARY. Two parallel planes or a line and a plane which are parallel, are everywhere equally distant.

780. If two straight lines AB, CD, are intersected by three parallel planes MN, PQ, RS, their corresponding segments are proportional:

$$\frac{AI}{IB} = \frac{CO}{OD}. \tag{693}$$

781. If two intersecting lines are each parallel to a given plane, the plane of these lines is also parallel to that plane.

782. If two angles not in the same plane have their sides parallel and lie in the same direction: *First,* Their planes are parallel; *Second,* The angles are equal (630).

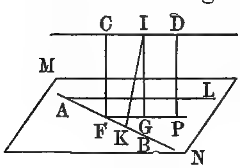


Fig. 112

783. Two straight lines AB, CD, not in the same plane being given: *First,* A perpendicular CF can be drawn common to both these lines; *Second,* Only one can be drawn; *Third,* This perpendicular is *the shortest distance between the two lines,* that is, it is the shortest line that can be drawn from any point in the first to any point in the second; thus, $CF < IK$.

784. The opening between two intersecting planes M, N, is called a *dihedral angle.* The planes M, N, are the *faces* of the angle, and the intersection AB is the *edge* (611).

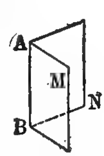


Fig. 113

Thus the *magnitude* of a dihedral angle is independent of that of its faces, and a clear idea of the magnitude may be obtained by supposing the planes at first to coincide and then to turn one about the edge AB, as one opens a book: the dihedral angle, at first zero, increases as the faces are separated. Thus, a dihedral angle is generated by a plane rotating about a straight line drawn in the plane. In the movement of the plane each of the

points describes an arc of a circle the center of which is on the edge of the dihedral angle.

A dihedral angle is designated by the letters AB of the edge, or to avoid confusion, when there are several dihedral angles which have the same edge, by the four letters $MABN$ of the faces, placing the edge in the middle.

785. *Two dihedral angles coincide* when having the same edge their faces coincide (596). *Two dihedral angles are equal* when they can be made to coincide.

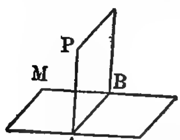


Fig. 114

786. *Two dihedral angles PABM and PABN are adjacent* if they have the same edge AB and a common face PAB between them (576).

787. A plane P is perpendicular to another plane MN if it forms with this second plane a right dihedral angle (614).

A plane PQ is oblique to another MN (Fig. 115) when the first forms two unequal adjacent dihedral angles $PABM$, $PABN$, with the second.

788. When a plane meets another plane and makes adjacent dihedral angles equal, each of these angles is called a *right dihedral angle* (Fig. 114).

All right dihedral angles are equal.

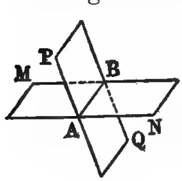


Fig. 115

All dihedral angles $PABM$ smaller than a right dihedral angle is are *acute dihedral angles*, and all dihedral angles $PABN$ larger than a right dihedral angle is are *obtuse dihedral angles* (616).

789. When two planes cut each other, the angles formed which are not adjacent are *vertical dihedral angles*. Such are:

$$PABM \text{ and } QABN.$$

If two planes intersect, their vertical dihedral angles are equal (613).

790. The sum of the two adjacent dihedral angles, formed by the intersection of two planes, is equal to two right dihedral angles (618).

The sum of all the consecutive dihedral angles formed on the same side of a plane MN about a given edge AB is equal to two right dihedral angles, and the sum of all the consecutive dihedral angles about the same edge is equal to four right dihedral angles.

791. Two dihedral angles are *complementary* and *supplementary* under the same conditions as plane angles are complementary and supplementary (**617**).

It is the same with *alternate-interior* or *alternate-exterior* angles (**624**, **799**).

792. *The plane angle of a dihedral angle* AB (Fig. 116) is the angle CBD formed by the perpendiculars BD and BC, erected in each of the faces at the same point B in the edge.

The plane angles of two equal dihedral angles are equal, and conversely.

According as a dihedral angle is right, acute, or obtuse (72), its plane angle is right, acute, or obtuse, and conversely.

793. Two dihedral angles AB, $A'B'$, are to each other as their plane angles CBD, $C'B'D'$, and conversely (**709**).

794. When a straight line AB is perpendicular to a plane P (Fig. 117), every plane Q passed through the line is perpendicular

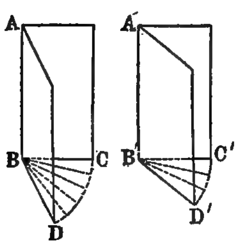


Fig. 116

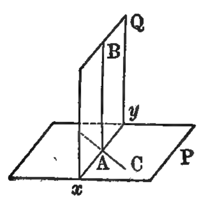


Fig. 117

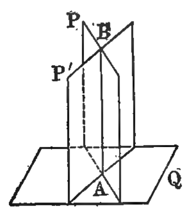


Fig. 118

to the first plane (**787**). All planes parallel to AB are also perpendicular to the plane P.

795. Through a straight line AC not perpendicular to a plane MN (Fig. 105), one plane ACB, and only one, which is perpendicular to the first plane, can be drawn. The intersection BC of the perpendicular plane is the projection of the line AC on the plane (**763**).

796. If two planes P, Q, are perpendicular to each other, a straight line AB drawn in one of them perpendicular to their intersection xy is perpendicular to the other (**762**).

797. If two planes P, Q, are perpendicular to each other, every straight line AB perpendicular to one of the planes is parallel to the other or wholly contained in it.

798. Any plane Q which is perpendicular to two others P, P',

which intersect, is perpendicular to their intersection AB (Fig. 118).

799. When two parallel planes P, P', are cut by a third plane Q (Fig. 110), we have the same relations for the dihedral angles as those given for plane angles in article (**625**). The converse statements are also true when the intersections of the first two planes by the third are parallel (**791**).

When the transverse plane is perpendicular to one of the parallel planes, it is also perpendicular to the other.

800. Two dihedral angles whose faces are parallel each to each are equal or supplementary (**782**).

801. If two lines are drawn through a given point in space perpendicular to the faces of a dihedral angle, the angle of the perpendiculars and the plane angle of the dihedral angle are equal or supplementary (**792**).

802. Every point in the plane which bisects a dihedral angle is equidistant from the faces of the angle (**609**).

BOOK II

POLYHEDRAL ANGLES — POLYHEDRONS — SYMMETRY

803. A *polyhedral angle* is the opening of three or more planes which meet at a common point. The common point S is called the *vertex* of the polyhedral angle.

The successive intersections SA, SB, . . . of the planes which form the polyhedral angle are the *edges;* the portion of the indefinite plane ASB included between the edges is a *face;* the angle ASB formed by two consecutive edges is a *face angle;* and each angle formed by the consecutive faces is a *dihedral angle.*

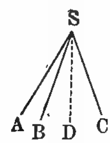


Fig. 119

A polyhedral angle is designated by the letter S at its vertex, or, to avoid confusion when there are several polyhedral angles which have the same vertex, by the letters $SABCD$ of its edges commencing with the vertex.

Remark. We will consider only the *convex polyhedral angles,* that is, those in which any section made by a plane cutting all its faces is a convex polygon (**648**).

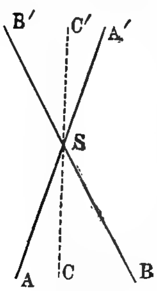


Fig. 120

804. A polyhedral angle is called a *trihedral, tetrahedral, pentahedral,* etc., according as it has three, four, five, etc., faces (**632**).

805. A trihedral angle is bi-rectangular or tri-rectangular according as it has two or three right-dihedral angles.

806. Two polyhedral angles coincide when they have the same vertex and their faces coincide (**596**).

Two polyhedral angles which coincide are equal.

807. *Two polyhedral angles $SABC$, $SA'B'C'$, are symmetrical* when one is formed by prolonging the faces of the other through the vertex (**789**).

808. In any trihedral angle:

1st. Any one of the face angles is smaller than the sum and greater than the difference of the two others (**637**).

2d. If two dihedral angles are equal, the opposite face angles are equal, and conversely (**635**).

3d. The smallest dihedral angle is opposite the smallest face angle, and conversely (**638**).

809. *Two trihedral angles S and S' are equal:*

1st. When a dihedral angle and the adjacent face angles of one are equal respectively to a dihedral angle and the adjacent face angles of the other and are situated in the same order:

$$SA = S'A', \quad ASB = A'S'B', \quad ASC = A'S'C';$$

2d. When two dihedral angles and the included face angle of one are equal to two dihedral angles and the included face angle of the other and are situated in the same order:

$$ASB = A'S'B', \quad SA = S'A', \quad SB = S'B';$$

3d. When three face angles of one are equal to the three face angles of the other and are situated in the same order:

$$ASB = A'S'B', \quad BSC = B'S'C', \quad CSA = C'S'A';$$

4th. When the three dihedral angles of one are equal to the three dihedral angles of the other and are situated in the same order:

$$SA = S'A', \quad SB = S'B', \quad SC = S'C'. \qquad \textbf{(654)}$$

810. *Any two polyhearal angles are equal:*

1st. When the dihedral and face angles are equal each to each and placed in the same order;

2d. When their edges are parallel each to each and situated in the same order.

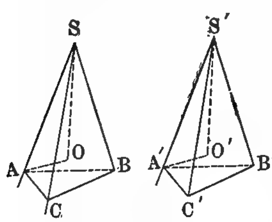


Fig. 121

811. When two trihedral angles have two face angles equal each to each, but the included dihedral angle of the first smaller than that of the second, then the third face angle of the first is smaller than that of the second.

Conversely, if the third face angle is smaller in the first trihedral angle than in the second, the dihedral angle included between the two equal face angles is smaller in the first than in the second (**658**).

812. In any two vertical polyhedral angles the dihedral and

face angles are equal each to each (807), but arranged in reverse order. Therefore, they are not equal, that is, they cannot be made to coincide.

813. The sum of the face angles of any polyhedral angle is less than four right angles.

The sum of the dihedral angles of any trihedral angle is less than six and greater than two right-dihedral angles.

814. Having three face angles such that their sum is less than four right angles and each one of them is less than the sum of the two others, a trihedral angle may be constructed (808).

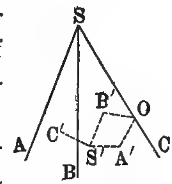


Fig. 122

815. The planes which bisect the three dihedral angles of a trihedral angle intersect in a straight line, which is the geometrical locus of the points included by the angle and equidistant from its faces (609).

816. If from a point S' within a trihedral angle $SABC$ (Fig. 122) perpendiculars $S'A'$, $S'B'$, $S'C'$, are drawn to the respective faces BSC, ASC, ASB, of this trihedral angle, a second trihedral angle $S'A'B'C'$ is formed with its faces $B'S'C'$, $A'S'C'$, $A'S'B'$, perpendicular to the edges SA, SB, SC, of the first. Furthermore, the trihedral angles S and S' are supplementary, that is, the face angles of one are supplementary to the plane angles of the dihedral angles of the other (792). Thus, $\angle A'S'B'$ is the supplement of the plane angle $A'OB'$ of the dihedral angle SC; and $\angle ASB$ is the supplement of the plane angle of the dihedral angle $S'C'$.

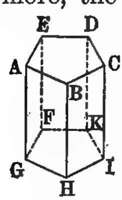


Fig. 123

817. A solid bounded on all sides by polygons is a *polyhedron* (631). These polygons are the *faces* of the polyhedron, the intersections of the faces are the *edges*, and the intersections of the edges are the *vertices* of the polyhedrons.

A straight line joining any two vertices not in the same face is a *diagonal* of a polyhedron.

818. A polyhedron is called respectively a *tetrahedron*, a *pentahedron*, a *hexahedron*, . . . according as it has 4, 5, 6 . . . faces (632).

819. A prism is a polyhedron of which two opposite faces, *called bases*, are parallel, and the other faces, called *lateral faces*, intersect in parallel lines, called *lateral edges*. The *altitude* of

the prism is the distance between the bases (779). In any prism (Fig. 123) the lateral edges AG, BH, CI, . . . are equal (779), and the lateral faces $ABHG$, $BCIH$, . . . are parallelograms (640).

A prism is a *right* or an *oblique* prism, according as its lateral edges are perpendicular or oblique to the planes of the bases (762).

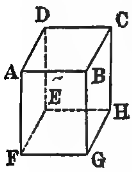


Fig. 124

A prism is *triangular, quadrangular, pentagonal,* . . . according as its bases are triangles, quadrilaterals, pentagons . . . (632).

A *regular prism* is a right prism whose bases are regular polygons (740).

820. The sections of a prism made by parallel planes are equal polygons; thus the bases of a prism are equal, and any section made by a plane parallel to the bases is equal to the bases.

A section of a prism made by a plane perpendicular to the lateral edges is a *right section.*

821. A *truncated prism* is that part of a prism included between one base and a section made by a plane not parallel to the base. This base and the section are called the *bases* of the truncated prism (894).

822. A prism whose bases are parallelograms $EFGH$, $DABC$, (Fig. 124), is a *parallelopiped* (640). Thus, a parallelopiped is a hexahedron made up of six parallelograms, which are equal in pairs.

Any face may be the *base of the parallelopiped,* and the distance between the base and the opposite face is the *altitude.*

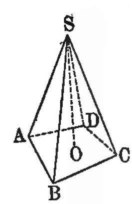


Fig. 125

A *rectangular parallelopiped* is one whose faces are all parallelograms. The three edges ED, EH, EF, which meet in any one vertex E, are perpendicular to each other.

The three dimensions of a rectangular parallelopiped are the two dimensions of its base and its altitude, that is, the three adjacent edges which meet in any vertex.

823. The *cube* is a rectangular parallelopiped whose faces are squares. All its edges are equal.

824. A *pyramid* is a polyhedron (Fig. 125) of which one face $ABCD$, called the *base,* is a polygon, and the other faces SAB, SBC, . . . called *lateral faces,* are triangles having a common

vertex S, called the *vertex* of the pyramid. The intersections of the lateral faces are called *lateral edges*. Such are: SA, SB, . . .

The *altitude* is the perpendicular drawn from the vertex to the base. A pyramid is triangular, quadrangular, pentagonal, . . . according as its base is a triangle, quadrilateral, pentagon, . . . (632). A *pyramid* is *regular* when its base is a regular polygon and its lateral edges are equal. The lateral faces are equal isosceles triangles, the altitude of which is called the *slant height* of the pyramid.

825. A plane P parallel to the plane of the base $ABCDE$ of a pyramid (Fig. 126):

1st. Divides the edges SA, SB, . . . and the altitude Sh proportionally. Thus,

$$\frac{SA}{SA'} = \frac{SB}{SB'} \cdots\cdots = \frac{Sh}{Sh'}.$$

2d. The section $A'B'C'D'E'$ is similar to the base, and the ratio of the two polygons is equal to the ratio of the squares of the lateral edges and altitude. Thus,

$$\frac{ABCDE}{A'B'C'D'E'} = \frac{\overline{SA}^2}{\overline{SA'}^2} = \frac{\overline{Sh}^2}{\overline{Sh'}^2} \qquad (699, 726)$$

826. If two pyramids of the same altitude are cut by planes parallel to their bases, and at equal distances from their vertices, the sections will have the same ratio as their bases. If the bases are equal or equivalent, the sections are also.

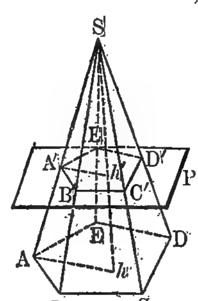


Fig. 126

827. The *frustum* of a pyramid is the portion of a pyramid included between the base and a section made by a plane parallel to the base. The base of the pyramid and the section are the *bases of the frustum* (Fig. 126) (895).

828. A *polyhedron* is *convex* when it is situated totally on one side of the plane of any one of its faces (648).

829. Two *polyhedrons* are of the *same kind* when their surfaces are composed of the same number of triangles, quadrilaterals, pentagons, . . . placed in the same order. Thus, two pyramids

or prisms are of the same kind when their bases have the same number of sides.

830. *Two tetrahedrons are equal* (**818**): *First,* when three adjacent edges and the included polyhedral angle of one are equal to three adjacent edges and the included polyhedral angle of the other and placed in the same order; *Second,* when two faces and the included dihedral angle of one are equal to two faces and the included dihedral angle of the other and placed in the same order; *Third,* when one face and the three adjacent dihedral angles of one are equal to one face and the three adjacent dihedral angles of the other and arranged in the same order; *Fourth,* when the edges of one are equal to the edges of the other and are arranged in the same order (**809**).

831. *Two prisms are equal* if three faces, including a trihedral angle of one, are respectively equal to three faces, including a trihedral angle of the other, and are similarly placed. Two right prisms of the same base and altitude are equal. All cubes which have an equal side are equal.

832. In any polyhedron the number of vertices plus the number of faces is equal to the number of edges plus 2. Thus,

$$V + F = E + 2,$$

wherein V is the number of vertices, F the number of faces, and E the number of edges.

833. The number of conditions necessary for the equality of two polyhedrons of the same kind (**829**) is equal to the number E of edges.

834. The sum of all the face angles of a polyhedron is equal to as many times four right angles as there are vertices in the polyhedron less two. Thus,

$$s = 4\,(V - 2);\ \text{for } V = 8,\ s = 4\,(8 - 2) = 24\ rt\ \measuredangle s, \qquad \textbf{(652)}$$

wherein s is the sum of the face angles expressed in right angles, and V the number of vertices.

835. *In any parallelopiped* (**822**): *First,* the diagonals bisect each other; *Second,* the sum of the squares of the diagonals is equal to the sum of the squares of the sides.

Thus, A, B, C, D, being the diagonals, and a, b, c, the three adjacent sides, we have:

$$A^2 + B^2 + C^2 + D^2 = 4\,a^2 + 4\,b^2 + 4\,c^2. \qquad \textbf{(739)}$$

In any rectangular parallelopiped, the four diagonals are equal, and we have:

$$4\,D^2 = 4\,a^2 + 4\,b^2 + 4\,c^2 \text{ or } D^2 = a^2 + b^2 + c^2,$$

that is, the square of one diagonal is equal to the sum of the squares of three sides.

If the *parallelopiped is a cube,* the three sides are equal, and we have:

$$D^2 = 3\,c^2, \text{ and } \frac{D}{c} = \sqrt{3}. \tag{731}$$

Thus the ratio of the diagonal D to one side c of the cube is equal to the square root of three $\sqrt{3}$.

836. *Two points are symmetrical with respect to a third point* if this third point bisects the straight line which joins them.

Two points are symmetrical with respect to a line or plane when this line or plane bisects at right angles the line which joins the two points.

837. *Two straight lines are symmetrical with respect to a point, a line, or a plane* when their extremities are symmetrical with respect to the point, line, or plane. The point, line, and plane are respectively called *center of symmetry, axis of symmetry,* and *plane of symmetry.*

838. *Two polygons or two polyhedrons are symmetrical with respect to a point, a line, or a plane* when each vertex of one has a symmetrical vertex in the other with respect to the point, the line, or the plane.

839. Two straight lines, two polygons symmetrical with respect to a straight line, are equal each to each.

Two straight lines, two polygons symmetrical with respect to a point or a plane, are equal.

Two polyhedral angles, or two polyhedrons symmetrical with respect to a point or a plane, have homologous dihedral angles equal and arranged in inverse order. In general they cannot be made to coincide.

BOOK III

THE CYLINDER—THE CONE—THE SPHERE

840. *A right circular cylinder,* or *cylinder of revolution,* is a solid generated by a rectangle $ABCD$, which makes one entire revolution about one of its sides as an axis. The side AB which serves as axis is called the *axis of the cylinder. The bases of the cylinder* are the circles described by the sides AB and DC perpendicular to the axis. *The altitude of the cylinder* is the distance AB between the two bases. *The lateral surface of the cylinder* is the surface generated by the side CD parallel to the axis. CD is called the *generatrix.* Any position of the generatrix is an *element* of the surface.

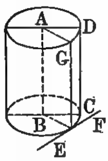


Fig. 127

841. *A right circular cone, or cone of revolution,* is a solid generated by the revolution of a right triangle ABS about one of its legs as an axis.

The side SB which serves as an axis is called the *axis or the altitude of the cone. The base of the cone* is the circle generated by the side AB perpendicular to the axis. The *slant height of the cone* is the hypotenuse of the generating triangle. *The vertex of the cone* is the point where the lateral surface meets the axis. The *lateral surface* is generated by the hypotenuse SA. SA is the *generatrix.* Any position of the generatrix is an element of the surface.

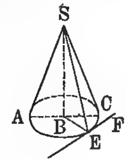


Fig. 128

842. The section of a right circular cylinder made by a plane: *First,* parallel to the bases is a circle equal to the bases; *Second,* parallel to the axis is a rectangle whose opposite sides are two elements of the cylinder.

843. The section of a right circular cone made by a plane: *First,* parallel to the base is a circle; *Second,* passing through the vertex perpendicular to the base is an isosceles triangle whose sides are two elements of the cone.

844. *The frustum of a cone* is that part of a cone included between the base and a section parallel to the base. The base of the cone and the section are *the bases of the frustum.*

The slant height of the frustum of a cone of revolution is that part AB of the generatrix included between the two bases (Fig. 138), and the *altitude* is the distance CD between the bases (827).

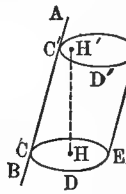


Fig. 129

845. A *cylindrical surface* is a curved surface generated by a moving straight line AB, called a *generatrix*, which moves parallel to itself and constantly touches a fixed curve CDE called the *directrix.*

When the directrix is a closed plane curve, all sections made by planes cutting the surface which are parallel to the plane of the directrix are equal to the directrix, and a *cylinder* is a solid $CDEC'D'E'$ included by the parallel planes, which are limited by the curves equal to the directrix and that portion of the cylindrical surface included between these parallel planes.

The bases of the cylinder are the two parallel planes CDE and $C'D'E'$, and the distance HH' between the bases is the *altitude.*

A cylinder is *right* or *oblique*, according as the generatrix is or is not perpendicular to the plane of the bases.

In a right circular cylinder the directrix is a circle (840).

The *right section of a cylinder* is a section made by a plane perpendicular to the generatrix (820).

846. A *prism and a cylinder are inscribed in or circumscribed about one another*, according as their bases are inscribed in or circumscribed about one another (673, 677). Just as a circle, or in general any plane surface limited by a curve, may be regarded as the limit approached by any inscribed or circumscribed polygon when the number of sides is indefinitely increased (601), the cylinder may be considered as being the limit approached by any inscribed or circumscribed prisms which have these polygons for bases. Thus, the right cylinder may be considered as a right prism, and an oblique cylinder as an oblique prism. Therefore all properties of the surfaces or volumes of prisms apply as well to cylinders, provided that these properties are independent of the number of sides, and

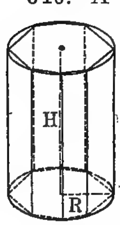


Fig 130

that the bases and altitude of the cylinder are substituted for the bases and altitude of the prism.

847. The development of the lateral surface of a prism is a plane surface.

If the prism is a right prism, the development is a rectangle, whose altitude is the altitude of the prism and whose base is the perimeter of the base of the prism. Likewise, the development of the lateral surface of a cylinder is a plane surface, and when the cylinder is a right cylinder, it is a rectangle whose altitude is the altitude of the cylinder and whose base is the perimeter of the base of the cylinder.

848. A conical surface is the surface generated by a moving straight line SA, called the *generatrix*, passing through a fixed point S, called the *vertex*, and constantly touching a fixed curve BCD, called the *directrix*.

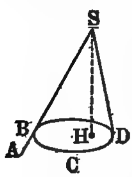


Fig. 131

When the directrix BCD is the boundary of a plane surface, the solid $SBCD$, included between this surface and the vertex, is called a *cone*. The plane surface is the *base of the cone*, and the altitude is the distance SH from the vertex to the plane of the base.

When the directrix is a circle, and the vertex lies on a perpendicular erected at its center, the cone is a *right circular cone* (**841**). When these conditions are not fulfilled the cone is *oblique*.

849. A *pyramid and a cone are inscribed in or circumscribed about one another*, according as, having the same vertex, their bases are inscribed in or circumscribed about one another.

The cone may be considered as the limit of inscribed or circumscribed pyramids when the number of sides is indefinitely increased (**846**). Thus the right circular cone (**841**) may be considered as a regular pyramid (**824**) whose slant height is the side of the cone, and whose base is a circle; and, in general, any cone may be considered as being a pyramid. Therefore all properties of surfaces or volumes of pyramids apply as well to cones, provided that they be independent of the number of sides of the base of the pyramid.

850. The development of the lateral surface of a pyramid is a plane surface, as is also that of the lateral surface of a cone.

When the cone is one of revolution, the development of the lateral surface is the sector of a circle whose radius is the side

of the cone, and whose base is an arc equal to the circumference of the base of the cone (760).

851. A *plane is tangent to a cylinder or to a cone of revolution* when it touches only one element of the surface of the solid, that is, when it contains a tangent EF to the base and the element (840, 841) which passes through the point of contact E (Figs. 127 and 128). The above statement applies to any cone or cylinder whose base is a convex polygon.

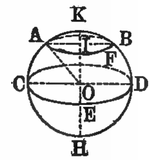


Fig. 132

Any plane tangent to a cylinder or to a cone of revolution is perpendicular to a plane passing through the axis of the cone and the element (841) of the surface at the point of contact.

852. A *sphere* is a solid bounded by a surface every point of which is equally distant from a point O called the *center* (665).

A sphere may be considered as being generated by a semicircle KCH, revolving on its axis KH.

All straight lines OA, drawn from the center to the surface, are called radii. A straight line AB, which has its extremities in the surface of the sphere, is a *chord*. A chord CD which passes through the center is a *diameter*. All diameters are equal to two radii and consequently equal to each other.

All sections CED, made by planes passing through the center, are called *great circles*.

A quarter $CE = ED$ of a great circle is called a *quadrant* (222).

All great circles divide the sphere into two equal parts (666).

A section AFB, made by a plane which does not pass through the center, is a *small circle*.

853. In the same sphere or in equal spheres two circles equally distant from the center are equal, and of two circles unequally distant from the center, the smaller one is the farther. The converse statements of the above are also true (672).

854. The distance between two points on the surface of a sphere is the arc of the great circle joining these two points.

855. The extremities H and K of the diameter perpendicular to the plane of a circle AFB are the *poles of this circle.*

Each of the poles H and K of a circle AFB is equally distant from all points in the circumference of the circle, that is, all the

arcs of the great circles passing through the pole and the circumference are equal.

Conversely, if all points on a line drawn on the surface of a sphere are equidistant from one fixed point in the circumference, the line is the circumference of the circle which has this point for its pole.

856. The angle formed by the arcs AB, AC, of two great circles which meet in a point A, is called a *spherical angle.* The point of meeting is the *vertex,* and the arcs the *sides.*

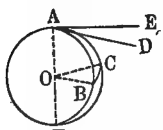


Fig. 133

857. A *lune* is a portion $ABFCA$ of the surface of a sphere, bounded by two semi-circumferences of great circles. The *angle of the lune* is the angle DAE between the semi-circumferences which form its boundaries.

A *spherical wedge* is a portion $AOFBC$ of a sphere bounded by a lune and two great semicircles. The dihedral angle formed by the planes of the semicircles is the angle of the wedge. The plane angle of this dihedral angle is the angle DAE (**792**).

A *spherical lune or wedge is right, acute, or obtuse,* according as its angles are right, acute, or obtuse (**788**).

Two great circles the planes of which are perpendicular to each other divide the sphere into four equal right wedges, and the surface into four equal right lunes.

858. A part ABC of the surface of a sphere bounded by three or more arcs of great circles is called a *spherical polygon.*

The arcs are the *sides of the polygon.*

859. A *spherical triangle* is *right, isosceles,* or *equilateral,* under the same conditions as a plane triangle (**633**, **635**, **636**).

A *spherical triangle* is *bi-rectangular* or *tri-rectangular* according as it has two or three right angles.

860. A *spherical triangle is the polar triangle of another* when the vertices of the second are the poles of the first (**855**).

861. A *spherical pyramid* is a solid $OABC$, bounded by a spherical polygon ABC, and the circular sectors OAB, OAC, OBC, whose bases are the different sides of the polygon and whose vertex is the center of the sphere (Fig. 133). The polygon ABC is the *base of the pyramid,* and the center of the sphere is the *vertex.*

A *spherical pyramid* is *bi-rectangular* or *tri-rectangular* accord-

ing as its base is a bi-rectangular or tri-rectangular triangle (859).

Three great circles, such that the plane of each is perpendicular to the planes of the two others, divide the sphere into eight tri-rectangular pyramids equal each to each, and the surface into eight equal tri-rectangular spherical triangles.

862. In any spherical triangle any side is less than the sum of the other two and greater than their difference (601). Articles (635, 636, 638, 658) apply as well to spherical triangles as to plane triangles.

863. The sum of the sides ot any spherical polygon is less than the circumference of a great circle.

864. The angle of two arcs of great circles (856) is equal to the plane angle of the dihedral angle formed by the planes of the two arcs.

The angles of a spherical polygon are the plane angles of the dihedral angles formed by the planes of the sides (792).

865. The sum of the angles of a spherical triangle are less than six and greater than two right angles (813).

866. *Two spherical triangles on the same or equal spheres are equal: First,* when two sides and the included angle of one are equal to two sides and the included angle of the other and similarly placed; *Second,* when one side and the adjacent angles of one are equal to one side and the adjacent angles of the other and similarly placed; *Third,* when they have three sides equal each to each and similarly placed; *Fourth,* when they have three angles equal each to each and similarly placed (654, 809).

867. *A spherical triangle may be constructed: First,* when two sides and the included angle are given; *Second,* when one side and the adjacent angles are given; *Third,* when three sides are given; *Fourth,* when three angles are given (663).

868. A *zone* is that portion of the surface of a sphere included between two parallel planes *CED*, *AFB* (Fig. 132). The *bases* of the zone are the two circumferences *CED* and *AFB*, which include the zone. When one of the two planes is tangent to the sphere, the zone has only one base.

The distance between the bases is the *altitude* of the zone.

869. A line is inscribed in a sphere when it terminates in the surface of the sphere. Such is *AB* (Fig. 132).

A *polyhedron is inscribed in a sphere* when all its sides are

inscribed in the sphere. A sphere is circumscribed about a polygon when the polygon is inscribed in the sphere (673).

870. *A sphere, and only one, may be passed through four points* in space not in the same plane (680).

The six planes drawn perpendicular to the middles of the edges of a tetrahedron meet in a single point equally distant from the four vertices of the tetrahedron. *This point is the center of a sphere, which may be circumscribed about the tetrahedron* (688).

871. *A straight line AE and a sphere O are tangent* when they have only one point A in common (Fig. 133).

A plane DAE is tangent to a sphere O when they have but one point A in common.

Any plane DAE perpendicular to a radius OA at its extremity is tangent to the sphere (673). Any straight line AD perpendicular to the radius OA is tangent to the sphere, and lies in the plane which is tangent to the sphere at that point A.

The perpendicular OA erected to the tangent plane DAE at the point of contact is normal to the sphere O. Any line normal to the surface passes through the center of the sphere, and all radii are normal to the surface of the sphere. The shortest and longest distances from a given fixed point to the surface of a sphere is the normal to the surface of the sphere passing through the point (675).

872. *A polyhedron is circumscribed about a sphere* when each of its faces is tangent to the surface of the sphere.

A sphere is inscribed in a polyhedron when the polyhedron is circumscribed about the sphere (677).

873. The six planes which bisect the dihedral angles of a tetrahedron meet in a single point equally distant from the four faces of the tetrahedron. *This point is the center of a sphere which may be inscribed in the tetrahedron* (687).

874. *Two spheres are tangent* when they have but one point in common (675). Two spheres which have their common point on the line of centers are either *tangent externally or internally,* according as the point is situated between the centers or on the prolongation of the line of centers.

Articles (681 to 683) apply to the surfaces of spheres as well as to circles, except that the surfaces cut each other in circles.

BOOK IV

SIMILAR POLYHEDRONS AND THE MEASUREMENT OF ANGLES

875. *Two polyhedrons are similar* when their dihedral angles are equal each to each and are similarly placed, and the homologous faces are similar (695).

876. A plane P (Fig. 126), drawn parallel to the plane of the base of a pyramid, cuts off a pyramid $SA'B'C'D'E'$, which is similar to the original pyramid $SABCDE$ (825).

877. *Two tetrahedrons are similar: First,* when they have an equal polyhedral angle included between proportional edges and similarly placed; *Second,* when they have an equal dihedral angle included between two faces similar each to each and similarly placed; *Third,* when they have a similar face and three adjacent dihedral angles equal each to each and situated in the same order; *Fourth,* when their edges are proportional each to each and similarly placed (700).

878. *Two prisms or two pyramids are similar* when they have an equal dihedral angle at the base included between two faces similar each to each and similarly placed.

Two prisms or two regular pyramids (819, 824) *are similar* when their bases are similar polygons and their altitudes to each other as the sides of their bases, or as the radii of the circles inscribed in or circumscribed about the bases.

Two right prisms are similar when their bases are similar and their altitudes are to each other as the homologous sides of the bases.

Two rectangular parallelopipeds are similar when their dimensions are proportional. *All cubes are similar.*

879. *Two polyhedrons* composed of the same number of tetrahedrons similar each to each and similarly placed, are similar; and the converse is also true (702).

Two polyhedrons similar to a third are similar to each other.

880. *All dihedral angles are measured by their plane angles* (**792**), that is, they contain as many right dihedral angles as their plane angles contain right plane angles.

881. *A spherical lune is measured by twice its angle* (**857**), that is, it contains as many tri-rectangular spherical triangles (**861**) as twice its angle contains right plane angles (**918**).

A spherical wedge is measured by twice its plane angle, that is, it contains the tri-rectangular spherical pyramid as many times as its plane angle contains right angles (**857**, **861**, **928**).

882. Taking the tri-rectangular spherical triangle and the right triangle as units (**881**):

1st. *A spherical triangle is measured by the excess of the sum of its angles over two right angles.*

2d. *Any spherical polygon is measured by the excess of the sum of its angles over as many times two right angles as there are sides less two* (**858**, **864**, **919**).

883. Taking the spherical tri-rectangular pyramid and the right angle as units (**881**):

1st. *A spherical triangular pyramid is measured by the excess of the sum of its angles over two right angles.*

Any spherical pyramid is measured by the excess of the sum of the angles of its base over as many times two right angles as there are sides to the base less two (**861**, **864**, **929**).

884. *Any trihedral angle is measured by the excess of the sum of its plane angles over two right angles.* Taking the tri-rectangular trihedral angle and the plane right angle as units (**792**, **805**).

BOOK V

MENSURATION OF POLYHEDRONS (781)

885. The *volume* of a body is the ratio of that body to another taken as unity (**216**). Thus, supposing a cube whose side is equal to one foot is taken as unity, when a body, of any form whatever, contains the tenth part of the foot cube twelve times, the volume of the body is equal to 1.2 cubic feet.

886. *The product of a surface and a line* is the product of the area of the surface by the length of the line (**713**). The area is expressed in units of surface one side of which is the unit of length.

887. *The volume of a rectangular parallelopiped is equal to the product of its base and its altitude, or the product of its three dimensions* (**822**).

The volume of a cube is equal to the cube of its edge (**823**).

888. Two parallelopipeds are to each other as the products of their three dimensions, or as the products of their bases and altitudes. If they have an equal dimension, they are to each other as the products of their other two dimensions, and if they have two dimensions equal they are to each other as their third dimension (**717**).

Two cubes are to each other as the cubes of their edges (**823**).

889. The volume of a prism is equal to the product of its base and its altitude (**819**). When the prism is a right prism, the altitude is equal to one of the lateral edges.

The volume of a prism is also equal to the product of its right section and one of its lateral edges (**820**).

Any parallelopiped, being simply a special case of the prism, is measured the same as a prism (**887**).

Any two prisms are to each other as the products of their bases and their altitudes, and according as two prisms have equivalent bases or equal altitudes they are to each other as their altitudes or their bases. They are equivalent if they have the same altitudes and equivalent bases.

890. *The lateral surface of a right prism* is equal to the perim-

eter of the base times the altitude, and the *lateral surface of any prism* is equal to the perimeter of its right section times one of the lateral edges (820).

891. *The volume of any pyramid is equal to one-third the product $B \times H$ of the base and the altitude.* It is equal to one-third the volume of a prism of equivalent base and equal altitude (824, 889).

Any two pyramids are to each other as the products of their bases and their altitudes, and according as the two pyramids have the equivalent bases or the same altitude they are to each other as their altitudes or their bases. They are equivalent if they have the same altitudes and equivalent bases.

892. *The lateral surface of a regular pyramid* is equal to half the product of the perimeter of the base and the altitude of one of the lateral faces (824).

893. Two tetrahedrons, triangular prisms, or parallelopipeds, which have an equal polyhedral angle, are to each other as the products of the sides which include the equal angle (725).

894. *The volume of a truncated triangular prism ABCDEF*

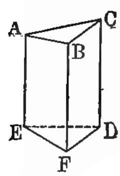


Fig. 134

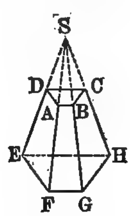


Fig. 135

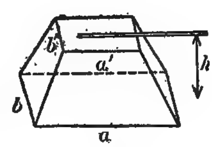


Fig. 136

(821) is equal to the sum of the volumes of the three pyramids whose common base is the lower base of the prism and whose vertices are the vertices A, B, C, of the upper base of the prism.

$$V = \frac{1}{3} B\,(a + b + c),$$

wherein V is the volume of a truncated prism; B is the lower base; and a, b, c, the altitudes of the various vertices A, B, C, with respect to the base B.

895. The volume of the frustum of a pyramid *ABCDEFGH* (827) is equal to the sum of the volumes of three pyramids having an altitude equal to the altitude of the frustum and their

bases respectively, the lower base $EFGH$, the upper base $ABCD$, and a mean proportional between these two bases of the frustum (**344**). Thus,

$$V = \frac{1}{3}H \times B + \frac{1}{3}H \times b + \frac{1}{3}H\sqrt{Bb} = \frac{1}{3}H(B + b + \sqrt{Bb}),$$

wherein V is the volume of the frustum, B the lower base, and b the upper base.

896. *The volumes of two similar polyhedrons are to each other as the cubes of their homologous linear dimensions, and their surfaces are to each other as the squares of these dimensions.*

897. *The volume of a pile of stones or the capacity of dump-cart.* Suppose a pile of crushed stone to be piled so that its upper and lower bases are rectangles, then its volume is (Fig. 136):

$$V = \frac{h}{6}[b(2a + a') + b'(2a' + a)],$$

wherein h is the height of the pile, a and b the dimensions of the lower base, and a' and b' those of the upper. The same formula may be used for the calculation of the capacity of a dump-cart.

If b' should equal zero, as is sometimes the case, we have:

$$V = \frac{h}{6}b(2a + a').$$

When the bases are similar, the solid is the frustum of a pyramid, and its volume may be calculated from the formula in article (**895**).

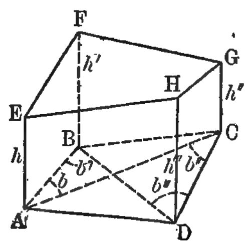


Fig. 137

898. Excavations. To calculate the total volume of an excavation, divide it into parts bounded laterally by vertical planes, on the bottom by any quadrilateral $ABCD$ (Fig. 137), and on the top by the surface of the soil, which has no geometrical form but which may be supposed to be generated by a straight line which moves on the two opposite lines EF and GH, or EH and FG, the points E, F, G, H, all being on the surface of the soil.

Since the area of a trapezium is expressed in triangles, and designating respectively the areas of the triangles

	$ABC,$	$ABD,$	$CDA,$	$CDB,$
by	$b,$	$b',$	$b'',$	$b''',$

the volume of the solid is equal to:

$$V=\frac{b(h+h'+h'')+b'(h+h'+h''')+b''(h+h''+h''')+b'''(h'+h''+h''')}{6}$$

When $ABCD$ is a trapezoid, AB being parallel to CD, we have $b = b'$ and $b'' = b'''$, and the preceding formula becomes:

$$V = \frac{b(2h + 2h' + h'' + h''') + b''(h + h' + 2h'' + 2h''')}{6}.$$

If $ABCD$ is a parallelogram, we have $b = b' = b'' = b'''$, and the formula becomes:

$$V = \frac{b(h + h' + h'' + h'''}{2} = B\,\frac{h + h' + h'' + h'''}{4},$$

$B = 2b$ being the total surface of the base $ABCD$. When the upper base $EFGH$ is plane, we have further $h + h'' = h' + h'''$, and therefore:

$$V = B\frac{h + h''}{2} = B\,\frac{h' + h'''}{2}.$$

When the base $ABCD$ is reduced to a triangle ABC, the solid becomes a truncated triangular prism, and we have (894), B being the surface of the triangle ABC,

$$V = B\frac{h + h' + h''}{3}.$$

It is possible that the upper base may become reduced to a single edge EF, the altitudes h'' and h''' becoming zero. In this case, according as the base is a trapezium, a trapezoid, or a parallelogram, we have respectively, making h'' and $h''' = 0$ in the preceding formulas:

$$V=\frac{h(b + b' + b'') + h'(b + b' + b''')}{6},$$

$$V=\frac{b(2h + 2h') + b''(h + h')}{6} = \frac{(h + h')(2b + b'')}{6},$$

$$V=\frac{b(h + h')}{2} = B\frac{h + h'}{4}.$$

Finally, if the upper base become reduced to a single point E, we have a pyramid, and the volume is:

$$V = B\frac{h}{3}.$$

BOOK VI

REGULAR POLYHEDRONS AND THE MENSURATION OF CYLINDERS, CONES, AND SPHERES

899. *A regular polyhedral angle* is one which has all its dihedral angles equal and all its face angles equal (**803**).

900. A *regular polyhedron* is one whose dihedral angles are all equal and whose faces are regular polygons, equal each to each (**740, 817**). Thus all cubes are regular polyhedrons (**823**).

The *center and the radius of a regular polyhedron* are the center and the radius of the sphere circumscribed about the polyhedron. The *apothem of a regular polyhedron* is the radius of the sphere inscribed in the polyhedron (**743, 869, 872**).

901. In any regular polyhedron a single sphere may be inscribed, and about any regular polyhedron a single sphere may be circumscribed (**900**).

902. Two polyhedrons of the same kind are always similar (**829, 875**).

903. The volume of a regular polyhedron is equal to its surface times one-third its apothem (**900**).

Table of Five Regular Polyhedrons

Giving the number and kind of their faces, their surfaces, and their volumes; their edges being taken as unity (**745**).

POLYHEDRONS.	FACES.	SURFACE.	VOLUME.
Tetrahedron	4 triangles.	1.732051	0.117851
Cube or hexahedron	6 squares.	6.000000	1.000000
Octahedron	8 triangles.	3.464102	0.471404
Dodecahedron	12 pentagons.	20.645779	7.663119
Icosahedron	20 triangles.	8.660254	2.181695

From this table an octahedron whose edge is 2.5 feet has respectively

$$3.464102 \times (2.5)^2 = 21.6506 \quad \text{sq. ft.}$$
$$0.471404 \times (2.5)^3 = 7.365687 \text{ cu. ft.}$$

for its surface and volume.

904. Two cylinders or cones of revolution (**804**, **805**) are similar when the altitude h and radius r of the base of the first are proportional to the altitude h' and the radius r' of the base of the second, that is, when

$$h : h' = r : r'.$$

905. Two spheres are always similar.

906. *The lateral surface of a cylinder of revolution* (**840**, **845**) is equal to the perimeter of the base times the altitude. Thus, for a circular cylinder:

$$S = 2\,\pi RH,$$

wherein S is the surface, $2\,\pi R$ the perimeter of the base, R the radius of the base, and H the altitude of the cylinder.

The lateral surface of any cylinder is equal to the perimeter of its right section times its generatrix (**845**, **890**).

907. *The volume of any cylinder* is equal to its base times its altitude. Thus, for a circular cylinder,

$$V = \pi R^2 H,$$

wherein V is the volume, R the radius of the base, πR^2 the area of the base, and H the altitude of the cylinder.

908. *The lateral surface of a cone of revolution* is equal to half the product of the circumference of its base by its slant height (**718**, **841**). Thus, for a circular cone,

$$S = \pi RC,$$

wherein S is the surface, R the radius of the base, πR half the circumference of the base, and C the slant height.

909. *The volume of any cone* is equal to one-third the product of its base and its altitude. Thus, for a circular cone,

$$V = \frac{1}{3}\,\pi R^2 H,$$

wherein V is the volume, R the radius of the base, πR^2 the area of the base, and H the altitude of the cone.

Thus the volume of a cone is one-third that of a cylinder of an equivalent base and the same altitude (**907**).

910. Two cylinders or two cones are to each other as the products of their bases and their altitudes. If they have the same

altitudes they are to each other as their bases; if they have equivalent bases they are to each other as their altitudes, and if they have equal altitudes and equivalent bases they are equivalent (889, 891, 907, 909).

911. Two similar cylinders or cones of revolution (904) are to each other as the cubes of any of their homologous linear dimensions. Thus,

$$\frac{V}{V'} = \frac{H^3}{H'^3} = \frac{C^3}{C'^3} = \frac{R^3}{R'^3} = \frac{D^3}{D'^3},$$

wherein V is the volume, H the altitude, C the slant height, R the radius of the base, and D the diameter of the base.

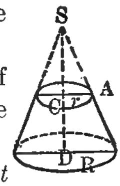


Fig. 138

The lateral surfaces and the total surfaces of similar cones or cylinders are to each other as the squares of these same dimensions.

912. *The lateral surface of the frustum of a right cone* (836) is equal to the slant height, times half the sum of the circumferences of its bases.

Thus,

$$S = C\frac{2\pi R + 2\pi r}{2} = C\pi(R + r),$$

wherein S is the surface, C the slant height, R the radius of the lower base, and r the radius of the upper base.

913. The volume of the frustum of a cone is equal to the sum of the volumes of three cones which have a common altitude equal to the altitude of the frustum, and their bases equal respectively to the lower base, the upper base, and the mean proportional between the two (895).

$$V = \frac{1}{3}\pi R^2 H + \frac{1}{3}\pi r^2 H + \frac{1}{3}\sqrt{\pi r^2 \times \pi R^2}\, H = \frac{1}{3}\pi H(R^2 + r^2 + Rr),$$

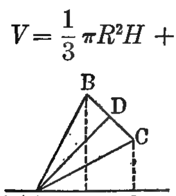


Fig. 139

wherein V is the volume, R the radius of the lower base, r the radius of the upper base, and H the altitude of the frustum.

914. The surface generated by the base BC of an isosceles triangle ABC, revolving about an axis MN, which passes through the vertex external to the triangle and in the same plane, is equal to the projection $PQ = p$ of the base upon the axis MN, times the cir-

cumference $2\pi r_1$ of the circle whose radius is equal to the altitude, $AD = r_1$ of the triangle. Thus,

$$S = p \times 2\pi r_1.$$

The surface generated by a sector of a regular polygon under the same conditions is found in the same manner, p being the projection of the entire base upon the axis.

915. *The surface of a zone* is equal to the altitude H of the zone times the circumference $2\pi R$ of a great circle (852, 868). Thus,

$$S = 2\pi RH.$$

916. On the same or equal spheres, two zones are to each other as their altitudes, and on spheres of different radii two zones of the same altitude are to each other as the radii or diameters of the spheres (915).

917. *The surface of a sphere* of radius $R = \frac{D}{2}$, when considered as a zone, is equal to (915),

$$S = 2\pi R \times 2R = 4\pi R^2 = \pi D^2.$$

Thus the surface of a sphere is equal to the area of four great circles, or of a circle whose radius is equal to the diameter of the sphere (753). The surfaces S and s of two spheres are to each other as the squares of their radii R and r or their diameters D and d. Thus,

$$S = 4\pi R^2 \quad \text{and} \quad s = 4\pi r^2,$$

and

$$S : s = R^2 : r^2 = D^2 : d^2.$$

918. *The surface of a spherical lune* is equal to the arc a corresponding to its angle α times the diameter $2R$ of the sphere (881). Thus:

$$a = 2\pi R \frac{\alpha}{360},$$

and

$$S = 2Ra = 4\pi R^2 \frac{\alpha}{360}.$$

919. *The surface of any spherical triangle* is equal to the radius of the sphere times the excess of the sum of the arcs a, b, c, corresponding to the angles over the semi-circumference (882). Thus the surface of a triangle is

$$S = R(a + b + c - \pi R).$$

The area of any spherical polygon is equal to the radius of the sphere times the excess of the sum of the arcs corresponding to its angles over as many times a semi-circumference as there are sides less two.

920. The volume of a solid generated by the revolution of any triangle ABC about a straight line MN, drawn through its vertex in the same plane and external to the triangle, is equal to the surface generated by the base BC times a third of the altitude $AD = h$ of the triangle.

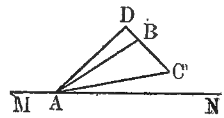


Fig. 140

The surface generated by the base of a triangle is the lateral surface of the frustum of a cone (Fig. 140) (**912**); it is that of a cone when AC or AB coincide with MN (**908**), and that of a cylinder when BC is parallel to MN (**906**). In any case this surface may be measured, and if it be represented by S, the volume generated by the triangle ABC is:

$$V = \frac{1}{3} Sh.$$

921. *The volume of a solid generated by an isosceles triangle ABC* (Fig. 139) revolving about a straight line drawn through its vertex, in its plane and external to it, is equal to the projection p of the base BC on the axis multiplied by two-thirds of the area of a circle whose radius is the altitude $AD = r_1$ of the triangle. Thus,

$$V = p \times \frac{2}{3} \pi r_1^2.$$

The volume of a solid generated by the revolution of a sector of a regular polygon about a straight line MN drawn through the vertex, in the same plane and external to it, is equal to the projection p of the base on the axis times two-thirds the area of a circle inscribed to the base. The sector may be a semi-polygon revolving on its diameter. In any case, r_1 being the radius of the inscribed circle, the generated volume is

$$V = p \times \frac{2}{3} \pi r_1^2.$$

922. *The volume generated by the revolution of a regular polygon about one of its sides as an axis:* expressed in terms of its radius R, and in terms of its side c (**745**):

Triangle	$\frac{3}{4}\pi R^3\sqrt{3}$	$\frac{1}{4}\pi c^3$
Square	$2\pi R^3\sqrt{2}$	πc^3
Pentagon	$\frac{5}{4}\pi R^3\sqrt{5+2\sqrt{5}}$	$\frac{1}{4}\pi c^3(5+2\sqrt{5})$
Hexagon	$\frac{9}{2}\pi R^3$	$\frac{9}{2}\pi c^3$
Octagon	$2\pi R^3\sqrt{4+2\sqrt{2}}$	$2\pi c^3(3+2\sqrt{2})$
Decagon	$\frac{5}{2}\pi R^3\sqrt{5}$	$\frac{5}{2}\pi c^3(5+2\sqrt{5})$
Dodecagon	$\frac{3}{2}\pi R^3(\sqrt{6}+\sqrt{2})$	$3\pi c^3(7+4\sqrt{3})$

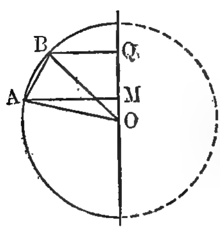


Fig. 141

923. A *spherical sector* is a solid generated by the revolution of a circular sector OAB about a diameter OQ, external to the sector and in the same plane with it. The *base of the spherical sector* is the zone described by the base AB of the circular sector (**868**).

The *volume of a spherical sector* is equal to the altitude $H = MQ$ of the zone, which serves as base, times two-thirds the area of a great circle of radius R. Thus,

$$V = \frac{2}{3}\pi R^2 H.$$

924. Considering the sphere as a spherical sector whose altitude is equal to the diameter of the sphere $2R = D$, from the preceding article, we have the *volume of the sphere* equal to its diameter, times two-thirds the area of a great circle.

$$V = 2R \times \frac{2}{3}\pi R^2 = \frac{4}{3}\pi R^3 = \frac{1}{6}\pi D^3.$$

925. The *volume of any spherical sector* is equal to one-third of the area of the zone, which serves as base, times the radius (**891**, **915**). Thus,

$$V = \frac{1}{3} \times 2\pi RH \times R = \frac{2}{3}\pi R^2 H.$$

926. The volume of a sphere is also equal to one-third of the product of its surface and its radius. Thus,

$$V = \frac{1}{3} \times 4\pi R^2 \times R = \frac{4}{3}\pi R^3.$$

Writing R in terms of the surface S (917), we have:

$$S^3 = 36\,\pi V^2.$$

927. *Two spheres are to each other as the cubes of their radii or diameters.* V and v being the volumes of the two spheres, we have (924):

$$V = \frac{4}{3}\pi R^3 \text{ and } v = \frac{4}{3}\pi r^3;$$

then $$V : v = R^3 : r^3 = D^3 : d^3. \qquad (917)$$

928. *The volume of a spherical wedge* is equal to the arc a corresponding to its angle α times two-thirds the square of its radius R. Thus,

$$V = \frac{2}{3} aR^2. \qquad (881, 918)$$

929. *The volume of any spherical pyramid* is equal to the product $B \times \frac{1}{3} R$ of the base, times one-third the radius (883, 919).

930. The volume of a solid generated by the revolution of a circular segment CDm, about a diameter AB, external to the segment, is equal to the projection $PQ = p$ of its base $CD = b$ upon the axis, multiplied by one-sixth of the area of a circle whose radius is equal to the base b. Thus,

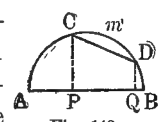


Fig. 142

$$V = p \times \frac{1}{6}\,\pi b^2.$$

931. *The volume of any spherical segment* is equal to half the sum of its bases, times its altitude, plus the volume of a sphere whose diameter is equal to the altitude of the segment. Thus, H being the altitude, and r and r' the radii of the bases, we have (753, 868, 924):

$$V = \frac{\pi r^2 + \pi r'^2}{2}\,H + \frac{1}{6}\,\pi H^3 = \frac{\pi H}{2}\,(r^2 + r'^2) + \frac{1}{6}\,\pi H^3.$$

When the segment has only one base, half the sum of the bases is replaced by half the base; thus,

$$V = \frac{1}{2}\,\pi r^2 H + \frac{1}{6}\,\pi H^3.$$

Considering the sphere as being a segment the altitude H of which is equal to the diameter $2\,R = D$ of the sphere, the first

term in the second member of the above equation becomes zero, and we have:

$$V = \frac{4}{3}\pi R^3.$$

932. *A right cylinder is equilateral* when its height is equal to the diameter of its base (**840**).

A right cone is equilateral when its slant height is equal to the diameter of its base (**841**).

A right cylinder is inscribed in a sphere when its bases are little circles of the sphere (**852**).

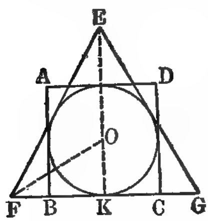


Fig. 143

An equilateral cylinder ADBC is circumscribed about a sphere (Fig. 143) when its axis is a diameter of the sphere.

A cone is inscribed in a sphere when its vertex and the circumference of its base lie on the surface of the sphere. *An equilateral cone EFG is circumscribed about a sphere* (Fig. 143) when its axis is the altitude of an equilateral triangle circumscribed about a great circle of the sphere.

933. *The total surfaces of a sphere, of a circumscribed cylinder of a circumscribed equilateral cone, are to each other as the numbers* 4, 6, 9; *and their volumes are to each other as these same numbers* (**906, 907, 908, 909, 917, 924**).

Remark 1. The lateral surface of the cylinder is equivalent to the total surface of the sphere.

Remark 2. The total surface of the cylinder is a mean proportional between that of the sphere and the cone (**344**).

Remark 3. The volume of the cylinder is a mean proportional between that of the cylinder and the cone.

The total surfaces of the sphere, of the inscribed cylinder and equilateral cone, are to each other as the numbers 16, 12, 9; *and their volumes are to each other as the numbers* 32, $12\sqrt{2}$, 9.

Thus the total surface of the cylinder is the mean proportional between that of the sphere and the cone; and its volume is also a mean proportional between those two solids.

PROBLEMS IN GEOMETRY

DRAWING OF THE FIGURES

934. Figures which are drawn simply to aid in following the demonstration of a problem, may be done free hand; but when measurements are to be obtained by a certain construction, the figures must be drawn accurately and to scale. In order to do this, instruments are necessary.

935. All the instruments which are necessary to construct all the figures of elementary geometry are the *rule* and the *compass.*

The first is used for drawing straight lines, the second for describing circles, and both of them in combination for constructing angles.

Besides these two instruments we have several others, which, though not necessary, are almost indispensable; these are: the T-square, the triangles, the protractor, the reducing compass.

The T-square is used for drawing parallel horizontal lines.

The triangles, which are generally, one 60° and one 45°, right triangle, are used to draw parallels and perpendiculars.

The protractor is used for laying off and measuring angles.

The reducing compass is used for constructing similar figures according to a given proportion, having one figure given.

Remark. When a point is to be determined by the intersection of two lines, these lines should intersect as nearly at right angles as possible.

ANGLES — TRIANGLES — PERPENDICULARS — PARALLELS

936. *To construct an angle equal to a given angle E* (Fig. 144), from the point E as a center, with any radius EG, describe the arc GH; from the point O on the line AB, with the same radius, describe the indefinite arc CL; take $CD = GH$ and draw the side OD; then the angle DOC is equal to the angle E.

Remark. The angle may be constructed by aid of the protractor or with the triangles, by drawing lines parallel to the sides and intersecting in the point O (630).

To construct an angle equal to the sum of two given angles A, B, construct angle GOE = angle A, then angle HOG = angle B, and then angle HOE is equal to angle A plus angle B.

In the same manner the sum of any number of angles may be constructed, and, in general, the angles may be added or subtracted.

To construct the supplement of a given angle GOE, prolong one side EO, then the angle GOF is the supplement (617).

To construct the complement of a given angle GOE, erect a perpendicular OH to one side OE at the vertex O, and the angle GOH is the complement.

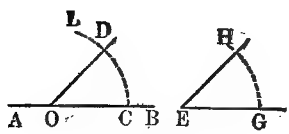


Fig. 144

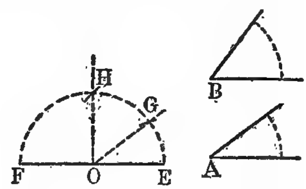


Fig. 145

Two angles, A and B, of a triangle being given to find the third, construct the angle HOE equal to the sum of A and B, then HOF is the required angle (652).

937. *To draw a straight line AB through a given point A, so as to make a given angle ABC with another line BC.* Through any point D, taken on the straight line BC, draw the line DE, making the angle CDE equal to the given angle (Fig. 146); then draw the line AB through A parallel to ED (625).

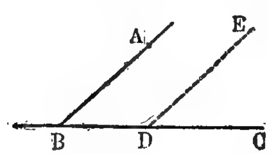


Fig. 146

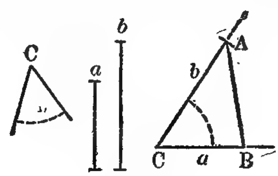


Fig. 147

938. *Two sides a and b and the included angle C of a triangle being given to construct the triangle* (663). Construct an angle equal to the given angle C; lay off a distance on one leg equal to a, and on the other equal to b; then join the two by the line AB which completes the triangle ABC (654).

In the same manner a parallelogram may be constructed when two sides and the included angle are given.

939. *One side a, and the two adjacent angles B and C, being given to construct the triangle* (663). Draw BC equal to a; then at the extremities construct the angles $ABC = B$ and $ACB = C$; the point A where the prolonged sides of these angles meet determines the triangle ABC (654).

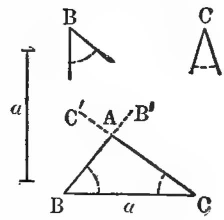


Fig. 148

If the angle opposite the side had been given, the third angle would have been determined according to article (936), and the problem would be the same as the one preceding.

940. *The three sides a, b, c, of a triangle being given to construct the triangle* (663). Draw the line BC equal to the side a; then from the extremities with b and c respectively as radii, arcs of circles are described, and their point of intersection A determines the triangle; drawing AB and AC, we have the required triangle ABC (Fig. 149) (654).

941. *Two sides a and b, and an opposite angle A, of a triangle being given to construct the triangle.* Construct the given angle A (Figs. 150 to 152); on one of the legs of this angle lay off $AC = b$; with C as center describe an arc of radius equal to a which cuts the line AB in B and B'; joining these two points to the vertex C we have one or two triangles which satisfy the conditions (663).

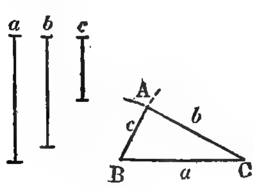


Fig. 149

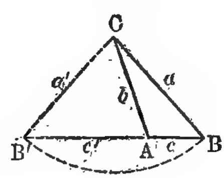


Fig. 150

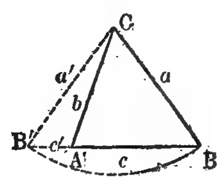


Fig. 151

1st. When the angle A is right or obtuse, angle B is acute (652), and $a > b$ (638); the arc BB' cuts AB in two points, but the triangle ABC is the only one which satisfies the conditions, because the angle CAB' is less than a right angle.

2d. If the angle A is acute and $a > b$ (Fig. 151) $\angle A > \angle B$, there is still but one solution, and that is the triangle ABC. In case $a = b$ there is still but one solution, because the point B' falls upon the vertex of the angle A.

3d. When A is acute and $a < b$, we have $\angle A < \angle B$, that is, $\angle B$ may be either acute or obtuse, and there are two solutions (Fig. 152): in the triangle ABC, which satisfies the conditions, the angle B is acute; in the triangle $AB'C$, which also satisfies the conditions, the angle B' is obtuse.

There are two solutions when $a < b$ is greater than the perpendicular DC. When $a < b$ is equal to CD, the arc BB' is tangent to AB at the point D, and the two triangles ABC and $AB'C$ coincide with the right triangle ADC, which is the only solution.

942. *Construct a right triangle,* having given: *First,* the hypot-

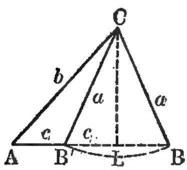


Fig. 152

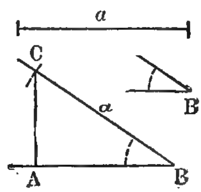


Fig. 153

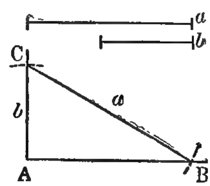


Fig. 154

enuse a and an acute angle B; *Second,* the hypotenuse a and a leg b:

1st. Construct the angle $CBA = B$ (937); take $BC = a$, and from the point C draw a perpendicular CA to the line AB; the triangle ABC satisfies the conditions (655).

2d. Construct a right angle; on one of its sides take $AC = b$; from the point C with a radius equal to a describe an arc cutting

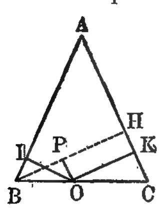


Fig. 155

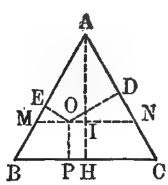


Fig. 156

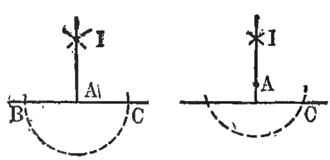


Fig. 157

AB in B, and draw CB, which completes the triangle ABC, satisfying the conditions (655).

943. *The sum $OI + OK$ of the perpendiculars drawn from a point O in the base BC of an isosceles triangle,* (Fig. 155) *to the legs, is constant and equal to the perpendicular BH.* Draw OP parallel to AC, that is, perpendicular to BH; then $OK = PH$

(626); and $OI = BP$, because the right triangles OBP and OIB are equal, having the same hypotenuse and angles POB and OBI equal each to each, both being equal to the angle C (625, 635).

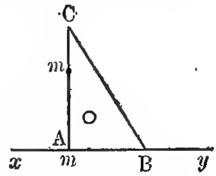


Fig. 158

The sum $OP + OE + OD$ *of the perpendiculars drawn from any point* O, *taken inside the equilateral triangle* ABC, (Fig. 156) *to the three sides of the triangle, is constant and equal to the altitude* AH *of the same triangle.* Draw MN through the point O parallel to BC; then OP equals IH (631); since the triangle AMN is isosceles as well as equilateral, $OE + OD = AI$; therefore

$$OP + OE + OD = AH.$$

944. *To erect a perpendicular to a given straight line* BC, (Fig. 157) *passing through a point* A, *which may be in or external to the line.* From the point as a center describe an arc, cutting the line in two points B and C, equally distant from A; with these points as centers and a radius longer than half the distance between the points, describe two arcs which intersect in I; I is also equally distant from B and C, therefore the line AI is the required perpendicular (621).

To solve the same problem with the triangle, m being the point through which the perpendicular to the line xy is to be drawn, place one edge of a T-square or rule parallel to the line xy, then, using this as a guide, slide the triangle along the edge until the point m coincides with one edge of the triangle, and then draw the line AC, which is the required perpendicular.

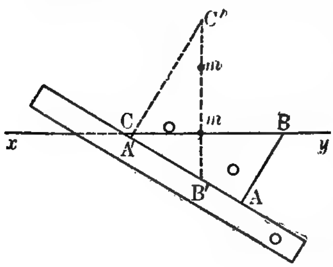


Fig. 159

It is preferable to make the hypotenuse of the triangle coincide with xy; place the edge of the rule against the leg, and, holding the rule fast, place the triangle with its other leg against the rule and its hypotenuse on the point m, and then draw the perpendicular $C'B'$.

From the construction given in Fig. 162 we have the method of drawing the perpendicular bisector of a line AB.

945. *To erect a perpendicular at the extremity* B *of a line*

AB which can not be prolonged. From any point O without AB as a center and OB as a radius, describe an arc DBC (Fig. 160); from D draw a diameter DOC of the circle, then the line BC is the required perpendicular (684). The perpendicular may also be drawn with the triangle (944).

ANOTHER CONSTRUCTION. With B as a center and any con-

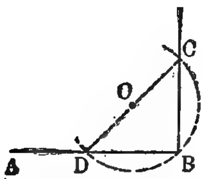


Fig. 160

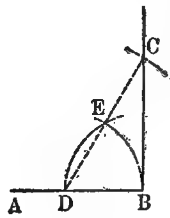


Fig. 161

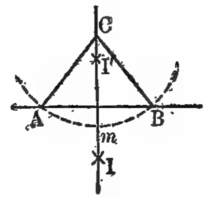


Fig. 162

venient radius, describe an arc of a circle; from the point D as center, with the same radius, describe an arc which cuts the first in E; draw the line DEC and lay off with the compass the distance $EC = DE = EB$; connecting C with B, we have the required perpendicular BC. For, if from E as center a semicircle were described with radius equal to EC, all three points, D, B, C, would lie on the circumference; therefore the angle DBC is inscribed in a semicircle and is a right angle (684).

946. To bisect: *First,* a straight line AB; *Second,* an arc AmB; *Third,* an angle at the center ACB, corresponding to the arc AmB. From A and B as centers describe arcs intersecting in I' and I; draw the line II', which is the perpendicular bisector of AB, and fulfills the three conditions (621, 671, 672).

Repeating the same construction, each half of AB may be bisected, which will divide the line into four equal parts; these parts may also be bisected, and so on; therefore this construction may be used to *divide a line into 2^n equal parts,* n being a whole number (967).

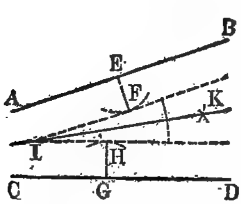


Fig. 163

947. *To bisect an angle whose sides AB, CD, do not intersect.* At any distance $EF = GH$, draw parallels to the sides AB and CD; the angle between these lines is the same as that between the given lines AB, CD (630), therefore the bisector IK fulfills the conditions of the problem (946).

948. *Through a point A, exterior to a given line CD, draw a parallel to the given line.* With A as a center and any convenient radius describe an arc BE; then, with the same radius and the point B on the line CD as center, describe the arc AC; on the arc EB lay off $EB = AC$ and draw a line AE through the given point A and the point E; this line is parallel to CD (625 672).

If the line CD is long enough (Fig. 165), the arc described from the point B as a center may be prolonged to cut CD again in D; then taking $ED = AC$ and drawing AE, we have the required parallel (676).

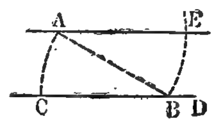


Fig. 164

The solution of the same problem with the triangle. Make the hypotenuse coincide with the line CD; place the rule against one leg and slide the triangle along the rule until the hypotenuse comes to the point A, then draw $E''B''$, which is the required parallel (625).

Any other position $E'B'$, taken by the hypotenuse during its movement from CD to $E''B''$, is also parallel to CD.

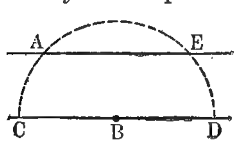


Fig. 165

949. *Through a given point A* (Fig. 167), *to draw a line through the vertex of an angle whose sides BC, DE, do not intersect.* Join A to two points B and D, taken on the sides of the angle, and draw BD; then drawing CE parallel to BD, CF to BA, and EF to DA, the line which joins A and F passes through the vertex. This construction may also be used when the point A is not included by the sides.

950. *To find the point C common to the line DE and the broken line ACB, which is the shortest distance from A to B by way of the line CD* (Fig. 168). From the point A drop a perpendicular to DE and take $DA' = DA$; then the straight line $A'B$ determines the point C. Thus, $AC + CB < AC' + C'B$.

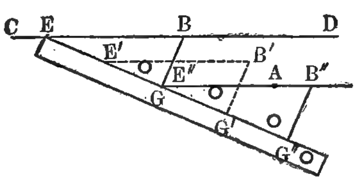


Fig. 166

Having $AC = A'C$ and $AC' = A'C'$ (621), we have $AC + CB = A'B$ and $AC' + C'B = A'C' + C'B$; and since $A'B < A'C' + C'B$ (601), $AC + CB < AC' + C'B$.

An elastic body or a ray of heat or light coming from A and

being reflected by DE to B takes the shortest path ACB. It is to be noted that CA and CB make equal angles with DE, therefore the perpendicular CF erected at C bisects the angle ACB. The angle ACF is called the *angle of incidence*, and is equal to the *angle BCF of reflection.* To hit a billiard ball A (Fig. 168) with another B, by shooting the ball against the cushion DE, the player constructs mentally $DA' = DA$, and aims at A'.

If DE is the bank of a river from which two factories A and B (Fig. 168) are to receive water through a single intake, C is the location of the intake which will require a minimum length of pipe.

If it is desired to hit the cushion twice with the ball B before hitting the ball A as shown in Fig. 169, wherein MN and NP

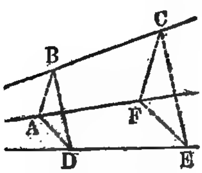


Fig. 167

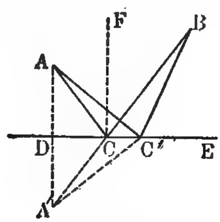


Fig. 168

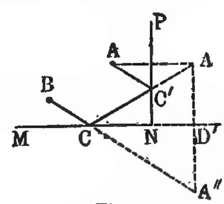


Fig. 169

are the cushions: on AA' perpendicular to NP take $DA' = DA$, and on $A'A''$ perpendicular to MN take $D'A'' = D'A'$; aiming at A'', the ball is reflected at C toward A', and then from C' to A.

CIRCLES—TANGENTS

951. *The circumference of a circle cannot be developed geometrically* (752); but with the following construction, by adding three times the diameter and one-fifth of one side of the inscribed square, gives a straight line MN equal to the circumference by less than two ten-thousandths of the diameter.

Commencing at the point A, lay off the radius of the circle six times in the direction of the diameter AB; draw OC perpendicular to AB; and AC, the side of the inscribed square; join C to the 5 in the line AN and draw DO parallel to $C5$, then lay off $AM = AD$, a fifth of AC.

Since

$$AC = \frac{AB}{2}\sqrt{2} = 1.414\cdots \times \frac{AB}{2} \quad (709), \qquad AM = 0.1414\cdots \times AB,$$

and consequently $\quad MN = 3.1414\cdots \times AB.$

952. *Describe a circle passing through three given points A, B, C, not in a straight line* (680). Draw AB and BC; erect perpendiculars at their middle points E and F, which intersect in O; the circle described with O as a center and a radius equal to AO fulfills the conditions (946).

To find the center of a circle draw two chords AB and BC,

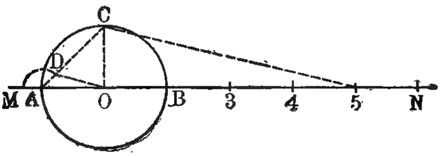


Fig. 170

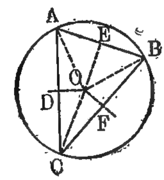


Fig. 171

and the perpendiculars erected at the middle points will intersect in the center of the circle.

In the same manner the center and the radius of an arc of a circle may be determined.

The above construction furnishes a means of circumscribing a circle about a given triangle ABC (688).

953. *To inscribe a circle in a given triangle ABC*, draw in the bisectors AO and BO of two angles A and B of the triangle (946); from the point of intersection O of these bisectors drop a perpen-

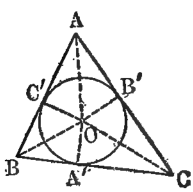


Fig. 172

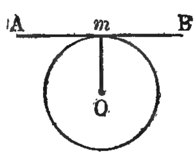


Fig. 173

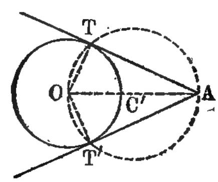


Fig. 174

dicular OC' to one of the sides AB of the triangle and OC', is the radius of the inscribed circle and O the center (622, 687).

In drawing the bisectors of the exterior angles of a triangle (Fig. 65), the centers of three escribed circles are found and their radii are determined in the same manner as that of the inscribed circle.

954. *Draw a tangent to a circle:* First, *through a point in the circumference;* Second, *through a point taken outside the circumference.*

1st. Draw a radius Om (Fig. 173), passing through the given

point m; the perpendicular AB erected to this radius at the point m is the required tangent (675, 944).

2d. Join the given point A to the center (Fig. 174); on OA as a diameter describe a circle cutting the given circle in T and T', then AT and AT' are the two tangents which satisfy the conditions. Drawing the radii OT and OT', the angles OTA and $OT'A$ are right angles, being inscribed in a semicircle (684), and therefore AT and AT' are tangents to the circle (1st).

955. *Draw a tangent to a circle making a certain angle with a given straight line.*

1st. If the angle is zero, that is, if the tangent is parallel to the given line, draw a diameter perpendicular to the given line, and the perpendiculars erected at the extremities of this diameter will satisfy the given conditions (954).

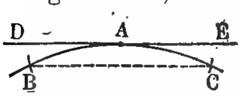


Fig. 175

2d. If the given angle is not zero, draw a line making the required angle with the given line, then draw two tangents parallel to this line as in (1st).

3d. If the tangent is to be perpendicular to given line, we have a special case of (2d), where the given angle is a right angle.

956. *Through a given point in an arc of a circle, the center of which is not known, draw a tangent to the arc.* Find two points B and C (Fig. 175) equally distant from the point A; drawing a straight line DE through A parallel to BC, we have the required tangent.

957. *Draw a tangent common to two circles C and C'.* About the center C of the larger circle describe a concentric circle having a radius equal to the difference of the radii of the two given circles; through the point C' draw the two tangents $C'T$ and $C'K$ to the constructed circle; draw radii through the points of contact and prolong them to the circumference of the given circle A and B; drawing AA' parallel to $C'T$ and BB' parallel to $C'K$, we have the two common tangents, since they are perpendicular to the radii CA, $C'A'$, and CB, $C'B'$.

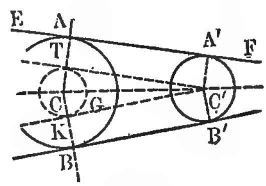


Fig. 176

If CT is taken equal to $CA + C'A'$, and Fig. 177 is constructed according to the same method as the above, the internal tangents are obtained (696).

REMARK. When the two circles are externally tangent, the

two internal tangents coincide and become one, and there are only three solutions of the problem.

If the circles are internally tangent, there is only one solution, and that is the common exterior tangent at the point of contact of the circles.

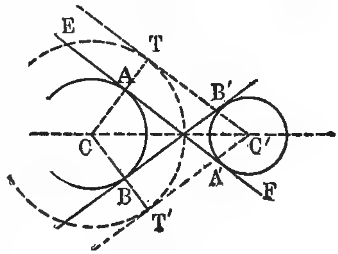


Fig. 177

When the circles intersect, there are no internal tangents, but the two external remain.

Another construction for drawing a common tangent to two circles. Draw two radii parallel and in the same direction, CE and $C'E'$; through the points E and E' draw EE', intersecting the line of centers in O, and the tangents OA' and OB' to one of the circles are tangent to the other. The radius $C'E''$ being opposite in direction from the one CE parallel to it, drawing the line EE'', the tangents to one circle which pass through the point O' are also tangent to the other.

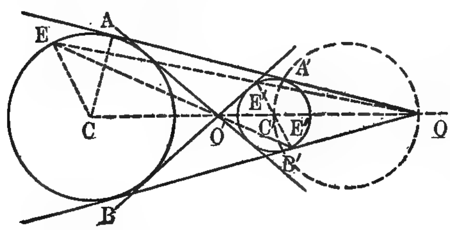


Fig. 178

Each of the circles described as OC', OC, $O'C'$, $O'C$, as diameters, determines two points of contact of the four tangents.

The radii, such as CA, $C'A'$, drawn to the points of contact of the same tangent, are parallel to each other and perpendicular to the tangent.

When the two circles are equal, the external tangents are parallel to the line of centers CC', and perpendicular to the extremities of the diameters which are perpendicular to the line of centers CC'; their point of intersection is at infinity. As to the internal tangents, their point of intersection O' is equal to the distance between centers.

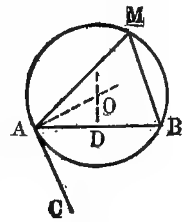


Fig. 179

958. *On a given straight line AB, construct a segment capable of containing a given angle.* At the point A form the angle BAC equal to the given angle (936); draw AO perpendicular to AC and DO perpendicular to AB at its middle point (946); with the point O as center, and OA for

radius, describe a circle, and the segment AMB is the required segment, since any angle M inscribed in this segment is clearly equal to BAC (684), the given angle.

In practice, *to construct a segment capable of containing a given angle AMB, or to describe an arc of a circle passing through three given points A, M, B* (952), an instrument is used, which is made of two rules hinged together and carrying a pencil at the joint M. Placing two pins in the points A and B, and spreading the instrument to correspond to the given angle AMB, the instrument is slid around, with its sides pressing against the pins, and in doing this the arc is described by the pencil in the vertex of the angle AMB.

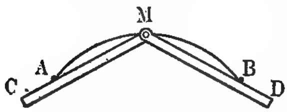


Fig. 180

959. *Describe a circle tangent to a given circle O, and to a straight line XY, in a given point P.*

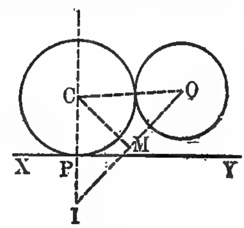


Fig. 181

At P erect a perpendicular PC to XY; take PI equal to the radius of the circle O, and draw IO; erect the perpendicular bisector MC of IO, which meets PC in C, the center of the required circle. The circle C is tangent to XY and to the circle O, since the distance $CO = CI$ of the centers equals the sum $CP + PI$ of the radii (681).

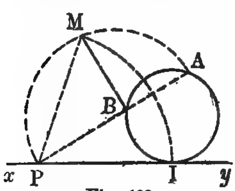


Fig. 182

960. *Draw a circle through two points A, B, tangent to a given straight line xy.* Draw AB and determine the mean proportional PM between PA and PB (970); take $PI = PM$, and draw a circle passing through the three points A, I, B, which fulfills the conditions of the problem. Having $PI^2 = PA \times PB$, xy is tangent to the circle in I (708).

961. *Draw a circle tangent to two given straight lines, and passing through a given point A.* The center of the circle is found to be on the bisector of the angle S, formed by the two given lines. The circumference passes also through a point A', symmetrical to A. Thus the problem is similar to the one preceding, and has two solutions, O and O'.

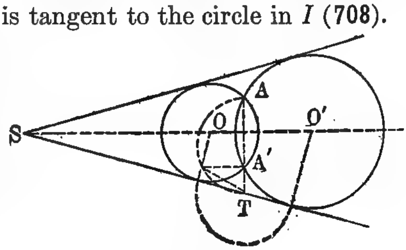


Fig. 183

962. *Draw a circle tangent to two given straight lines CD and EF, and to a given circumference O.*

At a distance $T'T''$ equal to the radius of the given circle O, draw parallels to the lines CD and EF, and thus determine the center I of the circle tangent to $C'D'$ and $E'F'$ and passing through the point O (**961**). I is also the center of the required circle, and IT its radius. The problem has four solutions: two in which the circumference O is externally tangent, as in Fig. 184, and two others where it is internally tangent, which are obtained by drawing the parallels $C'D'$, $E'F'$, inside of the given lines CD and EF.

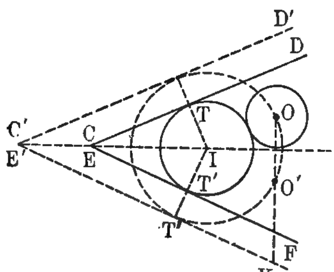


Fig. 184

963. *Draw a circle through a given point K, tangent to a given circle O and to a given straight line MN.*

SOLUTION 1. Suppose the problem solved, then (**708**)

$$BP : BI = BC : BK,$$

and from the similar triangles, BCA, BSI, we have

$$BA : BI = BC : BS.$$

These two proportions having the same means, the extremes give

$$BP \times BK = BA \times BS, \text{ and } BP = \frac{BA \times BS}{BK}.$$

Thus drawing a perpendicular to MN through the center O, and drawing BK; BA, BS, and BK, and the fourth proportional BP

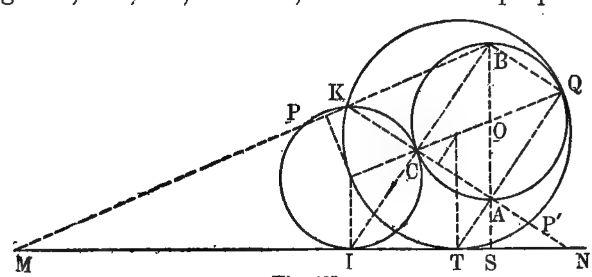


Fig. 185

to these three lines (**969**), give a second point P in the circumference of the required circle. The problem is therefore brought to that of article (**960**).

Solution 2. Supposing the problem solved, and joining
to the point K and to the point of contact Q, we have (707):

$$AP' : AT = AQ : AK,$$

and the two similar right triangles, AST and AQB, give:

$$AB : AT = AQ : AS.$$

From these two proportions,

$$AP' \times AK = AB \times AS, \text{ and } AP' = \frac{AB \times AS}{AK},$$

a relation which determines a second point P' in the circum ference of the required circle, and brings the problem to the so lution in article (960).

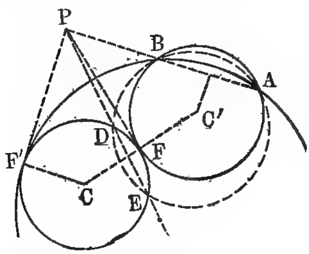


Fig. 186

964. *Draw a circle through tw given points A, B, and tangent to given circle C.* Draw a circle througl the two points A and B, and cuttin the given circle in any two point D and E. Draw the chords AB, ED and prolong them until they meet a P. From P draw a tangent PF to th circle C (954), then F is the point o contact of the circle C with the required circle C', which is con structed by passing a circle through the three points A, B, F (952)

Since (708):

$$\overline{PF}^2 = PE \times PD \quad \text{and} \quad PE \times PD = PA \times PB,$$
$$\overline{PF}^2 = PA \times PB,$$

therefore PF is also tangent to C' at F, and the circles are tan- gent to each other at that point. The tangent PF' gives a second solution, the circle passing through the three points A, B, F'.

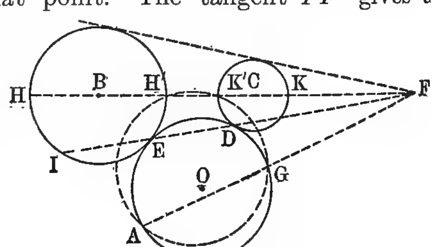


Fig. 187

965. *Draw a circle through a given point A, and tangent to two given circles B and C.* Deter- mine the center of sym- metry F, of the two circles B and C, and draw AF; pass a circle through H', K', and A (952), which cuts AF in a second point G; then, passing a circle O

through A, G, and tangent to one of the given circles B or C, it is tangent to the other and fulfills all the conditions. The point F and the points of contact E and D are on the same straight line, being three centers of symmetry (698).

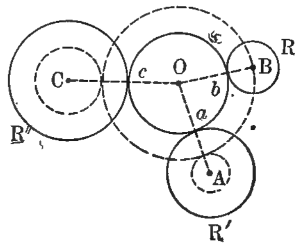


Fig. 188

966. *Draw a circle tangent to three given circles.*

From the points A and C as centers and with $R' - R$ and $R'' - R$ as radii, describe two circles; describe a third circle passing through B and tangent to the first two auxiliary circles (965). The center of this third circle is that of the required circle, which is described with a radius equal to $Oa = Ob = Oc$.

PROPORTIONAL LINES—SIMILAR POLYGONS

967. *Divide a straight line AB:* First, *into parts proportional to given lines E, F, G;* Second, *into parts proportional to given numbers;* Third, *into equal parts* (301).

1st. Through one of the extremities of AB draw an indefinite straight line AX, making any convenient angle with AB; on AX lay off $AR = E$, $RQ = F$, and $QP = G$; join PB, and through the points R and Q draw parallels to PB, then they divide the line AB in such a manner that,

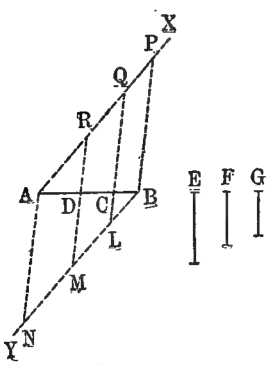


Fig. 189

$$\frac{AD}{AR} = \frac{DC}{RQ} = \frac{CB}{QP} \text{ or } \frac{AD}{E} = \frac{DC}{F} = \frac{CB}{G}. \text{(693)}$$

2d. Having chosen the length which is to represent unity, and having taken AR, RQ, ... proportional to the given numbers, by the same method as in (1st), the line AB is divided into parts proportional to these numbers.

3d. If the lengths laid off on AX are all equal, AB is divided into equal parts (946).

A convenient method of dividing a number of lines into any number of equal parts, is to divide a straight line into equal parts and draw perpendiculars to the line through these points

of division; then with a compass take some point on the first parallel and describe an arc of a radius equal to the given line, cutting the parallel whose number corresponds to the number of equal parts which it is desired to divide the given line into. In Fig. 190 the line AB is divided into seven equal parts.

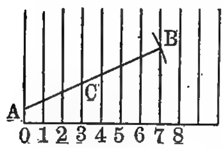


Fig. 190

In practice, to divide a straight line AB into a certain number of equal parts, 7 for example, the method of trial and error is most often used. The dividers are set to what one judges to be one-seventh the length of the given line AB and the distance stepped off on the line. Suppose C is the last point of division which shows that the opening of the dividers was too small, so the opening must be increased as near one-seventh of CB as the operator can judge. Let us now suppose that the seventh point falls outside in C', then the opening must be decreased as near one-seventh of BC' as the operator can judge, etc., until the seventh step of the dividers coincides with the extremity B of the line. With a little practice, three or four trials will give a result which is near enough exact.

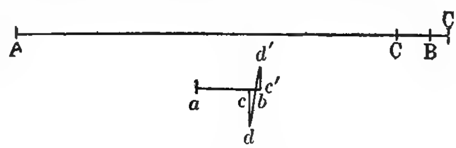


Fig. 191

Taking ac equal to the first opening of the dividers, and ac' equal to the second; then taking $cd = BC$ and $c'd' = BC'$, perpendicular to ac, and joining d and d', the length ab may be considered as one-seventh AB (Fig. 191).

Remark 1. This method of trial and error is used above all in dividing arcs or circumferences into any number of equal parts.

Remark 2. When the number of divisions may be reduced to several factors, such as $28 = 4 \times 7$, the line may be divided, first into 4 parts, and then each of these into 7.

968. To obtain any fraction of the length of a line, $\frac{3}{7}$ for example, divide the line into 7 equal parts and take 3 of them.

In Fig. 189, taking $AR = 3$ times and $AP = 7$ times some arbitrary length, we have $AD = \frac{3}{7}AB$.

When it is desired to construct a great number of lines which bear a constant ratio to a certain number of given lines, such as is the case when a figure is to be enlarged, the method of Fig. 190 will be found convenient.

Let it be required to construct a figure similar to another in a ratio of 7 : 3, placing any of its dimensions AB between the parallels O and 7, AC is the homologous dimension of the similar figure, and in this same manner all the dimensions are found.

The angle of reduction may be used to solve the same problem. For example, to reduce a figure in a ratio of 7 : 3, take AB

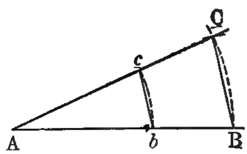


Fig. 192

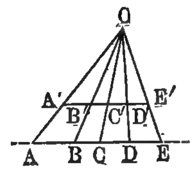


Fig. 193

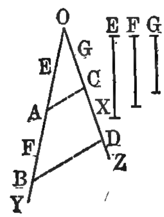


Fig. 194

$= 7$ times any convenient length; from the point A as center, with AB as radius, describe the arc BC, and lay off the chord $BC = 3$ times the length which is $\frac{1}{7}$ of AB; then draw AC. From the point A as center, and a radius Ab equal to one of the dimensions of the figure to be reduced, describe an arc bc. The chord of this arc, that is, the parallel bc to BC, is the homologous dimension of Ab. Thus we have (693):

$$bc : Ab = BC : AB = 3 : 7.$$

This problem may also be solved by the theorem (694) of a number of straight lines which meet in a point and are cut by two parallel lines AE, $A'E'$. The lengths OA and OA' being taken according to the ratio of symmetry, laying off the different dimensions on AE and drawing OB, OC, . . . the segments $A'B'$, $A'C'$, . . . are respectively the homologous dimensions of the first figure:

$$OA : OA' = AB : A'B' = AC : A'C' \ldots$$

969. *Find the fourth proportional X of three given lines E, F, G* (328) (Fig. 194).

On two straight lines OY and OZ, making a certain angle with each other, take $OA = E$, $AB = F$, and $OC = G$; join A and C, and draw a parallel BD to the line AC, then $CD = X$. Thus we have (693):

$$\frac{OA}{AB} = \frac{OC}{CD} \text{ or } \frac{E}{F} = \frac{G}{X}.$$

Instead of taking the lines one after the other, they may all be laid off from O. Thus OD is the fourth proportional of the three lines OA, OB, OC:

$$OA : OB = OC : OD.$$

The fourth proportional may also be obtained by means of

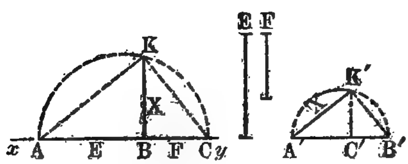


Fig. 195

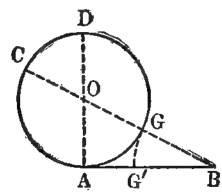


Fig. 196

the angle of reduction (Fig. 192); bc is the fourth proportional of the three lines AB, BC, Ab:

$$AB : BC = Ab : bc.$$

The figure (193) also gives the fourth proportional $A'B'$ of the three given lines OA, OA', AB:

$$OA : OA' = AB : A'B'.$$

970. *Find a mean proportional X between two given straight lines E and F* (305) (Fig. 195).

On a straight line xy, lay off $AB = E$ and $BC = F$; on AC as a diameter describe a semicircle, and the perpendicular $KB = X$. Thus, we have (706):

$$AB : KB = KB : BC, \text{ or } E : X = X : F \text{ and } X^2 = E \times F.$$

Taking $A'B' = E$ and $A'C' = F$, and describing a semicircle on $A'B'$ as a diameter, and erecting the perpendicular $C'K'$, we have $A'K' = X$ (706).

971. *Divide a straight line AB into extreme and mean ratio* (692) (Fig. 196).

At one of the extremities A of the given line erect a perpendicular $AO = \frac{AB}{2}$; from the point O as center, with OA as radius, describe a circle; draw BO, and taking $BG' = BG$ the point G' satisfies the conditions of the problem. Thus, we have (708):

$$BC : AB = AB : BG;$$

and

$$(BC - AB) : AB = (AB - BG) : BG, \qquad (349)$$

that is,

$$BG' : AB = AG' : BG',$$

or

$$AB : BG' = BG' : AG'; \qquad (345)$$

which was to be proved.

972. *On a side $A'B'$, given as the homologous side of a side AB of a polygon ABCDE, construct a second polygon, similar to the first* (695).

Construction 1. Make angle $B' = B$ (936); take $B'C'$ equal to the fourth proportional of the three sides AB, $A'B'$, BC (969); make angle $B'C'D' = BCD$; take $C'D'$ equal to the fourth proportional of the three sides AB, $A'B'$, CD, and so on.

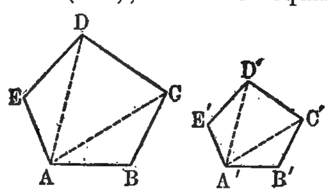


Fig. 197

The fourth proportionals may be obtained very rapidly by the methods shown in Figs. 192 and 193, or with a pair of reducing compasses. Having the lengths of the sides, the polygon may be constructed by drawing its sides, with the triangles, parallel to the homologous sides of the given polygon (948).

Construction 2. The construction of the equal angles and the fourth proportional may be avoided, by dividing the polygon into triangles and constructing the similar triangles in succession by drawing lines parallel to the sides of the original ones. Thus, drawing $A'B'$ parallel to AB, and $B'C'$ and $A'C'$ respectively parallel to BC and AC, the first triangle $A'B'C'$, similar to ABC, is completed, and in a like manner triangle ACD is constructed, and so on until the polygon $A'B'C'D'E'$ is completed (702).

CONSTRUCTION 3. Starting at A on AB, lay off $A'B'$; through the point B' draw a line $B'C'$ parallel to BC, and at the point where this line cuts the diagonal AC, draw a line $C'D'$, parallel to CD, and so on until the polygon is completed.

CONSTRUCTION 4. Sometimes it is desired to trace a polygon which has been surveyed, by locating the points C', D', E', with reference to one side $A'B'$. Take $A'B'$ as the common base, and C', D', E', as the vertices of the triangles which form the polygon.

REMARK 1. Supposing $A'B' = AB$, the preceding constructions give a polygon equal to the given polygon.

REMARK 2. The principle in Fig. 71 may be advantageously employed to construct a polygon similar to a given polygon.

973. *Find the greatest common measure of two given commensurable straight lines* AB and CD (213).

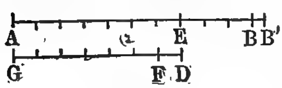


Fig. 198

Apply the rule in (102) to determine the greatest common divisor of two numbers. Thus the shorter line CD is laid off on the longer as many times as possible, which in this case is once plus the remainder $EB < CD$; EB is now laid off on CD, twice plus $FD < EB$; then the remainder FD, on EB; and since it is exactly contained in EB, FD is the greatest common measure.

$$EB = 3\,FD,$$
$$CD = 2\,EB + FD = 6\,FD + FD = 7\,FD,$$
$$AB = CD + EB = 7\,FD + 3\,FD = 10\,FD.$$

Therefore the ratio is:

$$\frac{AB}{CD} = \frac{10}{7}.$$

In the same manner the ratio of two commensurable arcs, having the same radius, may be found.

Find the ratio of two commensurable angles.

Draw the arcs corresponding to the same radius, and the ratio of these arcs will be the same as that of the angles.

974. *Find the ratio of two straight lines* AB', CD (Fig. 198), *correct to at least one-seventh, for example.* Divide CD into seven equal parts (967); laying off a seventh FD on AB', it is found

that it is contained 10 times plus a remainder $BB' < FD$; consequently we have:

$$\frac{AB'}{CD} > \frac{10}{7} \quad \text{and} \quad \frac{AB'}{CD} < \frac{11}{7}.$$

Both of these ratios satisfy the condition.

In the same manner the nearest value of the ratio of two arcs or angles may be found (973).

THE DIVISION OF CIRCLES INTO EQUAL PARTS—REGULAR POLYGONS

975. *Divide a circumference into* 2, 4, 8 . . . 2^n *equal parts.*

This may be done as described in article (946), but the following method with the triangle is much more expeditive.

Draw a diameter AB with the triangle, placing a rule or another triangle against the first; slide it clear of the circle, then, holding it fast, apply the leg of another triangle against the side, and draw the diameter CD perpendicular to AB, thus dividing the circumference into four equal parts. Having chosen a 45° triangle for the second diameter, rest its hypotenuse against the first triangle, and with its legs draw the diameters EF and GH, which make an angle of 45° with the others, thus dividing the circumference into eight equal parts.

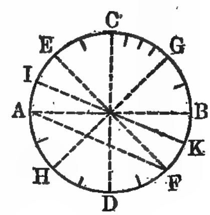


Fig. 199

Drawing the diameter IK parallel to the chord AF, and repeating the operations which were performed in dividing the circle into eight parts, the circumference will be divided into sixteen equal parts. Repeating the operation twice, starting from the diameter parallel to AK and then IF, the circumference will be divided into thirty-two equal parts. This division is indicated on the arc CG. In the same manner the circumference is divided into 2^n equal parts.

In practice, after having divided the circumference into 4 or 8 equal parts, it is often more convenient to make the subdivisions by trial and error with the dividers.

976. *Divide a right angle A or its corresponding arc BC into three equal parts.*

From the points B and C as centers, and $AB = AC$ for radius, describe arcs of a circle which cut the arc BC in the required points of division, and drawing AD and AE, these lines divide the angle A into three equal parts.

The triangle ACD being equilateral, the angle DAC is equal to two-thirds of a right angle, and therefore the angle BAD is equal to one-third of a right angle. For the same reason CAE is equal to one-third of a right angle, and the angle BAC is indeed divided into three equal parts, as is also the arc BC. The trisection of an angle is possible when the angle is a right angle (1017).

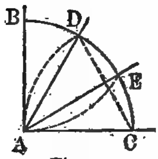


Fig. 200

977. *Divide a circumference into* 12 *equal parts.*

Draw two diameters AB, CD, perpendicular to each other (975), and from the extremities of these diameters with the radius of the circle describe arcs which divide each quadrant into three equal parts (976), and consequently the entire circumference into 12 equal parts.

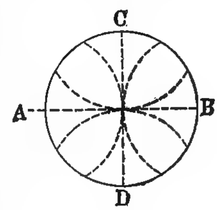


Fig. 201

978. *Divide a circumference into six equal parts.*

This is done by inscribing the radius six times in succession; the vertices of the inscribed hexagon form the points of division required (744). The circumference being divided into six equal parts, by taking every other one, it is divided into 3 equal parts.

Divide a circumference into 6 *equal parts with a* 60° *triangle.*

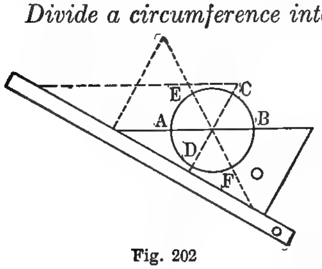


Fig. 202

Draw the first diameter with the hypotenuse of the triangle; then draw the diameter CD by placing the triangle in the position shown by the dash lines; reversing the triangle and giving it the position shown by the dotted lines, the diameter EF is drawn, which completes the division of the circumference into 6 equal parts. Drawing two diameters perpendicular to each other, and operating on each as was done with AB in the preceding demonstration, the circumference will be divided into 12 equal parts.

In practice, where a circumference is to be divided into a

number of equal parts, which number is a multiple of 3 or 6, it is convenient, after having divided it into 3 or 6 parts, to make the subdivisions by trial and error with a pair of dividers (967).

979. *Divide a circumference into 5 equal parts.*

Draw a diameter AB and a radius CD perpendicular to the diameter (944); from the middle point E, of CB as center and the distance ED as radius, describe an arc and draw the chord DF; using DF as a side, inscribe a regular pentagon in the circle. The vertices will then divide the circumference into 5 equal parts.

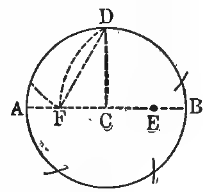


Fig. 203

In practice, it is preferable to use the dividers and the method of trial and error. The fifth of a circumference being exactly 72°, a protractor may also be used to good advantage.

980. The method of trial and error, with the dividers, may be employed to divide a circumference into any number of equal parts (967); but if the number is a multiple of 3, 4, or 6, it is used only for the subdivision.

When the number of parts divides 360 exactly, and if the decimal part of the quotient is equal to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ of a degree, the protractor may be used advantageously. Its center is made to coincide with the center of the circle, and arcs equal to 360 divided by the required number of divisions are laid off; then, with a rule, these points are joined to the center and divide the circumference into the same number of equal parts. This method is particularly advantageous where the number of divisions is great.

For the first trial in the method trial and error, the dividers should be set by the protractor as near the correct value as possible.

981. *Inscribe a square in a given circle.* Draw the diameters AC and BD perpendicular to each other, and joining the extremities of these diameters, the inscribed square $ABCD$ is obtained (740, 944).

The use of the 45° triangle (975) permits of a rapid solution of this problem. Resting one leg of the triangle against another triangle, one diameter may be drawn along the hypotenuse; then, reversing the triangle, the other diameter is drawn. The sides AD, BC, can be drawn along the other leg; then, using the 45°

triangle as a guide, the second can be slid up to draw the sides AB and DC.

982. *Construct a square whose side is given.* With the first triangle draw a straight line AB equal to the given side; then, by sliding on another triangle, lower the first parallel to itself; then with a 45° triangle, the figure is constructed as in the preceding problem. The proof is made by describing a circle, with the intersection O of the diagonals as a center, and OA as a radius; then the circumference of this circle should pass through vertices B, C, D.

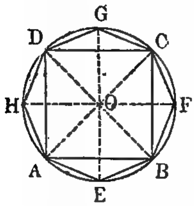


Fig. 204

983. *Inscribe a regular octagon in a given circle.* Having inscribed a square $ABCD$ in the circle (981), divide each of the arcs subtended by these sides into two equal parts (946), and, joining these points of division to the adjacent vertices of the square, the octagon $AEBFCGDH$ is obtained.

If an octagon had been inscribed in the circle, joining every other vertex, a square would have been obtained.

REMARK. From the above it may be deduced that, in general, *having a regular polygon inscribed in a circle, to inscribe a polygon of double the number of sides,* bisect the arcs subtended by the sides and connect these points to the extremities of the chords.

Having a polygon inscribed in a circle, to inscribe another of half the number of sides, connect every other vertex.

A regular octagon may be inscribed in a circle without inscribing a square. Operating as in (981) with the 45° triangle, the circumference may be divided into 8 equal parts, which is indicated by the 4 diameters HF, EG, AC, BD, although it is not necessary to draw them; the sides may also be drawn with the triangle; noting that the side AE is parallel to the chord HB, and commencing at this chord, the four sides AE, GC, HD, BF, are drawn; then, starting at the chord AF, the four other sides EB, DG, AH, FC, are drawn.

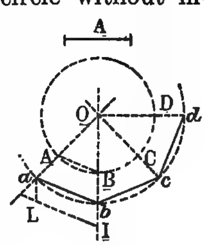


Fig. 205

984. *Draw a regular octagon when one side A is given.*

Describe a circle with any radius OA; in this circle inscribe a regular octagon (983), or simply one side AB of this octagon;

through the point I, taken on the prolongation of one of the radii OA, OB, draw IL parallel to AB and equal to the given side A of the octagon; then draw La parallel to OB and intersecting OA in a. From O as a center, and with Oa as a radius, describe a circle, and the octagon $abcd$ . . . inscribed in this circle is the one required.

This construction applies to all regular polygons which may be geometrically inscribed in a circle, but it may be greatly simplified for some polygons.

Thus for an octagon, after having drawn the straight lines OA and OB, making an angle of 45°, take $OA = OB$; through a point I draw IL parallel to AB, and continue as in the preceding example.

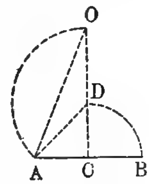


Fig. 206

Erecting a perpendicular CO at the middle of the side AB of the octagon which is to be constructed, take $CD = CB$ and $DO = DA$; the point O is the center of a circle which may be circumscribed about the octagon in question, which is then easily constructed (983). Angle $ODA = DCA + DAC = 90° + 45° = 135°$ (653);

$$\angle AOC = \frac{180° - 135°}{2} = \frac{45°}{2};$$

therefore $$\angle AOB = 45° = \frac{360°}{8}.$$

985. *Inscribe a regular hexagon in a given circle.*

Laying the radius of the given circle off successively as chord, these six chords will form the six sides of a regular inscribed hexagon $ABCDEF$ (978) (Fig. 207).

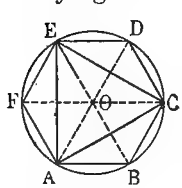


Fig. 207

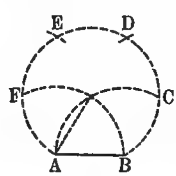


Fig. 208

A hexagon may be inscribed with a 60° triangle in the same manner that an octagon was inscribed with a 45° triangle (983). Draw the diameter FC with one triangle, then with another triangle slide this one parallel to itself until it is below the figure; then, resting the short side of the 60° triangle against the first, the diameters EB and AD are drawn, and, joining the extremities of these diameters, we

have the required hexagon; but, noting that each diameter is parallel to two sides of the hexagon, the sides may be drawn in directly with the triangles without drawing the diameters. It is thus that hexagonal bolt-heads and nuts are constructed.

986. *Construct a regular hexagon whose side is given* (Fig. 208).

Describe a circle with the given side for a radius, and inscribe a regular hexagon, which fulfills the conditions of the problem (985).

To construct a regular hexagon on a given straight line AB as a side, from the extremities A, B, as centers, with a radius equal to AB, describe the arcs BF and AC, which intersect in the center O of the circle circumscribed about the hexagon; with the points C and F as centers, and the same radius AB, describe two other arcs, thus obtaining the points D and E; then $DECBAF$ are the vertices of the required hexagon.

987. *Inscribe an equilateral triangle in a given circle.*

Inscribe first a hexagon and join every other vertex; thus the triangle ACE (Fig. 207) is obtained.

988. *Construct an equilateral triangle when one side is given.*

Operate as in (940) and make each side equal to the given side. The 60° triangle may also be used for constructing an equilateral triangle, the 60° angle being equal to the angle of the required triangle.

Having inscribed a hexagon or an equilateral triangle, polygons of 12, 24, 48, ... sides may be successively inscribed as indicated in (951).

989. If perpendiculars are dropped from the vertices of an equilateral triangle upon any diameter DE of the circumscribed circle (Fig. 209), the sum $AF + BG$ of the two perpendiculars on one side of the diameter is equal to the perpendicular CH on the other side.

Drawing the radius CO at C, it is perpendicular to the middle point of AB (621, 671). The rhombus $ALBO$ gives $OI = \frac{OL}{2} = \frac{OC}{2}$, and drawing IK perpendicular to DE, since the triangles IOK and COH are similar, we have $IK = \frac{CH}{2}$. But in the trapezoid $AFGB$ we have:

$$IK = \frac{AF + BG}{2} \text{ (662); therefore } AF + BG = CH.$$

990. *Construct a dodecagon whose side AB is given* (632).

Erect a perpendicular CO at the middle point of the given side AB; from A as a center, with AB as a radius, describe an arc DB, and take $DO = DB = AB$; the point O is the center of the circle circumscribed about the required dodecagon. Angle $ODA = DCA + DAC = 90° + 60° = 150°$ (653) (Fig. 210);

$$\angle AOC = \frac{180° - 150°}{2} = \frac{30°}{2}; \text{ therefore } \angle AOB = 30° = \frac{360°}{12}.$$

991. *Inscribe in a given circle:* First, *a regular decagon;* Second, *a regular pentagon;* Third, *a regular pentadecagon;* Fourth, *a regular polygon of* 30 *sides* (632).

1st. AB being the side of the decagon, the angle at the center $O = \frac{360°}{10} = 36°$. Drawing the bisector AG of the angle A, we have (704):

$$OA : OG = AB : GB.$$

Since $OA = OB$ and $AB = AG = OG$, we have:

$$OB : OG = OG : GB,$$

which shows that the side AB is equal to the longer segment OG of the radius OB, divided in extreme and mean ratio (971).

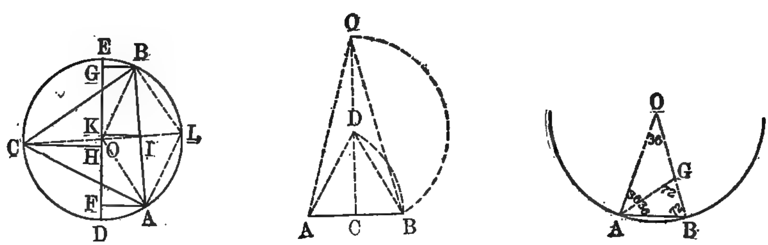


Fig. 209 Fig. 210 Fig. 211

To determine the side of a decagon, draw two radii OA, OB, perpendicular to one another; on OB as a diameter describe a circle; draw AO', and $AD = AC$ is the side of the required decagon, because it is equal to the longer segment of the radius OA divided in extreme and mean ratio.

Laying off the chord AD around the circumference, the required decagon is obtained.

2d. Joining every other vertex of the regular inscribed decagon, a regular inscribed pentagon is obtained (Fig. 213). If it is desired to obtain the side of the pentagon directly, the arc DC may be prolonged to E (Fig. 212), then DE is the required side.

3d. The difference between the arcs subtended by the sides of a regular inscribed hexagon and decagon being equal to $\frac{1}{6} - \frac{1}{10} = \frac{1}{15}$ of the circumference, the chord which subtends this difference is the side of a regular pentadecagon. Having the side and laying it off around the circumference of the circle, the required pentadecagon is obtained.

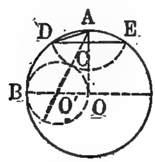


Fig. 212

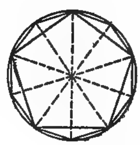


Fig. 213

4th. Having $\frac{1}{5} - \frac{1}{6} = \frac{1}{30}$, it is seen that the side of a regular inscribed polygon of 30 sides is the chord which subtends the arc equal to the difference of the arcs subtended by the sides of the regular inscribed pentagon and hexagon.

To construct a regular decagon on a given side AB, erect a perpendicular CO at the middle point of AB, and at B erect another perpendicular $BD = BC$; take $DE = DB$, and from the point A as center, and a radius equal to AE, describe an arc intersecting the perpendicular bisector of AB in O, the center of the circle which may be circumscribed about the decagon (Fig. 214).

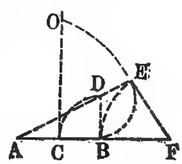


Fig. 214

Drawing EF perpendicular to AE, the two right triangles ABD and AEF are similar; BD being the half of AB, EF is half of AE; furthermore, since $FE = FB$, being tangents drawn from the point F to the same arc, from F as a center and a radius FE, describe an arc through B; thus it is seen that AB is equal to the longer segment of a radius AE divided in extreme and mean ratio.

992. *Inscribe a polygon of any number of sides in a circle.* Divide the circumference into as many parts as the polygon has sides, and join the points of division (967, 975), which will give the required polygon.

To construct a regular polygon of any number of sides, the same method as was used in Fig. 205, the construction of the regular octagon, may always be pursued.

993. *Circumscribe a regular polygon about a given circle.* Inscribe the required polygon in the given circle; draw tangents to the middle points of the arcs subtended by the sides of the inscribed polygon; these tangents are parallel to the sides of the polygon, and form the polygon $A'B'C'\ldots$ which was required. In general, the circumscribed polygon is constructed in the same manner as the inscribed, it being necessary only to divide the circumference into the required number of parts and draw the tangents.

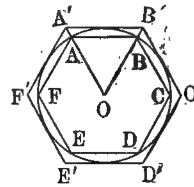


Fig. 115

994. *Inscribe a regular octagon in a given square ABCD.*

Draw the diagonals of the square, and from the vertices A, B, C, D, as centers, and radii equal to OA, describe arcs which determine the 8 vertices of the octagon on the sides of the square.

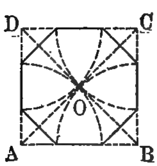


Fig. 116

995. *Cover a plane surface with regular polygons.* The sum of the consecutive adjacent angles which may be formed about a point in a plane being equal to 4 right angles or 360 (618), any regular polygon whose angle is contained a whole number of times in 4 right angles may be used to cover a plane surface (652). Therefore the following may be used:

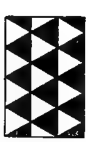


Fig. 217

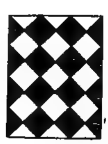


Fig. 218

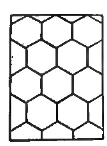


Fig. 219

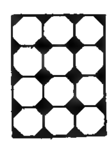


Fig. 220

1st. The equilateral triangle, whose angle $= \frac{2}{3} = \frac{4}{6}$ of a right angle (Fig. 217);

2d. The square, whose angle $= \frac{4}{4}$ of a right angle (Fig. 218);

3d. The regular hexagon, whose angle $= \frac{2 \times 4}{6} = \frac{4}{3}$ of a right angle (Fig. 219).

The angle of a regular octagon, being equal to $\frac{2 \times 6}{8} = \frac{3}{2}$ of a right angle, is not contained a whole number of times in 4 right angles, and consequently an octagon can not be used; but combining an octagon and a square in such a manner that two angles of the octagons and one of the square have the same vertex, we have $\frac{3}{2} \times 2 + 1 = 4$ right angles, which will cover the surface (Fig. 220).

AREAS OF POLYGONS AND CIRCLES

996. *Find the area of any polygon.* The polygon is divided into triangles by drawing all the diagonals through one vertex, or by joining a point taken within the polygon to all the vertices; find the area of each triangle (**718**), and the sum of these results will give the area of the polygon. Ordinarily the polygon is divided into right triangles and right trapezoids by drawing a diagonal and dropping perpendiculars from the vertices upon this diagonal.

997. *To change any polygon $ABCDE$ to an equivalent polygon having one less side.* Whether the polygon be convex (Fig.

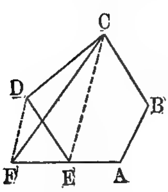


Fig. 221

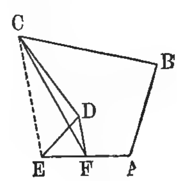


Fig. 222

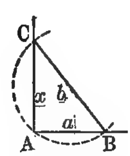


Fig. 223

221), or have a re-entrant angle (Fig. 222), join C and E, draw DF parallel to CE, and join C and F, then the triangles CED and CEF are equivalent (**720**), and consequently the polygons $ABCDE$ and $ABCF$ are also equivalent.

Remark. In this manner any polygon may be transformed into an equivalent triangle.

998. *Construct a square equivalent to the difference of two given squares.*

Draw two straight lines AB and AC perpendicular to each other; on one take $AB = a < b$, where a and b are the sides of

the given squares, and from B as a center, and b as a radius, describe an arc, cutting AC in C, thus determining the side x of the required square. From the right-angled triangle ABC (730):

$$\overline{AC}^2 = \overline{BC}^2 - \overline{AB}^2 = b^2 - a^2.$$

The same result would have been obtained by describing a semicircle on the side $BC = b$ as diameter, drawing in the chord $BA = a$ from B, and connecting A and C (684). Having the side AC, the square is constructed as in article (982).

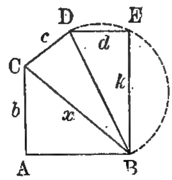


Fig. 224

999. *Construct a square equivalent to the sum or difference of any number of squares, a, b, c, d,* being the sides of the given squares.

Let k be the side of the equivalent square, and

$$k^2 = a^2 + b^2 + c^2 - d^2.$$

Draw two perpendiculars AB, AC, equal to a, b, and join c and B; at C draw $CD = c$ perpendicular to CB, join D and B; on BD as a diameter, describe a semi-circumference, and lay off $DE = d$ as chord; then, joining E and B, the required side k is determined. The successive right triangles give (730, 998):

$$\overline{BC}^2 = a^2 + b^2,$$
$$\overline{BD}^2 = \overline{BC}^2 + c^2 = a^2 + b^2 + c^2,$$
$$\overline{BE}^2 = BD^2 - d^2 = a^2 + b^2 + c^2 - d^2.$$

Having the side k, the square is constructed as in article 982.

1000. *Find the side x of a square which bears a given ratio $m : n$ to a given square a^2.*

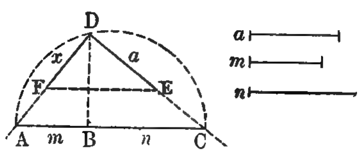


Fig. 225

Take $AB = m$ and $BC = n$; on AC as a diameter, describe a semicircle; at the point B erect a perpendicular BD to the line AC; draw DA and DC, on DC prolonged beyond C if it is necessary; take $DE = a$, and, drawing EF parallel to CA, we have $DF = x$. From (352, 699, 732):

$$x : a = DA : DC \text{ or } x^2 : a^2 = \overline{DA}^2 : \overline{DC}^2 = AB : BC = m : n.$$

If the ratio $m : n$ had been that of two numbers, 3 : 5 for example, take $AB = 3$ times and $BC = 5$ times some length taken as unity.

Construct a square which is a fractional part of a given square. Let $\frac{3}{5}$ be the fraction, that is, the squares are to each other as 3 is to 5. Instead of operating as above, describe a semicircle on AB as diameter, take $AE = \frac{3}{5}AB$ (967); at E erect a perpendicular EF to AB, and draw the chord AF, which is the side of the required square (Fig. 226).

Having (732) $\overline{AF}^2 = AB \times AE$,

$$\overline{AF}^2 = \overline{AB}^2 \times \frac{AE}{AB} \text{ and } \frac{\overline{AF}^2}{\overline{AB}^2} = \frac{AE}{AB} = \frac{3}{5}.$$

1001. *Two similar polygons p and p' being given, construct a third polygon P, which is similar to them and equivalent, 1st, to their sum; 2d, to their difference.*

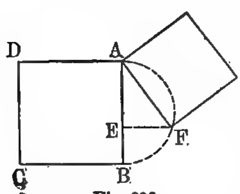


Fig. 226

1st. Construct a right triangle ABC (Fig. 224) with its legs equal to two homologous sides, a and b, of the polygons p and p', and then the hypotenuse will be equal to x, the homologous side to a and b of the similar polygon P; on this side the polygon P is constructed similar to p and p' (972) and is equivalent to their sum.

From (726),

$$p : p' = a^2 : b^2, \quad \text{and} \quad (p + p') : (a^2 + b^2) = p : a^2 \quad (349)$$
$$P : x^2 = p : a^2,$$

these two proportions having three equal terms, $x^2 = a^2 + b^2$, and we have $P = p + p'$.

2d. Taking the longer side, b, as the hypotenuse of the right triangle (Fig. 223), and constructing P on the leg $AC = x$, for the same reasons as in the first case we would have $P = p' - p$.

1002. *Construct a polygon p, similar to a given polygon P, and make the areas bear a given ratio, $m : n$, to each other.*

a being one of the sides of the polygon P, find the side x of the equare, such that $x^2 : a^2 = m : n$ (1000), and on x as a homologous side to a, construct a polygon p, similar to P (972).

In order that the perimeters of the polygons have the ratio $m : n$, we must have $x : a = m : n$ (703, 969).

In order that a circle of a radius x, bear a ratio $m : n$ to a circle

of given radius a, we must have $x^2 : a^2 = m : n$, *and for the circumferences to bear the same ratio,* we must have $x : a = m : n$.

1003. *Construct a square equivalent to a given parallelogram or triangle.* x being the side of the square, and b and h the base and altitude of the given figure, according as the figure is a parallelogram or a triangle, we have (718, 721):

$$x^2 = b \times h \text{ or } x^2 = b \times \frac{h}{2},$$

which shows that x is the mean proportional between the base and altitude in the first case and between the base and half the altitude in the second case (970).

REMARK. From this article and (997), a method for *constructing a square equivalent to any given polygon* may be deduced. Then article (999) gives the means of constructing of a square equivalent to any number of polygons combined in addition or subtraction.

1004. *Construct a rectangle on a given straight line c, equivalent to a given rectangle whose dimensions are a and b.*

The fourth proportional x, of the three lines c, a, b, is the second dimension of the required rectangle (969). From

$$c : a = b : x, \quad \text{we have} \quad c \times x = b \times a. \qquad (339)$$

1005. *Construct a rectangle equivalent to a given square, and the sum of whose dimensions is equal to a given line AB.*

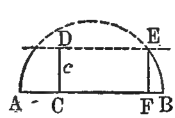


Fig. 227

On AB as a diameter, describe a semicircle; draw the perpendicular CD equal to the side c of the given square, then drawing DE parallel and EF perpendicular to AB, the two segments AF and BF are the dimensions of the required rectangle. From (706):

$$\overline{EF}^2 \text{ or } c^2 = AF \times BF.$$

The problem is only possible when $c < \frac{AB}{2}$, and it is seen that *of all the rectangles of the same perimeter the square is the maximum* (584).

1006. *Construct a rectangle equivalent to a given square, the difference of whose dimensions is equal to a given line AB.*

On AB as a diameter, describe a circle; at one extremity A, erect a perpendicular AC, equal to the side c of the given square,

and drawing CO, the dimensions of the required rectangle are CD and CE. From (708):

$$CD : c = c : CE,$$
$$CD \times CE = c^2.$$

1007. In any quadrilateral $ABCD$:

1st. *The middle points of the four sides are the vertices of a parallelogram MNPQ* (640, 699);

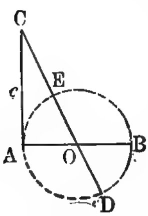


Fig. 228

2d. *The area of the parallelogram MNPQ is equal to one-half that of the quadrilateral ABCD.* This follows from the fact that the four triangles, OMN, ONP, OPQ, OQM, are respectively equivalent to the four triangles, BMN, CNP, DPQ, AQM, having the same base and equal altitudes.

1008. *The lunes of Hippocrates.*

Describing semicircles on the three sides, a, b, c, of a right triangle as diameters, the area of the two shaded lunes is equal to that, $\frac{bc}{2}$, of the triangle.

Noting that the area of the lunes is equal to the sum of the areas of the two semicircles described on the diameters b and c

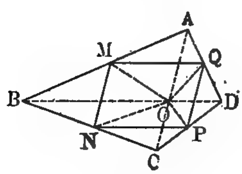


Fig. 229

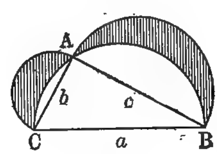


Fig. 230

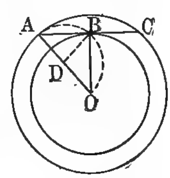


Fig. 231

and the triangle ABC less the area of the semicircle described on the diameter a, we have from (718, 730, 753):

$$S = \frac{bc}{2} + \frac{\pi b^2}{8} + \frac{\pi c^2}{8} - \frac{\pi a^2}{8} = \frac{bc}{2} + \frac{\pi}{8}(b^2 + c^2 - a^2) = \frac{bc}{2}.$$

There are other portions of a circle which may be measured exactly, but they are not contained a whole number of times in the entire circle; if such had been the case, the determination of the quadrature of a circle could have been easily solved (1017).

1009. *The area S of the ring included between the two concentric circles of radii OA and OB, is equivalent to the area* $\pi\overline{AB}^2$ *of a circle whose diameter is equal to the chord AC of the external circle tangent to the interior circle.*

From (730, 753):

$$S = \pi\overline{OA}^2 - \pi\overline{OB}^2 = \pi(\overline{OA}^2 - \overline{OB}^2) = \pi\overline{AB}^2.$$

From this it follows that *in order to divide a circle of radius OA into two equivalent parts by a concentric circle*, draw the chord AC, making an angle of 45° with the radius OA, and the perpendicular OB to AC will be the radius of the required circle.

To divide a circle of radius OA by a concentric circle in such a manner that they bear a certain ratio to each other, for example, so that the area of the internal circle be to that of the ring as 3 : 2, divide OA so that $OD : DA = 3 : 2$ (967); at the point D erect a perpendicular on OA and prolong it to the semi-circumference described on OA as a diameter, then OB is the radius of the internal circle. From (732, 749):

$$3 : 2 = OD : AD = \overline{OB}^2 : \overline{AB}^2 = \pi\overline{OB}^2 : \pi\overline{AB}^2.$$

In dividing OA into a certain number of equal parts and making the same construction for each point of division that has just been made for the point D, the circle of radius OA will be divided into the same number of equivalent parts by the concentric circles.

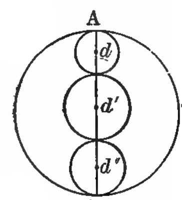


Fig. 232

1010. Dividing the diameter $AB = D$ of a circle into any number of parts, d, d', d'', equal or unequal, *the sum s of the circumferences of the circles which have the diameters d, d′, d″, is constant and equal to the circumference of the circle whose diameter is D.* From (752):

$$s = \pi d + \pi d' + \pi d'' = \pi(d + d' + d'') = \pi D.$$

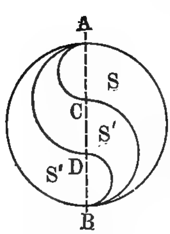


Fig. 233

This is also true for semicircles.

1011. Dividing the diameter $AB = D$ into a certain number of equal parts, 3 for example, upon which as diameters semicircles are described, as shown in Fig. 233, then the circle of diameter D is divided into the same number 3 of equal parts, the perimeter of each being equivalent to the circumference of the circle whose diameter is D (1010), and the area equal to $\frac{1}{3}$ that of the circle of diameter D. Thus, we have,

$$S = S' = S'' = \frac{1}{3}\,\frac{\pi D^2}{4}.$$

1st. Noting that S is equal to a semicircle of diameter $AC = \frac{D}{3}$, plus a semicircle of diameter $AB = D$, less a semicircle of diameter $CB = \frac{2}{3}\,D$, we have from (753):

$$S = \frac{1}{2} \times \frac{1}{4}\,\pi \left(\frac{D}{3}\right)^2 + \frac{1}{2} \times \frac{1}{4}\,\pi D^2 - \frac{1}{2} \times \frac{1}{4}\,\pi \left(\frac{2\,D}{3}\right)^2$$

$$= \frac{\pi D^2}{4}\left(\frac{1}{18} + \frac{1}{2} - \frac{4}{18}\right) = \frac{1}{3}\,\frac{\pi D^2}{4}.$$

2d. S' being equal to twice the remainder obtained in subtracting a semicircle of diameter $AC = \frac{D}{3}$ from a semicircle of diameter $AD = \frac{2}{3}\,D$, we have:

$$S' = 2\left[\frac{1}{2} \times \frac{1}{4}\,\pi \left(\frac{2\,D}{3}\right)^2 - \frac{1}{2} \times \frac{1}{4}\,\pi \left(\frac{D}{3}\right)^2\right] = \frac{\pi D^2}{4}\left(\frac{4}{9} - \frac{1}{9}\right) = \frac{1}{3}\,\frac{\pi D^2}{4}.$$

From (Fig. 233) it is seen that:

$$S'' = S = \frac{1}{3}\,\frac{\pi D^2}{4}.$$

REGULAR POLYHEDRONS AND SPHERES

1012. The figures shown below are the developments of five regular polyhedrons; they show clearly enough how these developments are drawn when a side of the polyhedron is given.

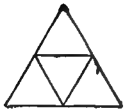


Fig. 234

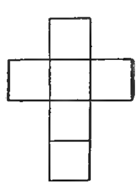


Fig. 235

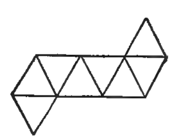


Fig. 236

For (Figs. 234, 236, and 238) the 60° triangle is used. As to the dodecahedron, after having constructed the pentagon P on the length given as one side, the sides of this polygon are

prolonged and a circle drawn through the points of intersection, and by drawing parallels to the sides of the pentagon P one-half of the development is determined. For the second half

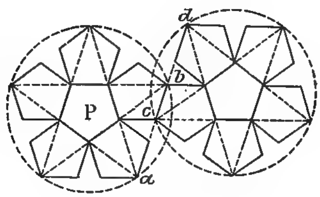


Fig. 237

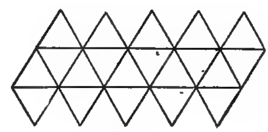


Fig. 238

prolong ab and take $cd = ab$; on cd as a chord describe a circle of the same radius as the first, and in this circle by drawing parallels to the sides in the first half of the development, the construction is completed.

1013. *A sphere being given, find its radius.* Take two points, A and B, on the surface of the sphere; from these points as centers, or rather as poles, with any convenient radius, describe two arcs which intersect in two points, D, D'; with another radius determine a third point, D''. D, D', and D'' being equally distant from the points A and B, they lie in the circumference of a great circle whose plane is perpendicular to the middle of AB, and it follows that if a triangle whose sides are equal to the distances between the three points, D, D', D'' (940), is constructed, that its circumscribed circle will be equal to the great circle of the sphere, and its radius will be equal to that of the sphere (952).

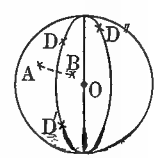


Fig. 239

1014. *Two points, A, B, on the surface of a sphere being given, describe a great circle through the points.*

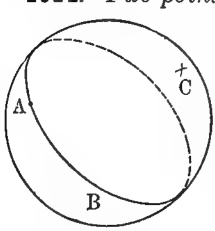


Fig. 240

From the points A and B as poles, with a radius equal to the chord of a quadrant (852, 916), describe two arcs which intersect in the point C, and from this point as a pole with the same radius describe a circle, which is the required great circle.

It is seen that the same construction may be used to find the poles of the circumference or an arc of a great circle.

1015. *Describe a small circle passing through three points, A, B, C, on the surface of the sphere.*

Operating as in (1013), two points, D, D', equidistant from A and B are determined, and through the points D, D', a great circle is described, whose plane is perpendicular to the line AB at its middle point, since it contains the points D, D', and whose center, O, is equidistant from the points A and B (768). In the same manner a great circle is determined whose plane is perpendicular to BC at its middle point; this circle intersects the first in the line PP', and the extremities P and P' are the poles from which as centers the required small circle may be described.

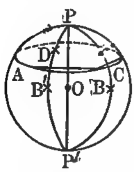


Fig. 241

1016. *Through a point A, taken on the surface of a sphere, draw a great circle perpendicular to the circumference or arc of another great circle BD.*

From the point A, taken on BD or outside of BD, as pole, and the chord of a quadrant as radius (1013), describe an arc of a great circle cutting the given circle in P; from the point P as pole, with the same radius, describe a great circle, which will pass through the point A. When the point A is the pole of BD, any great circle which passes through A satisfies the conditions, but in all other cases there is but one solution.

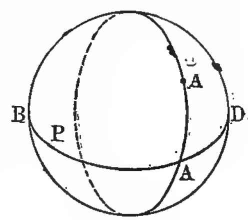


Fig. 242

1017. There are three problems which appear to belong to elementary geometry, and which may be solved with a rule and a compass (935). They are:

1st. *The trisection of an angle*, that is, the division of an angle or an arc into three equal parts (976).

By the following construction an angle C is obtained equal to a third of a given angle AOB; but the problem is not solved geometrically, since the method of trial and error is used to determine the line CDB.

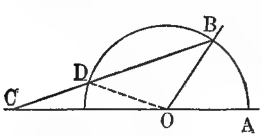


Fig. 243

From the vertex O as center, with a radius equal to OA, a semicircle is described; on the edge of a rule or a piece of paper CD is laid off equal to the radius OA, then the rule is so manipulated that the points C and D fall respectively upon the

line AC and the semi-circumference DBA, while the line CD extended will pass through B; when this is the case, draw CDB, and the angle C will be equal to $\frac{1}{3}$ of the angle AOB. Since the exterior angle $AOB = B + C$ (653), and $B = BDO$ (635), and $BDO = C + COD = 2\,C$, we have $\angle AOB = 3 \angle C$.

2d. *The quadrature of a circle,* which consists in constructing a square which has the same area as a given circle (1008).

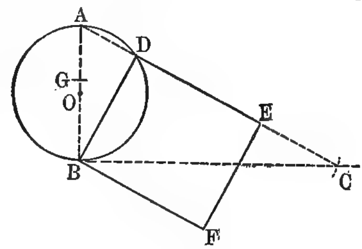


Fig. 244

The following method gives the solution correct to one decimal unit of the fifth order. Draw a diameter AB, and a tangent BC; take $OG = \frac{1}{6}$ of the radius OA; from the point G as a center and a radius equal to twice the diameter AB, describe an arc which cuts the tangent in C; join A and C, and the chord BD is the side of the required square $BDEF$.

3d. *Duplication of a cube,* which consists in finding the side of a cube which is double that of a given cube. The solution is obtained by calculation.

PART IV

TRIGONOMETRY

PLANE TRIGONOMETRY

1018. *The special object of trigonometry* is to furnish methods for the calculation of the unknown parts of a triangle (angles and sides) when enough is given to determine them (**938** to **942**).

Any polygon being composed of triangles, it follows that *the more general purpose of trigonometry* is to calculate the unknown parts of any polygon which is sufficiently determined.

DETERMINATION OF A POINT

1019. *The means of fixing the position of a point on a line.* Since from a certain point in a given line the same distance may be measured in two directions (**599**), it follows that it is not sufficient for the determination of a point to know its distance from a certain point in a given line, but the direction in which the distance is taken must also be known.

To simplify the expressions and facilitate the calculations, it is agreed to consider the distances measured in one direction as positive, and in the opposite direction as negative, and these are designated in the calculation by the usual signs + and − (**449**).

Generally the lines drawn from left to right and from down to up are considered positive, and those from right to left and up to down as negative.

The fixed point O of a line from which all distances on the line are measured is called the *origin.*

When the line on which the distances are measured is a straight line, it is called an *axis.*

The distance of any point on the axis to the origin is called the *abscissa;* it is generally designated by $+x$ or $-x$, according as it is measured in one direction or the opposite.

1020. Two directions, xx' and yy', perpendicular to each other being given, *the position of any point in the plane of these two*

directions is determined when the projections of the point on the straight lines xx' and yy' are known (**715**).

p and q being the projections of a point on the lines xx' and yy', erecting the perpendiculars pM and qM, each of these perpendiculars contains the point, therefore it must be at their intersection, M.

The point M being determined when its projections, p and q, upon two rectangular lines are known, and these projections being determined by their distance from the origin taken on the axes xx' and yy' (**1019**), therefore a point in a plane is determined when the abscissas of its projections upon two rectangular axes, drawn in the same plane, are known.

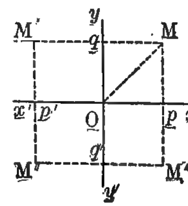


Fig. 245

The common origin of the two axes xx' and yy' is taken at their intersection O.

The axes are called *coördinate axes.*

The axis xx' is called the *x-axis.*

The axis yy' is called the *y-axis.*

The distances measured on the x-axis are called ***abscissas***, and those on the y-axis are called *ordinates.*

The abscissa Op of the projection p is also the abscissa of the point M. Since $Op = Mq$, it is seen that the abscissa of a point is the distance of the point from the y-axis.

The abscissa, which is designated by x, is positive or negative, according as it is measured on Ox or Ox'; that is, according as the point is at the right or the left of the y-axis.

In a like manner, since $Oq = Mp$, the ordinate of a point is the distance of the point from the x-axis. The ordinate is designated by y, and is positive or negative, according as the point is located above or below the x-axis.

The abscissa and ordinate of a point are the *coördinates of the point.*

Thus a point is determined by the algebraic values of its coördinates x and y (**450**).

For

$$\begin{aligned} &M, && x = +Op && \text{and} && y = +Oq; \\ &M', && x = -Op' && \text{and} && y = +Oq; \\ &M'', && x = -Op' && \text{and} && y = -Oq'; \\ &M''', && x = +Op && \text{and} && y = -Oq'. \end{aligned}$$

When $x = 0$, the point lies on the y-axis; when $y = 0$, it lies on the x-axis; and when both x and y are equal to 0, it lies on both, that is, at the origin.

Remark 1. That which has been said of the rectangular axes xx' and yy', holds likewise when the axes make any angle with each other; but then the lines Mp, Mq . . ., which remain parallel to the axes yy', xx', are oblique to the axes xx' and yy'.

Remark 2. In the case where the axes are rectangular, joining O and M, the right triangle OMp gives (730):

$$\overline{OM}^2 = x^2 + y^2.$$

The distance of any other point, M', M'' . . ., from the origin gives the same relation with the coördinates of the point considered.

1021. *Means of fixing the position of a point in space.*

In the same manner as a point in a plane is determined by its projections on two straight rectangular axes drawn in the plane (1020), *the position of any point in space is determined when its projections on three planes, each perpendicular to the other two, are known* (763).

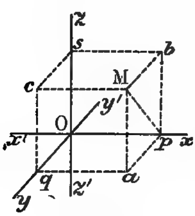


Fig. 246

a, b, and c being the projections of a point M on the three planes xOy, xOz, and yOz, determined by the rectangular axes xx', yy', and zz', which are the intersections of the planes, if at each of these points a perpendicular to the corresponding plane is erected, they will all three meet in the point M. Thus a point is clearly determined by its projections on the three planes.

Each of the projections, a, b, c, being determined when its respective projections, p and q, p and S, q and S, on two axes are known, it follows that these three projections, and consequently the point M, are determined when the points p, q, and s are known, which is nothing other than the projections of the point M upon the three axes, xx', yy', and zz' (715, 790).

The three points, p, q, s, on the axes, being determined by their abscissas with reference to the origin O (1019), *a point M is therefore determined when the abscissas of its projections on three rectangular axes are known.*

The three rectangular axes, xx', yy', and zz', are likewise called *coördinate axes*; xx' being the *x-axis*, yy' the *y-axis*, and zz' the *z-axis*.

The three planes determined by these axes are called the *coördinate planes.*

The abscissas Op, Oq, Os, of the projections of the point M on the axes, are called the *coördinates of the point* M; Op is the abscissa x, Op is the y-ordinate, and Oz the z-ordinate. *Thus a point is determined by its coördinates* (1020).

Since $Op = Mc$, $Oq = Mb$, and $Os = Ma$, the coördinates x, y, and z of a point are equal to the distances of this from the coördinate planes. These coördinates are positive or negative, according as the projections of the point upon the axes lie upon the parts Ox, Oy, and Oz, or upon Ox', Oy', and Oz'. Thus x will be positive or negative, according as the point M lies at the right or left of the plane yOz; y will be positive or negative, according as the point lies in front of or behind the plane xOz; and finally, z will be positive or negative, according as the point M is above or below the plane xOy.

When $x = 0$, the point is in the plane yOz; if $y = 0$ or $z = 0$, the point is respectively in the plane xOz or xOy.

When two of the coördinates are equal to zero, the point lies on one of the axes; thus, for $x = y = 0$, the point is on the z-axis. If $x = y = z = 0$, the point is on all three axes, and must be at the origin.

REMARK 1. That which has been said of planes or axes which are perpendicular to each other applies as well when they are inclined to each other, except that the perpendiculars Ma, Mb, Mc, to the planes of projection remain parallel to the axes. The projections, a, b, c, on the coördinate planes, or those, p, q, s, on the axes, instead of being *orthogonal projections,* are then *oblique projections.*

REMARK 2. The distance OM from the point M to the origin O being the diagonal of a parallelopiped whose edges are the coördinates of the point, in case the axes are rectangular, the parallelopiped is rectangular, and we have,

$$\overline{OM}^2 = x^2 + y^2 + z^2. \tag{835}$$

This relation exists no matter where the point is located about the origin.

DETERMINATION OF A STRAIGHT LINE

1022. *The position of a straight line is fixed by that of its extremities, and therefore by the coördinates of its extremities* (1021).

A straight line may also be defined by the conditions which determine: First, *one extremity;* Second, *its length;* Third, *its direction.*

1st. The position of one extremity of a straight line is determined by the algebraic values of the coördinates of this extremity.

2d. The length of a straight line is determined, without regard to the sign, by the ratio of it and the linear unit (713).

3d. It remains to fix the direction and sign of the line.

No matter what the position of the line with reference to the axes is, its direction and sign with reference to these axes will be known, when its direction and sign with reference to a system of axes parallel to the first and passing through the known extremity of the given line are known.

1023. *This last part of the question is therefore reduced to the determination of what is necessary to fix the direction and sign of a straight line with reference to a system of coördinate axes whose origin is at one extremity of the given line* (598, 599).

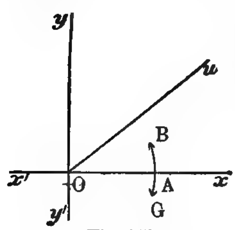


Fig. 247

At first, consider the most simple case, namely, where the straight line is in the same plane as the axes, that of xy for example (1021).

Let Ou be the straight line, then its sign is indicated by the order of its extremities O and u; the direction of this line will be determined when the angle uOx, which the line makes with the part Ox of x-axis, is known, and it is indicated upon which side of the x-axis this angle is to be taken because it is easily seen that two equal angles may be drawn with Ox as one side.

In order to dispense with the necessity of designating whether an angle is to be measured from one side or the other of Ox, a conventional system analogous to that in (1019) for fixing the position of a point has been adopted. Thus it has been agreed to consider as positive all the angles described by the straight line Ox in turning about the point O in the direction indicated by the arrow AB, and as negative all the angles described in turning in the opposite direction AC.

The positive angle is zero when Ou coincides with Ox; then it takes all the values between 0° and 90° in turning from Ox to Oy; when it coincides with Oy, it makes a positive angle of 90°

with Ox. In turning from Oy to Ox' it takes all the values from 90° to 180°, from Ox' to Oy' all the values from 180° to 270°, from Oy' to Ox all the values from 270° to 360°, and from Ox on, all the values from 360° up.

If Ou had revolved in the negative direction, it would have described all the negative angles just as it has the positive. It should be noted that the angles $+\alpha$, $+(360° + \alpha)$, $+(720 + \alpha)$, etc.; $-(360 - \alpha)$, $-(720 - \alpha)$, etc., all designate the same straight line, both in direction and sign.

REMARK. As the line Ou describes angles about the point O, the points in the line describe arcs corresponding to these angles (667), and according as these angles are positive or negative, the arcs are also positive and negative.

Thus an angle is determined when its corresponding arc is known, and vice versa; it is, of course, assumed that the arc is preceded by its sign + or −, according to the conventions adopted.

TRIGONOMETRIC EXPRESSIONS — THEIR USE FOR THE EXPRESSION OF THE VALUE OF ANY ANGLE OR ARC, POSITIVE OR NEGATIVE

1024. In the case where the straight line Ou has one of its extremities at the origin O, the line is determined when the algebraic values of the coördinates $y = Mp$, and $x = Mq$, of its other extremity are known (1020).

The ratios between the quantities x, y, and OM are constant, no matter what the position of M on Ou may be, that is, no matter what the value $OM = r$ may be. The quantity $OM = r$ is always positive since it is the distance of the point M from the origin O, and is measured in the positive direction along the generatrix Ou of the angle uOx. From this it follows that the direction of the line is determined when the algebraic values of two of the constant ratios between x, y, and r are known; because, assuming any value of r, these ratios give the corresponding values of x and y (516).

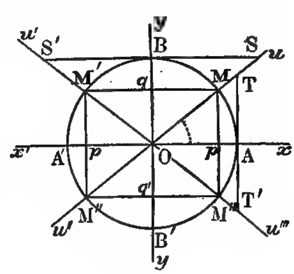


Fig. 248

Six different ratios or *trigonometric expressions* or *functions* may be formed with the quantities x, y, and r:

$\frac{y}{r}$. ratio of the ordinate Mp to the radius of the arc AB passing through the point M, is the *sine* of the angle $uOx = \alpha$, and of the arc AM, which is also designated by α. It has the same sign as the ordinate y (1020);

$\frac{x}{r}$, ratio of the abscissa Op to the radius, is the *cosine* of the angle and arc α. Its sign is the same as that of x;

$\frac{y}{x}$, ratio of the ordinate to the abscissa, is the *tangent* of the angle and arc α. It is positive or negative according as y and x have the same or opposite signs;

$\frac{r}{y}$, the reciprocal of the sine, of the same sign, is called the *cosecant* of the angle and arc α;

$\frac{r}{x}$, the reciprocal of the cosine, of the same sign, is called the *secant* of the angle and arc α;

$\frac{x}{y}$, the reciprocal of the tangent, is called the *cotangent* of the angle and arc α. It is positive or negative according as x and y have like or unlike signs; consequently it has the same sign as the tangent.

The above functions are written:

$$\sin\alpha = \frac{y}{r}, \quad \cos\alpha = \frac{x}{r}, \quad \tan\alpha = \frac{y}{x},$$

$$\csc\alpha = \frac{r}{y}, \quad \sec\alpha = \frac{r}{x}, \quad \cot\alpha = \frac{x}{y}.$$

1025. *Other forms of these functions. Trigonometric lines.*

1st. We have $\sin\alpha = \frac{y}{r} = \frac{Mp}{r}$, *ratio of the radius r to half Mp of the chord which subtends the arc corresponding to double the angle α.*

2d. $\operatorname{Cos}\alpha = \frac{x}{r} = \frac{Op}{r}$. As is shown in Fig. 248, *the cosine and the sine of α are respectively equal to the sine and cosine of the complement of the angle α.*

3d. Drawing the tangent AT (Fig. 248), the two similar triangles OAT, OpM, give (700, 1024):

$$\frac{AT}{p} = \frac{y}{x} = \tan\alpha.$$

Thus the tangent of an angle α is also represented by the ratio of the positive or negative tangent AT, drawn from the origin A of the arc described with the radius r, and prolonged to meet the other side of the angle α, to the radius r. This is why the expression $\frac{y}{x}$ is called *tangent*.

4th. The same similar triangles OAT and OpM give:

$$\frac{OT}{r} = \frac{r}{x} = \sec \alpha.$$

The secant is therefore represented by the ratio of that portion of the secant OT, measured on the second side of the angle and included between the center and the tangent, and the radius r. This gives the function its name *secant.*

5th. Drawing the tangent BS from the point B until it meets Ou, the two similar triangles OBS and OqM give:

$$\frac{BS}{r} = \frac{x}{y} = \cot \alpha,$$

which shows that the cotangent of an angle is represented by the ratio of the tangent BS to the radius.

This formula and the Fig. 248 show that a *cotangent of an angle is nothing other than the tangent of its complement.* This is where it gets its name *cotangent.*

6th. From the two similar triangles OBS and OqM:

$$\frac{OS}{r} = \frac{r}{y} = \csc \alpha.$$

Thus the cosecant of an angle is represented by the ratio of that portion OS of the secant to the radius.

From this formula and the figure, it is seen that *the cosecant of an angle is nothing other than the secant of its complement,* and hence its name *cosecant.*

We have therefore:

$$\sin \alpha = \frac{Mp}{r}, \quad \cos \alpha = \frac{Op}{r}, \quad \tan \alpha = \frac{AT}{r},$$

$$\csc \alpha = \frac{OS}{r}, \quad \sec \alpha = \frac{OT}{r}, \quad \cot \alpha = \frac{BS}{r}.$$

Putting $r = 1$,

$$\sin \alpha = Mp, \quad \cos \alpha = Op, \quad \tan \alpha = AT,$$
$$\csc \alpha = OS, \quad \sec \alpha = OT, \quad \cot \alpha = BS.$$

These last values of the trigonometric functions are represented by lines, and are called *trigonometric lines*.

1026. There are still two trigonometric functions which we will simply define, since they are not frequently used.

$\frac{r-x}{r}=\frac{Ap}{r}$ is the *versed sine* of the angle and arc α. For $r=1$, the versed sine, *vers sin* α, is equal to Ap.

$\frac{r-y}{r}=\frac{Bq}{r}$ is the *coversed sine* of the angle and arc α. For $r=1$, the coversed sine, *covers sin* α, is equal to Bq.

1027. *Signs of trigonometric functions.* Since the only variables which enter in the trigonometric functions of (**1024**) are the coordinates x and y, it is very easy to determine the signs of these variables no matter what the value of α may be (**487, 1020**).

For the values of α *between* $0°$ *and* $90°$, x and y are positive, x varies from r to 0, and y from 0 to r; therefore (**1024**):

$$\sin\alpha=+\frac{y}{r},\ \text{and varies from 0 to }+1;$$

$$\cos\alpha=+\frac{x}{r},\ \text{and varies from }+1\text{ to }0;$$

$$\tan\alpha=+\frac{y}{x},\ \text{and varies from 0 to}+\infty;$$

$$\csc\alpha=+\frac{r}{y},\ \text{and varies from }+\infty\text{ to }1;$$

$$\sec\alpha=+\frac{r}{x},\ \text{and varies from 1 to }+\infty;$$

$$\cot\alpha=+\frac{x}{y},\ \text{and varies from }+\infty\text{ to }0.$$

For the values of α *between* $+90°$ and $+180°$, y is positive and varies from r to 0, while x is negative and varies from 0 to $-r$; therefore:

$$\sin\alpha=+\frac{y}{r},\ \text{and varies from }+1\text{ to }0;$$

$$\cos\alpha=\frac{-x}{r}=-\frac{x}{r},\ \text{and varies from 0 to }-1;$$

$$\tan\alpha=\frac{+y}{-x}=-\frac{y}{x},\ \text{and varies from }-\infty\text{ to }0;$$

$$\csc\alpha=+\frac{r}{y},\ \text{and varies from 1 to }+\infty;$$

$$\sec \alpha = \frac{r}{-x} = -\frac{r}{x}, \text{ and varies from } \infty \text{ to } -1;$$

$$\cot \alpha = \frac{-x}{y} = -\frac{x}{y}, \text{ and varies from } 0 \text{ to } -\infty.$$

For the values of α *between* $+180°$ and $+270°$, y is negative and varies from 0 to $-r$, and x is also negative and varies from $-r$ to 0; therefore:

$$\sin \alpha = \frac{-y}{r} = -\frac{y}{r}, \text{ and varies from } 0 \text{ to } -1;$$

$$\cos \alpha = \frac{-x}{r} = -\frac{x}{r}, \text{ and varies from } -1 \text{ to } 0;$$

$$\tan \alpha = \frac{-y}{-x} = +\frac{y}{x}, \text{ and varies from } 0 \text{ to } +\infty;$$

$$\csc \alpha = \frac{r}{-y} = -\frac{r}{y}, \text{ and varies from } -\infty \text{ to } -1;$$

$$\sec \alpha = \frac{r}{-x} = -\frac{r}{x}, \text{ and varies from } -1 \text{ to } -\infty;$$

$$\cot \alpha = \frac{-x}{-y} = +\frac{x}{y}, \text{ and varies from } +\infty \text{ to } 0.$$

For the values of α *between* $+270°$ and $+360°$, y is negative and varies from $-r$ to 0, while x is positive and varies from 0 to $+r$; therefore:

$$\sin \alpha = \frac{-y}{r} = -\frac{y}{r}, \text{ and varies from } -1 \text{ to } 0;$$

$$\cos \alpha = +\frac{x}{r}, \text{ and varies from } 0 \text{ to } +1;$$

$$\tan \alpha = \frac{-y}{+x} = -\frac{y}{x}, \text{ and varies from } -\infty \text{ to } 0;$$

$$\csc \alpha = \frac{r}{-y} = -\frac{r}{y}, \text{ and varies from } -1 \text{ to } -\infty;$$

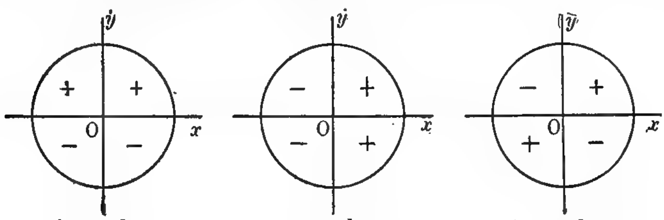
$$\sec \alpha = \frac{r}{x}, \text{ and varies from } +\infty \text{ to } +1;$$

$$\cot \alpha = \frac{+x}{-y} = -\frac{x}{y}, \text{ and varies from } 0 \text{ to } -\infty.$$

For values of α *greater than* $360°$, these values and signs are repeated and so on; thus, the trigonometric functions of the angles $(360° + 30)$, $(360° \times 2 + 30)$, etc., are the same as those of an angle of $30°$.

By inspection of Fig. 248 it is seen that *for any negative angle* $-\alpha$ (1023), *the trigonometric functions have the same values and the same signs as for the positive angle* $360 - \alpha$. From this it follows that if a table of the values for the negative angles were constructed, we would have the same as in the one given above, but in an inverse order. Thus, for the angles from 0° to − 90°, we would have the same values as for the positive angles from 360° to 270°.

The figure (249) below, indicates the signs of the trigonometric functions for the different values of the angle or the arc α.



sine and csc cos and sec tan and cot

Fig. 249

1028. It should be noted that the absolute values of the coordinates y and x, and therefore, those of the trigonometric functions of any angle uOx (**1024**), are equal to those of the acute angle which the line Ou makes with Ox or its prolongation Ox' (Fig. 248), this acute angle being always considered as positive.

From this it follows that in forming the table (**1071**) of the values of the trigonometric functions of all the positive angles included between 0° and 90°, it will contain also the absolute values of all the angles greater than 90°; having the absolute value, the sign may be prefixed which belongs to the given angle according to the table (**1027**) or the figure 249.

If it is desired to have the sine of the angle $uOx = +215°$, for example. Noting that Ou makes an angle of $215 - 180 = 35°$ with Ox'; look in the table (**1071**) for the sine 0.57358 of the angle of 35°, and prefixing the minus sign before this absolute value which corresponds to the angle 215°, we have: $\sin 215° = -0.57358$.

Any angle being given, the algebraic values of its trigonometric functions may be determined.

1029. A single trigonometric function does not determine the angle α, since for a given value $+S$ of the sine there are two

angles α and $180° - \alpha$, and for $\sin \alpha = -S$ there are two angles $180° + \alpha$ and $360° - \alpha$.

Since an acute angle α corresponds to a positive cosine, while its supplement $180° - \alpha$ corresponds to a negative cosine, an angle is determined when the value and sign of its sine and the sign of its cosine are given.

In the same manner there are two values of the angle for one value and sign of the cosine, and in order to determine an angle, the value and sign of its cosine and the sign of its sine must be known.

$+t$ being the value of the tangent of the angle α, we have $t = \frac{y}{x}$ and $t = \frac{-y}{-x}$, equations which may be satisfied by the two lines Ou and Ou'', directly opposed to one another and making the angles α and $180° + \alpha$ with the line Ox. Thus an angle is not determined by its tangent; but it becomes determined when besides its tangent the sign of one of its coördinates x or y, or, which is the same thing, its sine or cosine, is known.

If the given tangent were $-t$, we would have $-t = \frac{+y}{-x}$ and $-t = \frac{-y}{+x}$, which values are satisfied by the lines Ou' and Ou''', directly opposed to each other and making the angles $90° + \alpha$, and $270° + \alpha$ with Ox. Thus the angle is not determined, but will be when, besides the tangent, the sign of the sine or cosine is known.

In general, for each algebraic value of the principal trigonometric functions, sine, cosine, and tangent, there corresponds, for each of the two other functions, two equal values opposite in sign; this is shown in Fig. 249. It follows then that having the value of any one of the trigonometric functions, the angle is determined if the sign of one of the other two is known.

1030. *Designation of an angle by the words batter* and *grade.*

In masonry the *batter* of a wall is said to be so and so many feet per a certain number of feet in height, meaning that the face of the wall is inclined to the vertical by an angle whose tangent is equal to the ratio of the given numbers. For instance, if the batter of a wall is 1:10, the tangent of the angle is 0.1. The *grade* of a road is the height which the road rises from the horizontal in a given distance; it is generally expressed in per cent. Thus, a grade of $3\% = \frac{3}{100} = \tan \alpha = 0.03$ is expressed by the tangent

of the angle which the surface of the road makes with the horizontal. If the distance is taken on the surface of the road, this ratio is then the sine of the slope angle α, but in any case the slope is generally so small that there is little difference between the tangent and the sine.

1031. We have seen how, having a table containing the values of the trigonometric functions of the angles from 0° to 90°, the functions of any angles may be found (**1028**). Noting that the sine, the cosine, the tangent, the cotangent, the secant and the cosecant of an acute angle are respectively equal to the cosine, the sine, the cotangent, the tangent, the cosecant and the secant of its complement, it is seen that the functions of the angles from 0° to 45° are all that are necessary to determine those of all the angles. For example, if it is desired to have the sine of 70°, look for the cosine of $90° - 70° = 20°$ in the table (**1043**).

The absolute value of the cosine of an angle of 125° is 0.57358, the cosine of $180° - 125° = 55°$ (**1071**) and the sine of $90° - 55° = 35°$; its algebraic value is -0.57358 (**1027**).

General Rule. When the value of a trigonometric function of an angle between 90° and 180° is to be determined, find the value corresponding to the supplement of the angle and prefix the sign corresponding to the given angle (**1027**), which gives the required value. In practice, it is rarely required to find the functions of angles greater than two right angles, but, even if it should be, it offers no difficulties that have not been explained above.

1032. *Trigonometric tables.* In practice, use is scarcely ever made of functions other than the sine, cosine, tangent, and cotangent, and therefore the tables contain only these values.

The tables are so arranged that each absolute value may be read as a function of an angle and its complement. For instance, the sine of one angle is the cosine of its complement. Referring to the table (**1071**), the numbers in the second column are sines of the angle whose number of degrees is read at the top and minutes at the left in the first column, and at the same time these same values are the cosines of the angles (complements of the above) whose degrees are written at the bottom and minutes in the last column at the right. Reading from the top, the functions of all the angles expressed in minutes up to 45° are given, then reading from the bottom the functions of the angles from 45° to 90° are found.

1033. *Determination of the position of a straight line in space.* We have just seen how, by means of the trigonometric functions, the position of a line in the plane of the coördinates is fixed. Let us now examine the case where the straight line lies outside of these planes.

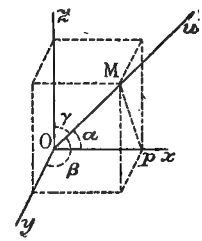


Fig. 250

Assume that one extremity of the line Ou lies at the origin O of the coördinate system. The position of the line will be determined when the coördinates x, y, and z of a point M situated in the line at any distance $+OM = r$ from the origin (1021). This position will, therefore, be determined when the ratios $\frac{x}{r}$, $\frac{y}{r}$, and $\frac{z}{r}$ are known. The signs of the ratios are determined by the signs of x, y, and z, because r is always positive.

Let the angles which the line Ou makes with the axes Ox, Oy, and Oz be respectively α, β, and γ. Mp being the perpendicular to Ox (770), Op is the abscissa x of the point M, and, in the plane Oux, we have:

$$\frac{Op}{OM} = \frac{x}{r} = \cos\alpha. \qquad (1024)$$

Likewise in the planes uOy and uOz we have:

$$\frac{y}{r} = \cos\beta \text{ and } \frac{z}{r} = \cos\gamma,$$

which shows that, knowing the cosines of the angles which the line makes with the coördinate axes, the algebraic ratios $\frac{x}{r}$, $\frac{y}{r}$, and $\frac{z}{r}$ are known, and therefore the line is determined.

1034. We have:

$$x^2 + y^2 + z^2 = \overline{OM}^2 = r^2; \qquad (1021)$$

therefore
$$\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1,$$

that is,
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \qquad (a)$$

which shows that *the sum of the squares of the cosines of the angles which a straight line makes with the rectangular axes of a system of coördinates is equal to one.*

REMARK 1. This relation shows that the cosines of the angles which a line makes with the three axes of a rectangular coördinate system cannot be arbitrarily chosen; but that the algebraic values of the cosines of two of the angles and the sign of the third cosine being given, the third cosine and the position of the line may be determined by means of the equation (a).

REMARK 2. The cosine of an angle which a straight line makes with an axis determines the surface of a cone of revolution of which the straight line is the generatrix. The cosines which the straight line makes with two axes of the coördinate system determine two lines, namely, the intersections of two conical surfaces of revolution, one line making an acute and the other an obtuse angle with the third axis; now if the sign of the cosine of the angle which the line makes with the third axis is known, it is determined which of the intersections is the required line, and thus the position of the line is fixed.

REMARK 3. If the line is situated in the plane of two of the axes, the formula (a) becomes,

$$\cos^2 \alpha + \cos^2 \beta = 1. \tag{1020}$$

1035. The circumference of a circle whose radius $r = 1$, being expressed by 2π (752), the quantity π corresponds to 180°, and it is evident that it may be used as a unit in measuring arcs and angles.

An arc α being expressed as a function of π, the value x of this same arc in degrees is

$$x = \alpha \frac{180}{\pi}. \qquad (a)$$

Conversely, if α is expressed in degrees, its value x in function of π is

$$x = \alpha \frac{\pi}{180}. \qquad (b)$$

Thus, according as

$$\alpha = \frac{\pi}{6}, \quad \frac{\pi}{5}, \quad \frac{\pi}{4}, \quad \frac{\pi}{3}, \quad \frac{\pi}{2}, \quad \frac{2\pi}{3}, \quad \pi, \quad \frac{3\pi}{2}, \quad 2\pi,$$

the same arc expressed in degrees is respectively:

30° 36° 45° 60° 90° 120° 180° 270° 360°.

PROJECTION OF STRAIGHT LINES

1036. A straight line having two directions (**599**), the length of a finite line will take the + or − sign, according as the length was taken in the positive or negative direction.

When a straight line is considered independently, either of its directions may be taken as positive, the opposite being negative. But when the line is referred to a given axis or system of axes, its sign is determined by its position with reference to these axes.

The direction of the projection of a straight line upon an axis is indicated by the order of the letters of two of its points, and the sign of each direction is the same as that for the same direction of the axis (**1019**).

To make this clear, the absolute length of the line $M'M''$ or $M''M'$ being 30 feet, the algebraic value of $M'M''$ is + 30 feet, and that of $M''M'$ is − 30 feet. In the same way the absolute value of the projection $p'p''$ or $p''p'$ of $M'M''$ on the axis Ox being 22 feet, the algebraic value of $p'p''$ is + 22, and that of $p''p'$ is − 22 feet.

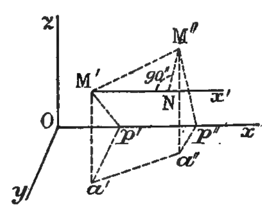


Fig. 251

1037. *The algebraic expression of the projection of a straight line upon an axis.* Having $Op'' = x''$, abscissa of the point M'', and $Op' = x'$, abscissa of the point M', it follows that

$$p'p'' = + (x'' - x'), \text{ and } p''p' = - (x'' - x').$$

Analogous expressions are obtained for the projections on each of the other axes Oy and Oz.

These expressions apply equally in the cases where x' and x'' have like or unlike signs.

Thus, the values of x' and x'' both being negative, which is the case when M' and M'' lie at the left of the yz plane, we have:

$$p'p'' = + [- x'' - (x')] = - (- x'' + x'),$$
$$p''p' = - [- x'' - (-x')] = -(-x'' + x'). \qquad (426)$$

If x' were negative and x'' positive, the preceding formulas would give:

$$p'p'' = + [+ x'' - (-x')] = + (x'' + x'),$$
and $$p''p' = - [+ x'' - (-x')] = - (x'' + x').$$

1038. *Relation between a straight line and its projections* (**1040**). If through the point M' (Fig. 281) axes parallel to the first system are drawn, the projections of $M'M''$ on these axes would be respectively equal to the projections on the first; furthermore, these projections would be the coördinates of the point M''.

If the axes are rectangular, taking the length of $M'M''$ equal to u, the formula of (1021) may be applied thus:

$$u^2 = (x'' - x')^2 + (y'' - y')^2 + (z'' - z')^2.$$

In case one of the projections is zero, which is the case when the line is situated in one of the coördinate planes or parallel to it, the preceding formula becomes,

$$u^2 = (x'' - x')^2 + (y'' - y')^2,$$

when the line is parallel to the xy plane. This formula is the same an given in (**1020**).

If the line were in the two planes yx and xz, for example, or parallel to them, it would coincide with the axis x or be parallel to it. Then its true length would be projected upon the x-axis, while the projections on the other two axes would be zero, and the preceding formula would become,

$$u^2 = (x'' - x')^2 \text{ or } u = (x'' - x'),$$

which is the same as in (**1037**).

1039. *The algebraic sum of the projections of the several portions of a broken line $ACDE$ on any axis, that is, the projection of the broken line on the axis, is equal to the projection of the line AE, which joins the extremities of the broken line, upon the same axis* (**1040**).

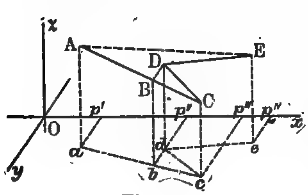


Fig. 252

x' being the abscissa of the point A, x'' that of the points B and D, x''' that of C, and x^{IV} that of E, we have successively (**1037**):

$$\begin{aligned}
\text{Projection of } \quad AB &= x'' - x', \\
\text{`` \quad ``} \quad BC &= x''' - x'', \\
\text{`` \quad ``} \quad CD &= x'' - x''', \\
\text{`` \quad ``} \quad DE &= x^{\text{IV}} - x''.
\end{aligned}$$

Adding all the projections, and reducing, we have (**458**):

$$\text{Projection of } \quad ACDE = x^{\text{IV}} - x',$$

which is nothing other than the projection of the straight line AE joining the extremities of the line $ACDE$.

REMARK. Considering a curved line as a broken line whose segments are infinitely small (601), it follows that the above statement applies also to curves, or, in general, any line.

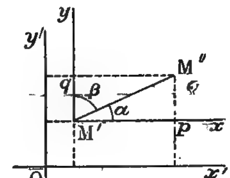


Fig. 253

1040. *Projection of a straight line, and, in general, any line, upon an axis, expressed in terms of its trigonometric functions* (1037).

1st. Let a straight line $M'M''$ be situated in the plane xy, with its extremity M' at the origin of the axes. From (1024), by representing the length of $M'M''$ by u, the projections $M'p$ and $M'q$ of the line on the axes by P_x and P_y, and noting that these projections are the coördinates of the point M'':

$$\frac{P_x}{u} = \cos \alpha, \qquad \text{and} \qquad \frac{P_y}{u} = \sin \alpha;$$

$$P_x = u \cos \alpha, \qquad \text{and} \qquad P_y = u \sin \alpha.$$

2d. These expressions apply also in the case where the line $M'M''$ being in the plane $x'y'$, does not have its extremity at the origin.

The angles α and β which the line $M'M''$ makes with the axes being the same as those which it makes with the parallel axes $M'x$ and $M'y$, and, moreover, since the projections $x'' - x'$ and $y'' - y'$ are respectively equal to P_x and P_y, we may write:

$$x'' - x' = u \cos \alpha, \quad \text{and} \quad y'' - y' = u \sin \alpha.$$

3d. It remains to consider the case where the line $M'M''$ is not in the plane of the axes (Fig. 253).

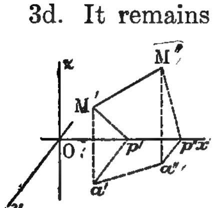


Fig. 254

The angle α which $M'M''$ makes with Ox is equal to the angle $M''M'x'$ which it makes with the axis Mx' parallel to Ox (611); moreover, the projection $M'N$ of $M''M'$ on $M'x'$ is equal to the projection $p'p'' = P_x$ of this same line of Ox, and we may write:

$$P_x = u \cos \alpha.$$

Thus, no matter what the position of a line with reference to an axis may be, the algebraic value of the projection of the line

upon the axis is equal to the absolute length of the line multiplied by the cosine of the positive angle included between the positive side of the axis and the line (1019, 1023).

REMARK. We have said (3d) that the projections $M'N$ and $p'p''$ were equal to each other.

PROOF. — The perpendiculars $M''N$, $M'p'$ and $M''p''$ drawn to the axes being in the planes which pass through $M'M''$ perpendicular to the parallel axes, since these planes cut the axes in M', N, p' and p'', and parallels comprehended between parallels are equal, we have $M'N = p'p''$.

4th. For a broken line $ACDE$ (Fig. 252), making $AC = u'$, $CD = u'' \ldots$, and designating the positive angles which AC, $CD \ldots$ make with Ox (3d) by α', $\alpha'' \ldots$, we have:

Projection of $AB = AB \cos \alpha'$,
" " $BC = BC \cos \alpha'$.

Adding, we have:

Projection of $u' = u' \cos \alpha'$,
" " $u'' = u'' \cos \alpha''$,
" " $u''' = u''' \cos \alpha'''$.

Adding all three, we have:

$$x^{\text{IV}} - x' = u' \cos \alpha' + u'' \cos \alpha'' + u''' \cos \alpha''',$$

$x^{\text{IV}} - x'$ being the projection of the line AE (2d) joining the extremities of the broken line.

Representing the sum of the products by $\Sigma\, u \cos \alpha$, the distance between the two extremities by U, and the angle which the line joining the extremities makes with the axis by α, the preceding equation becomes:

$$U \cos \alpha = \Sigma\, u \cos \alpha.$$

REMARK 1. Considering a curve as an infinite number of straight lines, this last equation applies also to curves.

REMARK 2. α' being the angle which AC makes with a parallel to Ox drawn through A, and not through C (Fig. 252), its value lies between 270° and 360°; α'' being the angle which CD makes with a parallel to Ox drawn through C, its value lies between 90° and 180°; α''' lies between 0° and 90°.

The angle α', formed by AC and Ox, being between 270° and 360°, its cosine is algebraically equal to that of the acute angle $360° - \alpha'$, which is the smaller of the two angles which the line

AC makes with Ox. Likewise the cosine of an angle α_1, which lies between 180° and 270°, is algebraically equal to that of the obtuse angle $360 - \alpha_1$, which is the smaller of the two angles which the line forms with the axis Ox. To determine the projection of a straight line or a series of straight lines on an axis, the calculations may be facilitated by taking the cosine of the smaller angle which the line makes with the axis Ox.

FORMULAS EXPRESSING THE RELATIONS BETWEEN THE TRIGONOMETRIC FUNCTIONS

1041. *Relations between the trigonometric functions of the same angle or arc α.*

From (1024):

1st. $\sin \alpha = \dfrac{y}{r}$, from which $y = r \sin \alpha$;

and $\cos \alpha = \dfrac{x}{r}$, from which $x = r \cos \alpha$.

Substituting these values of x and y in the equation

$$y^2 + x^2 = r^2, \tag{1020}$$

we obtain
$$r^2 \sin^2 \alpha + r^2 \cos^2 \alpha = r^2,$$

or
$$\sin^2 \alpha + \cos^2 \alpha = 1;$$

from which

$$\sin \alpha = \pm \sqrt{1 - \cos^2 \alpha} \text{ and } \cos \alpha = \pm \sqrt{1 - \sin^2 \alpha}.$$

2d. $\tan \alpha = \dfrac{y}{x} = \dfrac{r \sin \alpha}{r \cos \alpha} = \dfrac{\sin \alpha}{\cos \alpha}$;

from which

$$\tan \alpha = \frac{\sin \alpha}{\pm \sqrt{1 - \sin^2 \alpha}}, \qquad \text{or } \sin \alpha = \frac{\tan \alpha}{\pm \sqrt{1 + \tan^2 \alpha}},$$

$$\text{and } \tan \alpha = \frac{\pm \sqrt{1 - \cos^2 \alpha}}{\cos \alpha}, \qquad \text{or } \cos \alpha = \frac{1}{\pm \sqrt{1 + \tan^2 \alpha}}.$$

3d. $\cot \alpha = \dfrac{x}{y} = \dfrac{r \cos \alpha}{r \sin \alpha} = \dfrac{\cos \alpha}{\sin \alpha}$.

Thus, $\cot \alpha = \dfrac{1}{\tan \alpha}$, or $\tan \alpha = \dfrac{1}{\cot \alpha}$;

from which

$$\cot\alpha = \frac{\pm\sqrt{1-\sin^2\alpha}}{\sin\alpha}, \qquad \text{or } \sin\alpha = \frac{1}{\pm\sqrt{1+\cot^2\alpha}};$$

$$\text{and } \cot\alpha = \frac{\cos\alpha}{\pm\sqrt{1-\cos^2\alpha}}, \qquad \text{or } \cos\alpha = \frac{\cot\alpha}{\pm\sqrt{1+\cot^2\alpha}}.$$

$$\text{4th. } \sec\alpha = \frac{r}{x} = \frac{r}{r\cos\alpha} = \frac{1}{\cos\alpha}, \qquad \text{or } \cos\alpha = \frac{1}{\sec\alpha},$$

$$\sec\alpha = \frac{1}{\pm\sqrt{1-\sin^2\alpha}}, \qquad \text{or } \sin\alpha = \frac{\pm\sqrt{\sec^2\alpha-1}}{\sec\alpha},$$

$$\sec\alpha = \frac{1}{\cos\alpha} = \pm\sqrt{1+\tan^2\alpha}, \qquad \text{or } \tan\alpha = \pm\sqrt{\sec^2\alpha-1},$$

$$\sec\alpha = \frac{1}{\cos\alpha} = \frac{\pm\sqrt{1+\cot^2\alpha}}{\cot\alpha}, \qquad \text{or } \cot\alpha = \frac{1}{\pm\sqrt{\sec^2\alpha-1}}.$$

$$\text{5th. } \csc\alpha = \frac{r}{y} = \frac{r}{r\sin\alpha} = \frac{1}{\sin\alpha}, \qquad \text{or } \sin\alpha = \frac{1}{\csc\alpha},$$

$$\csc\alpha = \frac{1}{\pm\sqrt{1-\cos^2\alpha}}, \qquad \text{or } \cos\alpha = \frac{\pm\sqrt{\csc^2\alpha-1}}{\csc\alpha},$$

$$\csc\alpha = \frac{1}{\sin\alpha} = \frac{\pm\sqrt{1+\tan^2\alpha}}{\tan\alpha}, \qquad \text{or } \tan\alpha = \frac{1}{\pm\sqrt{\csc^2\alpha-1}},$$

$$\csc\alpha = \frac{1}{\sin\alpha} = \pm\sqrt{1+\cot^2\alpha}, \qquad \text{or } \cot\alpha = \pm\sqrt{\csc^2\alpha-1},$$

$$\csc\alpha = \frac{1}{\sin\alpha} = \frac{\sec\alpha}{\pm\sqrt{\sec^2\alpha-1}}, \qquad \text{or } \sec\alpha = \frac{\csc\alpha}{\pm\sqrt{\csc^2\alpha-1}}.$$

1042. *Relations between the trigonometric functions of two equal angles or arcs of unlike signs, α and $-\alpha$.*

For the same value of r, the lines making the angles α and $-\alpha$ with Ox will give (**1024**):

1st. For y, two values, y and $-y$, equal and of unlike signs; consequently the sines $\frac{y}{r}$ and $-\frac{y}{r}$ will be equal and of unlike signs; and

$$\sin(-\alpha) = -\sin\alpha.$$

Thus, *two equal angles of unlike signs have equal sines also of unlike signs.*

2d. For x, two values, equal and of the same sign; consequently the cosines will both be $\frac{x}{r}$ or $-\frac{x}{r}$; and

$$\cos(-\alpha) = \cos\alpha.$$

Thus, *two equal angles of like signs have the same cosines.*

3d. Since the values of x are equal and of the same sign, while those of y are equal and of unlike signs, it follows that the tangents $\frac{y}{x}$ and $-\frac{y}{x}$ are always equal and of unlike signs; and

$$\tan(-\alpha) = -\tan\alpha.$$

Thus, *two equal angles of unlike signs have equal tangents also of unlike signs.*

From the above we may deduce:

4th.
$$\csc(-\alpha) = -\csc\alpha;$$
$$\sec(-\alpha) = \sec\alpha;$$
$$\cot(-\alpha) = -\cot\alpha.$$

1043. *Relations between the trigonometric functions of two complementary angles or arcs,* that is, whose sum $\alpha + \alpha' = 90°$.

Let $\alpha = uOx$ and $\alpha' = uOy$.

y and x being the coördinates of the point M, and r being the radius OM, we have for angle α (1041):

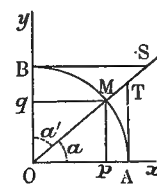


Fig. 255

$$Oq \text{ or } y = r\sin\alpha, \text{ and } Op \text{ or } x = r\cos\alpha.$$

On the contrary, for the positive angle α', the same values of x and y give:

$$y = r\cos\alpha' \text{ and } x = r\sin\alpha'.$$

Putting these two values of x and y equal to each other, and cancelling r, we have:

$$\sin\alpha = \cos\alpha' \text{ and } \cos\alpha = \sin\alpha'.$$

Dividing,

$$\frac{\sin\alpha}{\cos\alpha} = \frac{\cos\alpha'}{\sin\alpha'},$$

that is (1041, 2d),

$$\tan\alpha = \frac{1}{\tan\alpha'}, \quad \text{or} \quad \tan\alpha\tan\alpha' = 1.$$

Also (1041, 3d),

$$\tan\alpha = \cot\alpha'.$$

Thus, *the angles* α *and* α' *being complementary, the sines, cosines, and tangents of one are respectively equal to the cosines, sines, and cotangents of the other.* This is easily verified with the aid of Fig. 255 (1031).

1044. *Relations between the trigonometric functions of two angles or arcs, whose difference* $\alpha - \alpha' = 90°$.

Since two angles are complementary when their algebraic sum is equal to a right angle, by considering α' as negative we have the same case as the one preceding (**1043**).

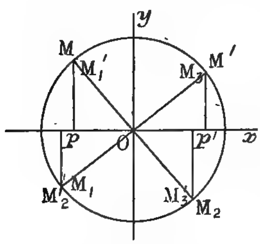


Fig. 256

Let $M'Ox = \alpha'$, the smaller of the two angles, and $MOx = \alpha$, the larger. The angles being measured in the positive direction from Ox, the angle $MOM' = \alpha - \alpha'$.

From the relations which exist between α and α', the value of the remainder MOM' must be a right angle; therefore, the right triangles MOp, $M'Op'$, are equal, and Mp or $y = Op'$ or x', and Op or $x = M'p'$ or y'.

Noting that y and x' have like signs and x and y' have unlike signs, no matter what the values of α and α' may be, that is, no matter what the position of the angle MOM' about the point O, as shown in the Fig. 256, $MOM' = M_1OM_1' = M_2OM_2' = M_3OM_3'$, may be, it follows that:

$$y = x' \text{ and } x = -y'.$$

Replacing, as in the preceding article, y, x, y' and x' by their values as given in article (**1041**),

$$\sin\alpha = \cos\alpha' \text{ and } \cos\alpha = -\sin\alpha'.$$

Thus, *for two angles whose difference is equal to a right angle, the sine of the greater is equal to the cosine of the smaller, and has the same sign, and its cosine is equal to the sine of the latter but has a different sign.*

Dividing the two equations,

$$\frac{\sin\alpha}{\cos\alpha} = -\frac{\cos\alpha'}{\sin\alpha'},$$

from which

$$\tan\alpha = -\frac{1}{\tan\alpha'}, \quad \tan\alpha\tan\alpha' = -1,$$

and

$$\tan\alpha = -\cot\alpha'.$$

EXAMPLE. What is the sine, cosine, and tangent of an angle of 165°?

The relation $\alpha - \alpha' = 90°$ becomes $165° - \alpha' = 90°$, and $\alpha' = 165° - 90° = 75°$.

From the table (1071), cos 75° = 0.25882, sin 75° = 0.96593, and the cot 75° = 0.26795, we have then, sin 165° = 0.25882, cos 165° = − 0.96593, and tan 165° = − 0.26795.

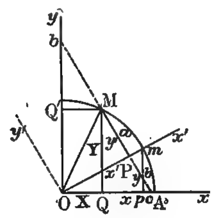


Fig. 257

1045. *Relations between the trigonometric functions of two angles or arcs a and b and those of their sum (a + b).*

Let $mOA = b$, $MOm = a$, and then $MOA = (a + b)$.

Investigating the relations which exist between the coördinates $MQ = Y$, $OQ = X$, and those $OP = x'$ and $MP = y'$ of the same point M with reference to the two systems of rectangular coördinates x, y, and $x'y'$, Y being the projection of OPM on Oy, and X being that of OPM on Ox, we find (1040):

$$Y = x' \cos POy + y' \cos PbO,$$
$$X = x' \cos b + y' \cos Mcx.$$

The angle $Mcx = y'Ox$, the difference between which and a right angle is equal to $x'Ox$ or b; then

$$\cos Mcx = -\sin b. \qquad (1044)$$

This relation exists no matter what the position of M may be, that is, regardless of the values of a and b.

The angle POy is the complement of the angle b; then

$$\cos POy = \sin b. \qquad (1043)$$

This relation exists no matter what value b may have; because, this angle being obtuse, the difference between it and a right angle is the angle POy, and we have again:

$$\cos POy = \sin b. \qquad (1044)$$

The angle $PbO = b$ (629); and

$$\cos PbO = \cos b.$$

Substituting these values of the cosines of Mcx, POy, and PbO in the equations of X and Y:

$$Y = x' \sin b + y' \cos b,$$
$$X = x' \cos b - y' \sin b.$$

Since (**1041**):

$$Y = r \sin (a + b), \qquad X = r \cos (a + b),$$
$$y' = r \sin a, \qquad x' = r \cos a,$$

the preceding equations become,

$$r \sin (a + b) = r \cos a \sin b + r \sin a \cos b,$$
$$r \cos (a + b) = r \cos a \cos b - r \sin a \sin b.$$

Cancelling r,

$$\sin (a + b) = \sin a \cos b + \cos a \sin b, \tag{1}$$
$$\cos (a + b) = \cos a \cos b - \sin a \sin b. \tag{2}$$

$$\tan (a + b) = \frac{\sin (a + b)}{\cos (a + b)} = \frac{\sin a \cos b + \cos a \sin b}{\cos a \cos b - \sin a \sin b}. \quad (\mathbf{1041})$$

Dividing both terms by $\cos a \cos b$, and substituting the tangent for the sine divided by the cosine,

$$\tan (a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}. \tag{3}$$

1046. *The trigonometric functions of the difference* $(a - b)$ *of two angles* a *and* b *expressed in terms of the functions of the two angles.* Retaining the same value of b, given in the formulas (1) and (2) of the preceding article, and making $(a + b) = a'$, which gives $a = (a' - b)$, we have:

$$\sin a' = \sin (a' - b) \cos b + \cos (a' - b) \sin b, \tag{1}$$
$$\cos a' = \cos (a' - b) \cos b - \sin (a' - b) \sin b. \tag{2}$$

Putting $a' = a$ and reducing the equation (2) (**511**):

$$\cos (a - b) = \frac{\cos a}{\cos b} + \sin (a - b) \frac{\sin b}{\cos b}. \tag{3}$$

Substituting this value in equation (1),

$$\sin (a - b) \left(\cos b + \frac{\sin^2 b}{\cos b} \right) = \sin a - \frac{\cos a \sin b}{\cos b}. \tag{4}$$

From (509, 4th):

$$\cos b + \frac{\sin^2 b}{\cos b} = \frac{\cos^2 b + \sin^2 b}{\cos b} = \frac{1}{\cos b}. \quad (\mathbf{1041})$$

Substituting this value in equation (4),

$$\frac{\sin (a - b)}{\cos b} = \frac{\sin a \cos b - \cos a \sin b}{\cos b};$$

that is, $\qquad \sin (a - b) = \sin a \cos b - \cos a \sin b.$

Substituting this value of $\sin(a - b)$ in equation (3),

$$\cos(a - b) = \cos a \cos b + \sin a \sin b.$$

Dividing one by the other,

$$\tan(a - b) = \frac{\sin(a - b)}{\cos(a - b)} = \frac{\sin a \cos b - \cos a \sin b}{\cos a \cos b + \sin a \sin b}.$$

Dividing both terms by $\cos a \cos b$,

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}.$$

1047. *Relations between the trigonometric functions of an angle a and those of one of twice its value $2a$.* Making $b = a$ in the values given for $\sin(a + b)$, $\cos(a + b)$, and $\tan(a + b)$ (**1045**):

1st. $\sin(a + b) = \sin 2a = \sin a \cos a + \cos a \sin a,$

that is, $$\sin 2a = 2 \sin a \cos a; \qquad (a)$$

2d. $$\cos(a + b) = \cos 2a = \cos^2 a - \sin^2 a. \qquad (1)$$

From (1041), $\cos^2 a = 1 - \sin^2 a.$

Substituting this value in equation (1),

$$\cos 2a = 1 - 2 \sin^2 a. \qquad (b)$$

If, instead of eliminating $\cos^2 a$ from equation (1), $\sin^2 a$ is eliminated:

$$\cos 2a = 2 \cos^2 a - 1; \qquad (b')$$

3d. $$\tan(a + b) = \tan 2a = \frac{2 \tan a}{1 - \tan^2 a}. \qquad (c)$$

1048. *Relations between the trigonometric functions of an angle a and those of another of half its value $\frac{a}{2}$.*

Substituting a for $2a$ and $\frac{1}{2}a$ for a in the formulas of the preceding article:

1st. Formula (a) gives:

$$\sin a = 2 \sin \frac{1}{2} a \cos \frac{1}{2} a.$$

From formula (b),

$$\cos a = 1 - 2 \sin^2 \frac{1}{2} a,$$

and (571),

$$\sin\frac{1}{2}a = \pm\sqrt{\frac{1-\cos a}{2}}.$$

2d. Formula (b') gives:

$$\cos a = 2\cos^2\frac{1}{2}a - 1,$$

and

$$\cos\frac{1}{2}a = \pm\sqrt{\frac{1+\cos a}{2}}.$$

3d. Formula (c) becomes:

$$\tan a = \frac{2\tan\frac{1}{2}a}{1-\tan^2\frac{1}{2}a}.$$

Transposing,

$$\tan^2\frac{1}{2}a + \frac{2}{\tan a}\tan\frac{1}{2}a = 1.$$

Solving,

$$\tan\frac{1}{2}a = -\frac{1}{\tan a} \pm\sqrt{\frac{1}{\tan^2 a}+1} = \frac{1}{\tan a}(-1\pm\sqrt{1+\tan^2 a}).$$

Also from (1041):

$$\tan\frac{1}{2}a = \frac{\sin\frac{1}{2}a}{\cos\frac{1}{2}a} = \pm\sqrt{\frac{1-\cos a}{1+\cos a}}.$$

1049. *To obtain the trigonometric functions of* $3a$ *in terms of those of* a, put $b = 2a$ in the formulas (1), (2), and (3) of (**1045**), which gives:

$$\sin 3a = \sin a\cos 2a + \cos a\sin 2a,$$

$$\cos 3a = \cos a\cos 2a - \sin a\sin 2a,$$

$$\tan 3a = \frac{\tan a + \tan 2a}{1-\tan a\tan 2a}.$$

Substituting the values of $\sin 2a$, $\cos 2a$, and $\tan 2a$ given in formulas (a), (b), and (c) (**1047**), and simplifying, we have:

$$\sin 3a = 3\sin a - 4\sin^3 a, \quad (1)$$

$$\cos 3a = 4\cos^3 a - 3\cos a, \quad (2)$$

$$\tan 3a = \frac{3\tan a - \tan^3 a}{1 - 3\tan^2 a}. \tag{3}$$

1050. *By making* $b = 3a$, *then* $b = 4a$, etc., *in the formulas of* (1045), *the relations which exist between the trigonometric functions of any multiple of* a *and those of* a *may be obtained.*

1051. *Changing* a *to* $\frac{1}{3}a$, *the formulas* (1), (2), *and* (3) *of* (1049) give:

$$\sin a = 3\sin\frac{1}{3}a - 4\sin^3\frac{1}{3}a,$$

$$\cos a = 4\cos^3\frac{1}{3}a - 3\cos\frac{1}{3}a,$$

$$\tan a = \frac{3\tan\frac{1}{3}a - \tan^3\frac{1}{3}a}{1 - 3\tan^2\frac{1}{3}a}.$$

These formulas express the relations which exist between the sine, cosine, and tangent of an angle, which is equal to three times another, and the sine, cosine, and tangent of the latter.

1052. *Other relations between the trigonometric expressions, which are frequently used in practice.*

1st. By addition and subtraction of the values of the sine and cosine of $(a + b)$ and $(a - b)$ (1045, 1046), we obtain:

$$\begin{aligned}
\sin(a+b) + \sin(a-b) &= 2\sin a\cos b,\\
\sin(a+b) - \sin(a-b) &= 2\cos a\sin b,\\
\cos(a-b) + \cos(a+b) &= 2\cos a\cos b,\\
\cos(a-b) - \cos(a+b) &= 2\sin a\sin b.
\end{aligned}$$

These formulas may be used to transform the product of two trigonometric expressions to a sum or difference.

2d. Putting $(a + b) = p$ and $(a - b) = q$ in the preceding formulas, from which (520) $a = \frac{1}{2}(p + q)$ and $b = \frac{1}{2}(p - q)$, we have:

$$\sin p + \sin q = 2\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q),$$

$$\sin p - \sin q = 2\cos\frac{1}{2}(p+q)\sin\frac{1}{2}(p-q),$$

$$\cos p + \cos q = 2 \cos \frac{1}{2}(p+q) \cos \frac{1}{2}(p-q),$$

$$\cos q - \cos p = 2 \sin \frac{1}{2}(p+q) \sin \frac{1}{2}(p-q).$$

These formulas are frequently used in logarithmic calculations, to change a sum or difference to a product.

3d. From these last formulas, by division; noting that

$$\frac{\sin A}{\cos A} = \tan A = \frac{1}{\cot A}: \tag{1041}$$

$$\frac{\sin p + \sin q}{\sin p - \sin q} = \frac{\sin \frac{1}{2}(p+q) \cos \frac{1}{2}(p-q)}{\cos \frac{1}{2}(p+q) \sin \frac{1}{2}(p-q)} = \frac{\tan \frac{1}{2}(p+q)}{\tan \frac{1}{2}(p-q)},$$

$$\frac{\sin p + \sin q}{\cos p + \cos q} = \frac{\sin \frac{1}{2}(p+q)}{\cos \frac{1}{2}(p+q)} = \tan \frac{1}{2}(p+q),$$

$$\frac{\sin p + \sin q}{\cos q - \cos p} = \frac{\cos \frac{1}{2}(p-q)}{\sin \frac{1}{2}(p-q)} = \cot \frac{1}{2}(p-q),$$

$$\frac{\sin p - \sin q}{\cos p + \cos q} = \frac{\sin \frac{1}{2}(p-q)}{\cos \frac{1}{2}(p-q)} = \tan \frac{1}{2}(p-q),$$

$$\frac{\sin p - \sin q}{\cos q - \cos p} = \frac{\cos \frac{1}{2}(p+q)}{\sin \frac{1}{2}(p+q)} = \cot \frac{1}{2}(p+q),$$

$$\frac{\cos p + \cos q}{\cos q - \cos p} = \frac{\cos \frac{1}{2}(p+q) \cos \frac{1}{2}(p-q)}{\sin \frac{1}{2}(p+q) \sin \frac{1}{2}(p-q)} = \cot \frac{1}{2}(p+q) \cot \frac{1}{2}(p-q).$$

From the first formula it is seen that *the sum of the sines of two angles is to their difference as the tangent of half the sum of these angles is to half their difference.*

4th. Some other convenient transformations of products, sums, and differences are given below:

$$\tan a \pm \tan b = \frac{\sin a}{\cos a} \pm \frac{\sin b}{\cos b} = \frac{\sin a \cos b \pm \sin b \cos a}{\cos a \cos b} = \frac{\sin (a \pm b)}{\cos a \cos b},$$

$$\sec a + \sec b = \frac{1}{\cos a} + \frac{1}{\cos b} = \frac{\cos a + \cos b}{\cos a \cos b} = \frac{2 \cos \frac{1}{2}(a+b) \cos \frac{1}{2}(a-b}{\cos a \cos b},$$

$$\sec a - \sec b = \frac{1}{\cos a} - \frac{1}{\cos b} = \frac{\cos b - \cos a}{\cos a \cos b} = \frac{2 \sin \frac{1}{2}(a-b) \sin \frac{1}{2}(a+b)}{\cos a \cos b},$$

$$\sin a + \cos b = \sin a + \sin (90^\circ - b) = 2 \sin \left(45^\circ + \frac{a-b}{2}\right) \sin \left(45^\circ + \frac{a+b}{2}\right),$$

$$\sin a + \cos a = 2 \sin 45^\circ \sin (45^\circ + a) = \sqrt{2} \sin (45^\circ + a),$$

$$\sin a - \cos b = \sin a - \sin (90^\circ - b) = -2 \sin \left(45^\circ - \frac{a+b}{2}\right) \sin \left(45^\circ - \frac{a-b}{2}\right),$$

$$\sin a - \cos a = -2 \sin (45^\circ - a) \sin 45^\circ = -\sqrt{2} \sin (45^\circ - a),$$

$$\sin^2 a - \sin^2 b = \sin (a + b) \sin (a - b),$$

$$\cos^2 a + \cos^2 b - 1 = \cos (a + b) \cos (a - b),$$

$$1 + \sin a = 1 + \cos (90^\circ - a) = 2 \cos^2 \left(45^\circ - \frac{a}{2}\right),$$

$$1 - \sin a = 1 - \cos (90^\circ - a) = 2 \sin^2 \left(45^\circ - \frac{a}{2}\right),$$

$$\sqrt{\frac{1 - \cos a}{1 + \cos a}} = \sqrt{\frac{2 \sin^2 \frac{a}{2}}{2 \cos^2 \frac{a}{2}}} = \tan \frac{a}{2},$$

$$\sqrt{\frac{1 - \sin a}{1 + \sin a}} = \sqrt{\frac{2 \sin^2 \left(45^\circ - \frac{a}{2}\right)}{2 \cos^2 \left(45^\circ - \frac{a}{2}\right)}} = \tan \left(45^\circ - \frac{a}{2}\right),$$

$$1 \pm \tan a = \frac{\sqrt{2} \sin (45^\circ \pm a)}{\cos a}.$$

For $a + b + c = \pi = 180°$, we have:

$$\tan a + \tan b + \tan c = \tan a, \tan b, \tan c,$$

$$\sin a + \sin b + \sin c = 4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2},$$

$$\cot \frac{a}{2} + \cot \frac{b}{2} + \cot \frac{c}{2} = \cot \frac{a}{2} \cot \frac{b}{2} \cot \frac{c}{2},$$

$$\sin^2 \frac{a}{2} + \sin^2 \frac{b}{2} + \sin^2 \frac{c}{2} + 2 \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2} = 1.$$

CALCULATION OF THE TRIGONOMETRIC TABLES

1053. The trigonometric tables were described in article (1032). It will now be shown how they are calculated.

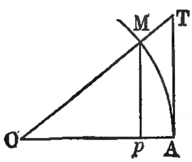


Fig. 258

1st. When an angle less than 90° is decreased, the ratio of the arc, which measures the angle, to the sine diminishes and approaches one as a limit (186).

Supposing OM or $r = 1$, we have (1025) $Mp = \sin \alpha$, $Op = \cos \alpha$, and $AT = \tan \alpha$. Letting a equal the length of the arc AM, we have,

$$a > \sin \alpha \quad \text{and} \quad a < \tan \alpha.$$

Since the $\sin \alpha$ or Mp is half the chord subtended by an arc twice as great as a, we have (649):

$$\alpha > \sin \alpha. \tag{1}$$

Furthermore, the surface of the sector OAM being less than that of the triangle OAT, we have:

$$\frac{1}{2} OA \times a < \frac{1}{2} OA \times \tan \alpha, \qquad (718, 760)$$

and

$$a < \tan \alpha \quad \text{or} \quad (1041)\ a < \frac{\sin \alpha}{\cos \alpha}. \tag{2}$$

From the inequalities (1) and (2), we have respectively:

$$\frac{a}{\sin \alpha} > 1 \quad \text{and} \quad \frac{a}{\sin \alpha} < \frac{1}{\cos \alpha},$$

which shows that the ratio of the length of the arc to the sine is included between 1 *and the quantity* $\frac{1}{\cos a}$ *always greater than* 1.

Since, as a decreases, $\frac{1}{\cos a}$ decreases and a approaches 1 as a limit, *it follows that* $\frac{a}{\sin a}$, *which is smaller than* $\frac{1}{\cos a}$, *may also be considered as having* 1 *for a limit.*

2d. From the inequalities

$$a < \tan a \text{ and } a > \sin a \text{ or } a > \tan a \cos a, \qquad (1041)$$

we deduce:

$$\frac{a}{\tan a} < 1 \quad \text{and} \quad \frac{a}{\tan a} > \cos a,$$

which shows that the ratio $\frac{a}{\tan a}$, *always greater than* $\cos a$, *lies between* 1 *and* $\cos a$, *and consequently has* 1 *for its limit.*

3d. It will now be shown that *the difference between the length* a *of the arc and the sine is less than one-fourth of the cube of the arc* a.

From the inequality (1st)

$$\frac{1}{2}a < \frac{\sin\frac{1}{2}a}{\cos\frac{1}{2}a},$$

we have:

$$\sin\frac{1}{2}a > \frac{1}{2}a\cos\frac{1}{2}a.$$

Multiplying this inequality by the equation

$$\sin a = 2\sin\frac{1}{2}a\cos\frac{1}{2}a, \qquad (1058)$$

and cancelling the common factor $\frac{1}{2}a$,

$$\sin a > a\cos^2\frac{1}{2}a,$$

or

$$\sin a > a\left(1 - \sin^2\frac{1}{2}a\right),$$

or

$$\sin a > a - a\sin^2\frac{1}{2}a,$$

and $$a - \sin a < a \sin^2 \frac{1}{2} a.$$

Multiplying this inequality by

$$\left(\sin \frac{1}{2} a < \frac{1}{2} a\right)^2 = \sin^2 \frac{1}{2} a < \frac{a^2}{4},$$

and cancelling the common factor $\sin^2 \frac{1}{2} a$,

$$a - \sin a < \frac{a^3}{4}.$$

EXAMPLE. Determine the error for an angle of 10″ in taking $\sin 10'' = a$, where a is the length of the arc.

The radius being 1, the arc corresponding to 180° is

$$\pi r = \pi = 3.1415926\ldots, \tag{751}$$

and the length of an arc corresponding to 10″ is (758)

$$a = \frac{3.1415926 \cdots \times 10}{180 \times 60 \times 60} = 0.000048481368110,$$

and $$\frac{a^3}{4} = 0.000000000000032 \cdots$$

Thus for an angle of 10″, in taking the arc for the sine, the error is less than about three-tenths of a decimal unit of the thirteenth order. Therefore we may write:

$$\sin 10'' = 0.0000484813681.$$

With the same degree of accuracy we may write:

$$\cos 10'' = \sqrt{1 - \sin^2 10''}, \tag{1041}$$
$$\cos 10'' = 0.9999999988248.$$

4th. With the help of sin 10″, cos 10″ and the following formulas,

$$\left.\begin{aligned} \sin (a + b) &= \sin a \cos b + \cos a \sin b \\ \cos (a + b) &= \cos a \cos b - \sin a \sin b \end{aligned}\right\} \tag{1045}$$

the sines and cosines of all the angles from 0° to 45° may be found.

The tangent and cotangent of each of these angles may be obtained from the formulas

$$\tan a = \frac{\sin a}{\cos a} \quad \text{and} \quad \cot a = \frac{\cos a}{\sin a}. \tag{1041}$$

5th. The trigonometric functions of the angles from 0° to 45° give those from 45° to 90°, as was shown in (1031) and (1043).

Finally, having the trigonometric functions for the angles up to 90°, from what was said in (1028), they can be determined for any angle larger.

It is evident that this method of calculating the trigonometric functions is long and fatiguing; it has been simplified by proceeding in another manner, but since it is not our purpose to calculate tables, this simpler method will not be given.

In practice, the engineer scarcely ever deals with angles smaller than 1′, therefore no angles smaller than 1′ are given in the tables (1071). In case it is desired to work with smaller angles, the method of interpolation as used in the logarithmic tables may be resorted to.

PRINCIPLES USED IN SOLVING TRIANGLES

1054. Remark. For the sake of simplicity in that which follows, the angles of the triangles will be represented by the letters A, B, and C written at the vertices and the sides respectively opposed to these angles by the letters a, b, and c, written near the middle of these sides. In the case of a right triangle the right angle is designated by A and the hypotenuse by a.

1055. Theorem 1. *In any right triangle, each leg is equal to the hypotenuse multiplied by the cosine of the adjacent angle.*

Since b and c may be considered as projections of a upon the legs, we have

$$b = a \cos C, \text{ and } c = a \cos B. \qquad (1040)$$

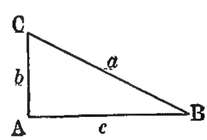


Fig. 259

The angles B and C being complementary, $\cos C = \sin B$, and $\cos B = \sin C$ (1043), and therefore

$$b = a \sin B, \text{ and } c = a \sin C.$$

Thus, *in any right triangle, each leg is equal to the hypotenuse multiplied by the cosine of the adjacent angle or the sine of the opposite angle.*

COROLLARY. The two equations $b = a \sin B$, and $c = a \cos B$, give:

$$\frac{b}{c} = \frac{\sin B}{\cos B} = \tan B, \tag{1041}$$

from which $$b = c \tan B$$

and $$c = b \tan C.$$

Since $\tan B = \cot C$, and $\tan C = \cot B$ (1043), we have:

$$b = c \cot C, \text{ and } c = b \cot B.$$

Thus, *in any right triangle. each leg is equal to the other multiplied by the tangent of the angle opposite the first leg or by the co-tangent of the adjacent angle.*

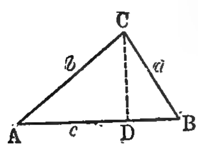


Fig. 260

1056. THEOREM 2. *In any plane triangle, the sines of the angles are to each other as the opposite sides.*

Dropping a perpendicular CD from the vertex C on the side c:

1st. In case this perpendicular falls upon c between the vertices A and B, from the right triangles ADC and BDC we have:

$$CD = b \sin A, \quad \text{and} \quad CD = a \sin B. \tag{1065}$$

Putting these two values of CD equal to each other,

$$b \sin A = a \sin B,$$

and $$\sin A : \sin B = a : b. \tag{340}$$

2d. In case the perpendicular falls on the side c extended, in the triangle ADC:

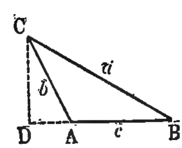


Fig. 261

$$CD = b \sin CAD,$$

and since the angles CAD and A are supplementary they have the same sine (1028), and:

$$CD = b \sin A.$$

From the triangle BDC,

$$CD = a \sin B.$$

Putting these two equal to each other:

$$\sin A : \sin B = a : b.$$

3d. In case the D should coincide with A, the triangle would be a right triangle, and we have directly (1st):

$$CD \text{ or } b = a \sin B;$$

and noting that $\sin A = 1$, we have,

$$b \sin A = a \sin B,$$

and

$$\sin A : \sin B = a : b.$$

If, instead of drawing the perpendicular from the vertex C, it had been drawn from A or B, we would have respectively:

$$\sin B : \sin C = b : c,$$

and

$$\sin C : \sin A = c : a.$$

These three equations prove that which was to be demonstrated, namely:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

1057. THEOREM 3. *In any triangle, the square of one side is equal to the sum of the squares of the other two, less twice their product times the cosine of the included angle.* Thus, for example (Fig. 291):

$$a^2 = b^2 + c^2 - 2\,bc \cos A. \tag{1}$$

It was demonstrated in geometry (734) that, in any triangle, the square of one side is equal to the sum of the squares of the other two plus or minus twice the product of one of these two sides and the projection of the other upon it, according as the angle opposite the first side is obtuse or acute.

Thus, Figs. 260 and 261 give respectively:

$$a^2 = b^2 + c^2 - 2\,c \times AD, \tag{2}$$
$$a^2 = b^2 + c^2 + 2\,c \times AD. \tag{3}$$

In the right triangle ADC (Fig. 290),

$$AD = b \cos A,$$

and in Fig. 291

$$AD = b \cos DAC, \text{ or } AD = -\,b \cos A,$$

A being the supplement of DAC (1028). These values of AD substituted in the formulas (2) and (3) reduce them to the same general form (1).

When the angle A is a right angle, its cosine is zero, and this general formula becomes (730):

$$a^2 = b^2 + c^2.$$

1058. Theorem 4. The algebraic sum of the projections of two sides of a triangle upon the third side is equal to the third side (1062). Thus, in Figs. 260 and 261,

$$c = a \cos B + b \cos A.$$

SOLUTION OF RIGHT TRIANGLES

1059. To solve a triangle having three of its six parts, angles or sides, given, is to find the remaining three parts. Three parts determine the triangle, but at least one of these parts must be a side; three angles do not determine a triangle (1018).

The three unknowns may be deduced in a general way from three following equations between the unknowns and the knowns (516). From (1057),

$$\begin{aligned} a^2 &= b^2 + c^2 - 2\,bc \cos A, \\ b^2 &= a^2 + c^2 - 2\,ac \cos B, \\ c^2 &= a^2 + b^2 - 2\,ab \cos C. \end{aligned}$$

The following system, which is equivalent to the above, may also be used (1058):

$$\begin{aligned} a &= b \cos C + c \cos B, \\ b &= a \cos C + c \cos A, \\ c &= a \cos B + b \cos A. \end{aligned}$$

The following relation often simplifies the calculations:

$$A + B + C = 180°.$$

1060. To say that a triangle is a right triangle determines one of its angles, therefore two other parts determine the triangle (1059).

Case 1. *The hypotenuse a* (Fig. 259) *and one of the acute angles B being given, find the angle C and the two sides b and c.* The triangle being a right triangle, the acute angles are complementary, and we have,

$$C = 90° - B.$$

Furthermore,

$$b = a \sin B, \quad \text{and} \quad c = a \cos B. \qquad \textbf{(1055)}$$

CASE 2. *The side b and the angle B being given, find C, a, and c.*

$$C = 90° - B.$$

From the relation $b = a \sin B$ (1055):

$$a = \frac{b}{\sin B}.$$

Also from (1055, corollary),

$$c = b \tan C, \text{ or } c = b \cot B = \frac{b}{\tan B}.$$

CASE 3. *The hypotenuse a and the side b being given, find c, B, and C.*

The triangle being a right triangle (730),

$$c = \sqrt{a^2 - b^2}.$$

If c is to be calculated by logarithms, reduce to the form,

$$c = \sqrt{(a + b)(a - b)}. \qquad (729)$$

From the relation $b = a \sin B$,

$$\sin B = \frac{b}{a}.$$

Having found B,

$$C = 90° - B, \text{ or } \cos C = \sin B = \frac{b}{a}.$$

CASE 4. *The sides b and c being given, find the hypotenuse and the angles B and C.*

Since $$b = c \tan B,$$

$$\tan B = \frac{b}{c}, \quad \text{also } \cot C = \tan B = \frac{b}{c}.$$

Having found B, $\quad C = 90° - B;$

then, from the relation $b = a \sin B$,

$$a = \frac{b}{\sin B};$$

or directly, $\quad a = \sqrt{b^2 + c^2}.$

Then $\quad b = a \sin B,$

and $\quad C = 90° - B.$

But this last method leads to longer calculations than the first.

SOLUTION OF PLANE TRIANGLES

1061. Case 1. *One side a and two angles A and B of the triangle ABC* (Fig. 260) *are given, to find the other two sides b and c, and the third angle C.*

In any triangle, the sum of the three angles being equal to two right angles,

$$C = 180° - (A + B).$$

From the theorem (**1056**), the sines of the angles of a triangle are proportional to the opposite sides,

$$\sin A : \sin B = a : b, \quad \text{and} \quad \sin A \sin C = a : c;$$

transposing,

$$b = a\frac{\sin B}{\sin A}, \quad \text{and} \quad c = a\frac{\sin C}{\sin A},$$

or

$$\log b = \log a + \log \sin B - \log \sin A,$$
$$\log c = \log a + \log \sin C - \log \sin A.$$

The area of the triangle can be calculated from the formula:

$$S = \frac{a^2 \sin B \sin C}{2 \sin A}, \tag{1065}$$

or $\quad \log S = 2 \log a + \log \sin B + \log \sin C - \log 2 - \log \sin A.$

Example. Let $a = 6789.24$ yds. $A = 42° 17' 23.4''$ and $B = 87° 24' 11.8''$ be given, to calculate the angle C, the sides c and b and the area S.

$$C = 180° - (A + B) = 50° 18' 24.8'';$$

calculation of b:

$$\begin{array}{rl} \log a = & 3.8318212 \\ \log \sin B = & \bar{1}.9995538 \\ -\log \sin A = - & \underline{\bar{1}.8279385} \\ \log b = & 4.0034365 \end{array} \qquad \begin{array}{rl} \log a = & 3.8318212 \\ \log \sin B = & \bar{1}.9995538 \\ -\log \sin A = & \underline{0.1720615} \\ \log b = & 4.0034365 \end{array} \quad (1032)$$

$$b = 10{,}079.44 \text{ yds.};$$

calculation of c:

$$\begin{array}{rl} \log a = & 3.8318212 \\ \log \sin C = & \bar{1}.8861953 \\ -\log \sin A = - & \underline{\bar{1}.8279385} \\ \log c = & 3.8900780 \end{array} \qquad \begin{array}{rl} \log a = & 3.8318212 \\ \log \sin C = & \bar{1}.8861953 \\ -\log \sin A = & \underline{0.1720615} \\ \log c = & 3.8900780 \end{array}$$

$$c = 7763.86 \text{ yds.};$$

calculation of S:

$$
\begin{array}{rlcrl}
2\log a = & 7.6636424 & \qquad & 2\log a = & 7.6636424 \\
\log\sin B = & \bar{1}.9995538 & & \log\sin B = & \bar{1}.9995538 \\
\log\sin C = & \bar{1}.8861953 & & \log\sin C = & \bar{1}.8861953 \\
-\log 2 = & -\ 0.3010300 & & -\log 2 = & \bar{1}.6989700 \\
-\log\sin A = & -\ \bar{1}.8279385 & & -\log\sin A = & 0.1720615 \\
\hline
\log S = & 7.4204230 & & \log S = & 7.4204230
\end{array}
$$

$$S = 26.328300 \text{ sq. yds.}$$

Two methods were followed in the logarithmic calculations. In the first the true logarithms were written down, and those which were to be subtracted were preceded by the sign $-$ minus. Applying the rule in (33, 2d), the successive figures of the logarithms which were to be subtracted were subtracted from the partial sums of the figures of the logarithms which were to be added.

Thus, $\log S$ was obtained by saying 4 and 8, 12 and 3, 15, less 5, 10; write zero in the result and add 1 to the next column, which gives $1 + 2 + 3 + 5 = 11$, less 8, 3; write 3 in the result and continue thus, observing the rule of subtraction (29).

For the characteristics which are negative, the rules for the addition and subtraction of algebraic quantities were followed (460, 461). Thus, they are subtracted if they belong to logarithms not preceded by the sign $-$; such are $\log\sin B$ and $\log\sin C$. On the contrary, they are added if they belong to logarithms preceded by the sign $-$; such is $\log\sin A$.

In the second method, the sign of each logarithm to be subtracted was changed, which left nothing but quantities to be added. Thus, in the calculation of S, having $\log 2 = 0.3010300$, we have $-\log 2 = -0.3010300 = \bar{1} + 1 - 0.3010300 = \bar{1}.6989700$. In the same manner, having $\log\sin A = \bar{1}.8279385$, we have $-\log\sin A = 1 - 0.8279385 = 0.1720615$. The value of logarithms whose signs have been changed is obtained according to the rule of (403) relating to the complement of a number.

1062. CASE 2. *Two sides a and b and an angle A opposite one of them being given, to find c, B, and C* (947).

We have, $$\sin A : \sin B = a : b, \tag{1056}$$

and $$\sin B = \frac{b\sin A}{a}; \tag{a}$$

then $$C = 180° - (A + B);$$

having C, $$\sin A : \sin C = a : c,$$

we have $$c = a \frac{\sin C}{\sin A},$$

and the area $$S = \frac{ab \sin C}{2}. \tag{1065}$$

REMARK. This solution needs some explanation. Since the same value (a) of $\sin B$ corresponds to two supplementary angles, one acute and one obtuse (1029), it is necessary to determine in what case B is obtuse and in what it is acute. This leads to the following remarks, based upon the fact that in any triangle there cannot be more than one right or obtuse angle (652), and that the greatest angle is opposite the greatest side (638).

1st. The given angle A being right or obtuse, the angle B is necessarily acute. Having $A > B$, we should also have $a > b$. There is always a solution, but there is only one, which may be seen from the Fig. 262.

2d. The given angle A being acute and $a > b$, then $A > B$, and it follows that B is acute, and there is but one solution. The

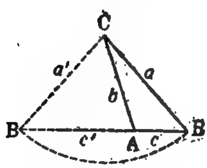


Fig. 262

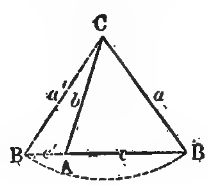


Fig. 263

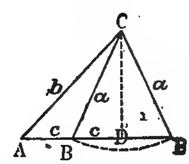


Fig. 264

Fig. 263 shows that the angle A would be obtuse in the second triangle $AB'C$ which has a and b for its sides.

In the case where A is acute and $a = b$, B' coincides with A, and the only solution is an isosceles triangle.

3d. The given angle A acute and $a < b$. In this case $B > A$ may be acute or obtuse, therefore there are two solutions, as indicated in (Fig. 264). In the triangle ABC, which satisfies the given conditions, the angle B is acute; in the triangle $AB'C$, which also satisfies the given conditions, $B' = 180° - B$ is obtuse.

There are two solutions when $a < b$ is greater than $CD = b \sin A$, that is, when

$$a > b \sin A \quad \text{or} \quad \frac{b \sin A}{a} < 1.$$

When $a = CD = b \sin A$, the arc BB' is tangent to AB at the point D, the two triangles ABC and $AB'C$ coincide with the right triangle ADC, and there is but one solution.

Finally, if $a < CD$ or $a < b \sin A$, the arc BB' would have no point common with AB, and there would be no solution. If, instead of commencing by determining the angles B and C, it had been desired to first determine the side c:

$$a^2 = b^2 + c^2 - 2\,bc \cos A, \qquad (1057)$$

from which $$c^2 - 2\,b \cos A \times c = a^2 - b^2,$$

and therefore, $$c = b \cos A \pm \sqrt{a^2 - b^2 + b^2 \cos^2 A}, \qquad (572)$$

or $$c = b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A}. \qquad (1041)$$

1063. CASE 3. *Having two sides a and b and the included angle C given, to find c, A, and B.*

1st. We have, $$c = \sqrt{a^2 + b^2 - 2\,ab \cos C}; \qquad (1057)$$

c being known, $$\sin A = \frac{a \sin C}{c}, \qquad (1056)$$

then $$B = 180 - (A + C).$$

2d. Commencing by determining A:

$$\sin A : \sin B = a : b.$$

In this proportion there are two unknowns, $\sin A$ and $\sin B$; one is eliminated by writing (349):

$$(\sin A + \sin B) : (\sin A - \sin B) = (a + b) : (a - b),$$

or substituting an equal ratio for the first member (1052, 3d):

$$\tan \frac{1}{2}(A + B) : \tan \frac{1}{2}(A - B) = (a + b) : (a - b).$$

$$\frac{1}{2}(A + B) = \frac{1}{2}(180 - C) = m^\circ$$

being known, this proportion contains only one unknown, namely, $\tan \frac{1}{2}(A - B)$, whose value is:

$$\tan \frac{1}{2}(A - B) = \frac{a - b}{a + b} \tan \frac{1}{2}(A + B).$$

Putting $\frac{1}{2}(A - B) = n^\circ$, then having half the sum m° and half

the difference n° of the angles A and B, from (520),

$$A = m^\circ + n^\circ \text{ and } B = m - n^\circ.$$

Having found A and B (1056),

$$c = \frac{a \sin C}{\sin A}.$$

This solution is to be preferred where logarithms are to be used (1061).

The area is given by the formula:

$$S = \frac{ab \sin C}{2}. \qquad (1065)$$

1064. CASE 4. *The three sides a, b, and c being given, to determine the three angles A, B, and C.*

Writing $$a^2 = b^2 + c^2 - 2\,bc \cos A, \qquad (1057)$$

we have, $$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}. \qquad (a)$$

Similar formulas will give B and C, or having determined A and B,

$$C = 180 - (A + B),$$

which in any case should be used as a check.

If logarithms are to be used, a more convenient formula than (a) can be used (1061), which is developed as follows:

$$2 \sin^2 \frac{1}{2} A = 1 - \cos A. \qquad \text{(1048, 1st)}$$

Substituting the value of cos A given in (a),

$$2 \sin^2 \frac{1}{2} A = 1 - \frac{b^2 + c^2 - a^2}{2\,bc} = \frac{a^2 - b^2 - c^2 + 2\,bc}{2\,bc}$$

$$= \frac{a^2 - (b - c)^2}{2\,bc} = \frac{(a + b - c)\,(a - b + c)}{2\,bc}, \qquad (728, 729)$$

from which

$$\sin \frac{1}{2} A = \sqrt{\frac{(a + b - c)\,(a - b + c)}{4\,bc}}.$$

This formula may be simplified by making the following substitutions:

$$a + b + c = 2\,p,$$

then $a + b - c = 2(p - c)$, and $a - b + c = 2(p - b)$, which gives

$$\sin\frac{1}{2}A = \sqrt{\frac{(p-b)(p-c)}{bc}}. \qquad (b)$$

$\frac{1}{2}A$ being necessarily an acute angle, its value is determined by its sine, as is likewise that of the angle A.

In the same manner,

$$\sin\frac{1}{2}B = \sqrt{\frac{(p-a)(p-c)}{ac}},$$

and

$$\sin\frac{1}{2}C = \sqrt{\frac{(p-a)(p-b)}{ab}}.$$

As proof we may write,

$$C = 180° - (A + B).$$

In the same manner the values of $\cos\frac{1}{2}A$ and $\tan\frac{1}{2}A$ may be found, since:

$$\cos\frac{1}{2}A = \sqrt{1 - \sin^2\frac{1}{2}A}. \qquad (1041)$$

Substituting the value given in (b) for $\sin\frac{1}{2}A$,

$$\cos\frac{1}{2}A = \sqrt{1 - \frac{(p-b)(p-c)}{bc}};$$

or reducing to the same denominator, and simplifying,

$$\cos\frac{1}{2}A = \sqrt{\frac{p(p-a)}{bc}}.$$

From (**1041**, 2d):

$$\tan\frac{1}{2}A = \frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A} = \frac{\sqrt{\dfrac{(p-b)(p-c)}{bc}}}{\sqrt{\dfrac{p(p-a)}{bc}}} = \sqrt{\frac{(p-b)(p-c)}{p(p-a)}}.$$

Analogous formulas may be obtained for the angles B and C, by proceeding in the same manner.

The area may be calculated from the formula which is developed in (3d) of the next article.

1065. *The area of a triangle* may be expressed in terms of two sides and the included angle, or one side and two angles or three sides.

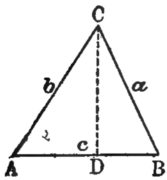


Fig. 265

1st. Letting S represent the area of the triangle,

$$S = \frac{c \times CD}{2}. \tag{718}$$

Substituting $b \sin A$ (1078) for CD,

$$S = \frac{bc \sin A}{2}, \tag{a}$$

which shows that *the area of a triangle is equal to half the product of any two sides and the sine of the included angle.*

2d. Writing $b = \dfrac{c \sin B}{\sin C}$ in the preceding expression (1056), we have:

$$S = \frac{c^2 \sin A \sin B}{2 \sin C} = \frac{c^2 \sin A \sin B}{2 \sin (A + B)}.$$

3d. Having

$$\sin A = 2 \sin \frac{1}{2} A \cos \frac{1}{2} A, \tag{1048}$$

substituting from (1087) for $\sin \frac{1}{2} A$ and $\cos \frac{1}{2} A$,

$$\sin A = 2 \sqrt{\frac{p (p - a) (p - b) (p - c)}{b^2 c^2}};$$

then substituting this value of $\sin A$ in equation (a),

$$S = \sqrt{p (p - a) (p - b) (p - c)}.$$

For $a = 200$ ft., $b = 180$ ft., and $c = 170$ ft.,

$$S = \sqrt{275 (275 - 200) (275 - 180) (275 - 170)} = 14{,}343 \text{ sq. ft.}$$

EXAMPLES

1066. In trigonometry all problems are reduced to the determination of triangles, or rather the sides and angles of these triangles.

1067. *Find the height CD of a building, the base of which is accessible.*

On the ground, which is level, measure the base DE, making it

about equal to the height of the building so as to avoid two small angles; let $DE = c = 10$ yards, place the instrument at E and measure the angle B, which is 41°, and let the height of the instrument be $BE = AD = 1.2$ yards.

This done, the problem is reduced to determining the side b of a right triangle ABC, when the side c and the angle B are known. Or, from (**1055**, corollary):

$$b = c \tan B = 10 \times \tan 41° = 10 \times 0.86929 = 8.693 \text{ yds.};$$

$$CD = 8.693 + 1.2 = 9.893 \text{ yds.}$$

In case the ground is not level, the point A can be determined and AD measured, then we have the same as in the first case.

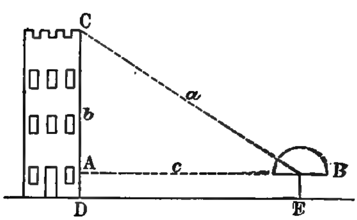


Fig. 266

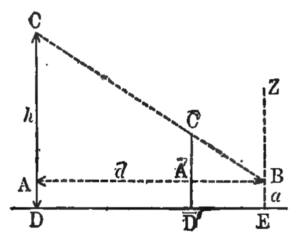


Fig. 267

Solution 2. At the extremity E of the base, a stake of known height BE is driven. Then at D' in line with D and E a second stake is held so that C' is in line with B and C, and measuring $A'C'$ and $A'B$, the two similar triangles ABC and $A'BC'$ give:

$$\frac{AC}{A'C'} = \frac{AB}{A'B}, \text{ and } AC = AB \times \frac{A'C'}{A'B};$$

or, making $CD = h$, $AB = d$, and $BE = a$,

$$h = d \times \frac{A'C'}{A'B} + a. \qquad (1)$$

If one has an instrument for measuring angles, the angle ZBC is measured, and we have:

$$AC = d \cot ZBC, \text{ and } h = d \cot zBC + a. \qquad (1a)$$

1068. *To find the distance AC from the point A to an inaccessible but visible point C.*

Lay off a base $AB = 100$ yards, for example; then measure the

angles $A = 65°$ and $B = 42°$. The problem is now reduced to determining the side b of an oblique triangle when one side c and the two adjacent angles A and B are known (**1061**).

First,

$$C = 180° - (A + B) = 180 - (65° + 42°) = 73°,$$

then $$\sin C : \sin B = c : b,$$

and $$b = \frac{c \sin B}{\sin C} = \frac{100 \times 0.669}{0.956} = 70 \text{ yds.}$$

1069. *To determine the height of a building or mountain, the base of which is inaccessible.*

In this case the angle $B = 43°$ is all that can be measured directly in the triangle ABC, and this is not sufficient for the cal-

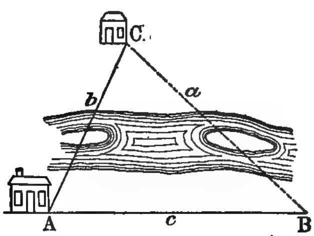


Fig. 268

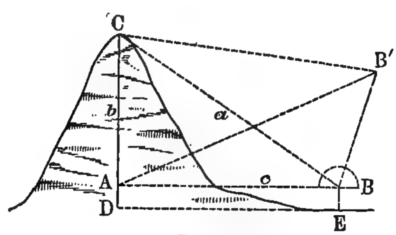


Fig. 269

culation of AC. Therefore the solution is commenced by determining the side BC, which is done as in the preceding case (**1068**). C is an inaccessible point whose distance from B is found from the relations in the triangle $BB'C$, BB' and the adjacent angles being known.

Having BC or $a = 500$ yards, for example (**1055**):

$$b = a \sin B = 500 \times 0.682 = 341 \text{ yards,}$$

and therefore $$CD = 341 + 1.2 = 342.2 \text{ yards.}$$

SOLUTION 2. The distance $AB = d$ from the accessible point B to the vertical passing through the inaccessible point C is determined by measuring the base BB', from B the angle between CB and BB' which the theodolite gives reduced to the horizontal, *i.e.*, ABB' and from B' the angle $AB'B$.

In the triangle ABB', the angle $BAB' = 180 - (ABB' + AB'B)$ and (**1056**),

$$AB = d = \frac{BB' \sin AB'B}{\sin BAB'}.$$

Substituting this value of d in formula (1067, 1a), the required height is obtained.

1070. *Find the distance between two inaccessible points C and C'.*

Determine the distances AC and AC' between the point A and each of the inaccessible points C and C', according to the method in article (1068); then measuring the angle CAC', in the triangle CAC', we have two sides AC and AC' and the included angle; therefore the side CC' may be found from (1063).

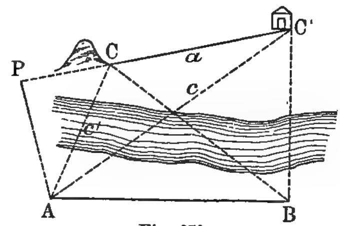


Fig. 270

1st. *Determination of AC.* Lay off the base AB = 100 yards, for example; then the angle $BAC = 66°$, and $ABC = 37°$; and $ACB = 180° - (66° + 37°) = 77°$. Then we have:

$$\sin ACB : \sin ABC = AB : AC,$$

and
$$AC = \frac{AB \sin ABC}{\sin ACB} = \frac{100 \times 0.6018}{0.9744} = 61.76 \text{ yds.} \qquad (a)$$

2d. *Determination of AC'.* Measure the angles $BAC' = 37°$ and $ABC' = 87°$; then

$$AC'B = 180° - (37° + 87°) = 56°.$$

In triangle ABC',

$$\sin AC'B : \sin ABC' = AB : AC',$$

and
$$AC' = \frac{AB \sin ABC'}{\sin AC'B} = \frac{100 \times 0.9986}{0.829} = 120.46 \text{ yds.}$$

3d. *Determination of the angle CAC'.* When the four points A, B, C, and C' are in the same plane, we have $CAC' = BAC - BAC' = 66° - 37° = 29°$. If these four points are not in the same plane, the angle is measured dire tly.

4th. *Determination of CC'.* In the triangle ACC' (1063),

$$CC' = \sqrt{\overline{AC}^2 + \overline{AC'}^2 - 2 \times AC \times AC' \times \cos CAC'},$$

or

$$CC' = \sqrt{61.76^2 + 120.46^2 - 2 \times 61.76 \times 120.46 \times 0.87462} = 72.88 \text{ yds.}$$

CC' might also have been determined by the method in (1063, 2d).

If logarithms are used in the solution, the following method is used. Let the angles of the triangle ACC' be designated by the letters A, C, and C'; and the sides opposite these angles by the letters a, c, and c'.

In the triangle ACC',

$$\frac{C + C'}{2} = \frac{180° - A}{2}; \tag{1}$$

then from (1063, 2d), noting that $\tan \frac{C + C'}{2} = \cot \frac{A}{2}$ (1043), that $\tan 45° = 1$, and making $\frac{c'}{c} = \tan \phi$,

$$\tan \frac{C - C'}{2} = \frac{c - c'}{c + c'} \cot \frac{A}{2} = \frac{1 - \frac{c'}{c}}{1 + \frac{c'}{c}} \cot \frac{A}{2}$$

$$= \frac{\tan 45 - \tan \phi}{1 + \tan 45° \tan \phi} \cot \frac{A}{2};$$

from (1046),

$$\tan \frac{C - C'}{2} = \tan (45° - \phi) \cot \frac{A}{2}. \tag{2}$$

Having measured the angle A, the equations (1) and (2) give C and C' (520).

From (1056),

$$a = \frac{c' \sin A}{\sin C'}. \tag{b}$$

With logarithms c' is calculated from the formula (a), and a from the formula (b).

Distance AP from an accessible point A to an inaccessible straight line CC' (Fig. 270). Since the angles ACP and ACC' are supplementary, $\sin ACP = \sin ACC'$ or $\sin C$; and consequently (1055),

$$AP = c' \sin C;$$

c' and C being calculated as was demonstrated above.

1071. *Table of the natural values of the trigonometric functions of the angles from 0° to 90°, for each minute.*

Starting at the tops of the pages, each angle in the last vertical column is the supplement respectively of the angle in the same horizontal row in the first vertical column; thus,

$$16° \, 51' + 163° \, 9' = 180°.$$

Likewise, commencing at the bottoms of the pages, each angle in the first vertical column is the supplement of the corresponding angle in the last column; thus,

$$73^\circ\ 52' + 106^\circ\ 8' = 180^\circ.$$

Since supplementary angles have the same functions, it follows that the table contains the functions of the angles from 0° to 180° for each degree; the sign of the functions may be obtained from (**1027**) or Fig. 249.

For angles between 180° and 360°, subtract 180°, thus obtaining an angle which is given in the table. Thus,

$$\tan 352^\circ\ 46' = \tan (352^\circ\ 46' - 180^\circ) = \tan 172^\circ\ 46'.$$

From the table the tangent is 0.12692, and according to (**1027**) the tan 352° 46′ is preceded by the sign −, so we have,

$$\tan 352^\circ\ 46' = -\ 0.12692.$$

The table also contains the lengths of the arcs which correspond to the angles from 0° to 90° when the radius $r = 1$. Thus the arc corresponding to the angle 23° 17′ is 0.40637, and the arc corresponding to the complement 66° 43′ of 23° 17′ is 1.16442.

According as an angle is greater than an angle in the table by 90°, 180° or 270°, its length is obtained by adding respectively:

$$\frac{1}{2}\pi = 1.5707963 \cong 1.57080,$$

$$\pi = 3.1415926 \cong 3.14160,$$

$$\frac{3}{2}\pi = 4.7123889 \cong 4.71240.$$

Thus the length of the arc corresponding to the angle 66° 43′ + 90° = 156° 43′ is 1.16442 + 1.57080 = 2.73522.

If the radius were 6 feet, the length of the arc in feet would be $6 \times 2.73522 = 16.41132$ feet.

Remark. With the aid of the table, an arc which is a given fraction of the radius or diameter may be found. Thus, if it is desired to find an arc equal to $\frac{2}{5}$ or $\frac{4}{10}$ of the radius, in the sixth column under arc, find the number which is nearest to 0.4. The number 0.39997, which corresponds to 22° 55′, is the nearest value. The next arc 0.40056 corresponds to 22° 56′; then by interpolation the angle which corresponds to the arc 0.4 is found to be 22° 55′ 6″.

0° = 0′ **Sup. 179° = 10740′**

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.0	1.0	0.0		0.0	1.5	
0	00000	00000	00000	Infinity	00000	70796	60
1	00291	00000	00291	3437.7467	00291	70505	59
2	00582	00000	00582	1718.8732	00582	70215	58
3	00873	00000	00873	1145.9153	00873	69924	57
		0.9					
4	01164	99999	01164	859.4363	01164	69633	56
5	01454	99999	01454	687.5489	01454	69342	55
6	01745	99998	01745	572.9572	01745	69051	54
7	02036	99998	02036	491.1060	02036	68760	53
8	02327	99997	02327	429.7176	02327	68469	52
9	02618	99997	02618	381.9710	02618	68178	51
10	02909	99996	02909	343.7737	02909	67887	50
11	03200	99995	03200	312.5214	03200	67597	49
12	03491	99994	03491	286.4777	03491	67306	48
13	03782	99993	03782	264.4408	03782	67015	47
14	04072	99992	04072	245.5520	04072	66724	46
15	04363	99990	04363	229.1817	04363	66433	45
16	04654	99989	04654	214.8576	04654	66142	44
17	04945	99988	04945	202.2187	04945	65851	43
18	05236	99986	05236	190.9842	05236	65560	42
19	05527	99985	05527	180.9322	05527	65269	41
20	05818	99984	05818	171.8854	05818	64979	40
21	06109	99981	06109	163 7002	06109	64688	39
22	06400	99979	06400	156.2591	06400	64397	38
23	06690	99978	06690	149.4650	06690	64106	37
24	06981	99976	06981	143.2371	06981	63815	36
25	07272	99974	07272	137.5074	07272	63524	35
26	07563	99971	07563	132.2185	07563	63233	34
27	07854	99969	07854	127.3213	07854	62942	33
28	08145	99967	08145	122.7740	08145	62651	32
29	08436	99964	08436	118.5402	08436	62361	31
30	08727	99962	08727	114.5886	08727	62070	30
	0.0	0.9	0.0		0.0	1.5	
31	09017	99959	09018	110.8920	09018	61779	29
32	09308	99957	09309	107.4265	09308	61488	28
33	09599	99954	09600	104.1709	09599	61197	27
34	09890	99951	09891	101.1069	09890	60906	26
35	10181	99948	10181	98.2179	10181	60615	25
36	10472	99945	10472	95.4895	10472	60324	24
37	10763	99942	10763	92.9085	10763	60033	23
38	11053	99939	11054	90.4633	11054	59743	22
39	11344	99936	11345	88.1436	11345	59452	21
40	11635	99932	11636	85.9398	11636	59161	20
41	11926	99929	11927	83.8435	11926	58870	19
42	12217	99925	12218	81.8470	12217	58579	18
43	12508	99922	12509	79.9434	12508	58288	17
44	12799	99918	12800	78.1263	12799	57997	16
45	13090	99914	13091	76.3900	13090	57706	15
46	13380	99910	13382	74.7292	13381	57415	14
47	13671	99906	13673	73.1390	13672	57125	13
48	13962	99902	13964	71.6151	13963	56834	12
49	14253	99898	14255	70.1533	14254	56543	11
50	14544	99894	14545	68.7501	14544	56252	10
51	14835	99890	14836	67.4019	14835	55961	9
52	15126	99885	15127	66.1055	15126	55670	8
53	15416	99881	15418	64.8580	15417	55379	7
54	15707	99877	15709	63.6567	15708	55088	6
55	15998	99872	16000	62.4992	15999	54797	5
56	16289	99867	16291	61.3829	16290	54507	4
57	16580	99862	16582	60.3058	16581	54216	3
58	16871	99858	16873	59.2659	16872	53925	2
59	17162	99853	17164	58.2612	17162	53634	1
60	17452	99848	17455	57.2900	17453	53343	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. 90° = 5400′ **89° = 5340′**

1° = 60′ **Sup. 178° = 10680′**

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1	17743	99843	17746	56.35059	17744	53052	59
2	18034	99837	18037	55.44152	18035	52761	58
3	18325	99832	18328	54.56133	18326	52470	57
4	18616	99827	18619	53.70859	18617	52179	56
5	18907	99821	18910	52.88211	18908	51889	55
6	19197	99816	19201	52.08067	19199	51598	54
7	19488	99810	19492	51.30316	19490	51307	53
8	19779	99804	19783	50.54851	19780	51016	52
9	20070	99799	20074	49.81573	20071	50725	51
10	20361	99793	20365	49.10388	20362	50434	50
11	20652	99787	20656	48.41208	20653	50143	49
12	20942	99781	20947	47.73950	20944	49852	48
13	21233	99774	21238	47.08534	21235	49561	47
14	21524	99768	21529	46.44886	21526	49271	46
15	21815	99762	21820	45.82935	21817	48980	45
16	22106	99756	22111	45.22614	22108	48689	44
17	22396	99749	22402	44.63860	22398	48398	43
18	22687	99742	22693	44.06611	22689	48107	42
19	22978	99736	22984	43.50812	22980	47816	41
20	23269	99729	23275	42.96408	23271	47525	40
21	23560	99722	23566	42.43346	23562	47234	39
22	23851	99715	23857	41.91579	23853	46943	38
23	24141	99708	24148	41.41059	24144	46653	37
24	24432	99701	24439	40.91741	24435	46362	36
25	24723	99694	24730	40.43584	24725	46071	35
26	25014	99687	25022	39.96546	25016	45780	34
27	25305	99680	25313	39.50589	25307	45489	33
28	25595	99672	25604	39.05677	25598	45198	32
29	25886	99665	25895	38.61774	25889	44907	31
30	26177	99657	26186	38.18846	26180	44616	30
	0.0	0.9	0.0		0.0	1.5	
31	26468	99650	26477	37.76861	26471	44326	29
32	26758	99642	26768	37.35789	26762	44036	28
33	27049	99634	27059	36.95600	27053	43744	27
34	27340	99626	27350	36.56266	27343	43453	26
35	27631	99618	27641	36.17760	27634	43162	25
36	27922	99610	27932	35.80055	27925	42871	24
37	28212	99602	28224	35.43128	28216	42580	23
38	28503	99594	28515	35.06955	28507	42289	22
39	28794	99585	28806	34.71511	28798	41998	21
40	29085	99577	29097	34.36777	29089	41708	20
41	29375	99568	29388	34.02730	29380	41417	19
42	29666	99560	29679	33.69351	29671	41126	18
43	29957	99551	29970	33.36619	29961	40835	17
44	30248	99542	30262	33.04517	30252	40544	16
45	30538	99534	30553	32.73026	30543	40253	15
46	30829	99525	30844	32.42129	30834	39962	14
47	31120	99516	31135	32.11810	31125	39671	13
48	31411	99507	31426	31.82052	31416	39380	12
49	31701	99497	31717	31.52839	31707	39090	11
50	31992	99488	32009	31.24158	31998	38799	10
51	32283	99479	32300	30.95993	32289	38508	9
52	32574	99469	32591	30.68331	32579	38217	8
53	32864	99460	32882	30.41158	32870	37926	7
54	33155	99450	33173	30.14462	33161	37635	6
55	33446	99441	33465	29.88230	33452	37344	5
56	33737	99431	33756	29.62450	33743	37053	4
57	34027	99421	34047	29.37111	34034	36762	3
58	34318	99411	34338	29.12200	34325	36472	2
59	34609	99401	34629	28.87709	34616	36181	1
60	34899	99391	34921	28.63625	34907	35890	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. 91° = 5460′ **88° = 5280′**

2° = 120′ Sup. **177°** = 10620′

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1	35190	99381	35212	28.39940	35197	35599	59
2	35481	99370	35503	28.16642	35488	35308	58
3	35772	99360	35795	27.93723	35779	35017	57
4	36062	99350	36086	27.71174	36070	34726	56
5	36353	99339	36377	27.48985	36361	34435	55
6	36644	99328	36668	27.27149	36652	34144	54
7	36934	99318	36960	27.05656	36942	33854	53
8	37225	99307	37251	26.84498	37234	33563	52
9	37516	99296	37542	26.63669	37525	33272	51
10	37806	99285	37834	26.43160	37815	32981	50
11	38097	99274	38125	26.22964	38106	32690	49
12	38388	99263	38416	26.03074	38397	32399	48
13	38678	99252	38707	25.83482	38688	32108	47
14	38969	99240	38999	25.64183	34979	31817	46
15	39258	99229	39290	25.45170	39270	31526	45
16	39550	99218	39581	25.26436	39561	31236	44
17	39841	99206	39873	25.07976	39852	30945	43
18	40132	99194	40164	24.89783	40143	30654	42
19	40422	99183	40456	24.71851	40433	30363	41
20	40713	99171	40747	24.54176	40724	30072	40
21	41004	99159	41038	24.36751	41015	29781	39
22	41294	99147	41330	24.19571	41306	29490	38
23	41585	99135	41621	24.02632	41597	29199	37
24	41876	99123	41912	23.85928	41888	28908	36
25	42166	99111	42204	23.69454	42179	28618	35
26	42457	99098	42495	23.53205	42470	28327	34
27	42748	99086	42787	23.37178	42761	28036	33
28	43038	99073	43078	23.21367	43051	27745	32
29	43329	99061	43370	23.05768	43342	27454	31
30	43619	99048	43661	22.90376	43633	27163	30
	0.0	**0.9**	**0.0**		**0.0**	**1.5**	
31	43910	99036	43952	22.75189	43924	26872	29
32	44201	99023	44244	22.60201	44215	26581	28
33	44491	99010	44535	22.45410	44506	26290	27
34	44782	98997	44827	22.30810	44797	26000	26
35	45072	98984	45118	22.16398	45088	25709	25
36	45363	98971	45410	22.02171	45379	25418	24
37	45654	98957	45701	21.88125	45669	25127	23
38	45944	98944	45993	21.74257	45960	24836	22
39	46235	98931	46284	21.60563	46251	24545	21
40	46525	98917	46576	21.47040	46542	24254	20
41	46816	98904	46867	21.33685	46833	23963	19
42	47106	98890	47159	21.20495	47124	23672	18
43	47397	98876	47450	21.07466	47415	23382	17
44	47688	98862	47742	20.94597	47706	23091	16
45	47978	98848	48033	20.81883	47997	22800	15
46	48269	98834	48325	20.69322	48287	22509	14
47	48559	98820	48617	20.56911	48578	22218	13
48	48850	98806	48908	20.44649	48869	21927	12
49	49140	98792	49200	20.32531	49160	21636	11
50	49431	98778	49491	20.20555	49451	21345	10
51	49721	98763	49783	20.08720	49742	21054	9
52	50012	98749	50075	19.97022	50033	20764	8
53	50302	98734	50366	19.85459	50324	20473	7
54	50593	98719	50658	19.74029	50615	20182	6
55	50883	98705	50950	19.62730	50905	19891	5
56	51174	98690	51241	19.51558	51196	19600	4
57	51464	98675	51533	19.40513	51487	19309	3
58	51755	98660	51824	19.29592	51778	19018	2
59	52045	98645	52116	19.18793	52069	18727	1
60	52336	98629	52408	19.08114	52360	18436	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **92°** = 5520′ **87°** = 5220′

3° = 180′ Sup. **176°** = 10560′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.0	**0.9**	**0.0**		**0.0**	**1.5**	
0	52336	98629	52408	19.08114	52360	18436	60
1	52626	98614	52699	18.97552	52651	18146	59
2	52917	98599	52991	18.87107	52942	17855	58
3	53207	98584	53283	18.76775	53233	17564	57
4	53498	98568	53575	18.66556	53523	17173	56
5	53788	98552	53866	18.56447	53814	16982	55
6	54079	98537	54158	18.46447	54105	16691	50
7	54369	98521	54450	18.36554	54396	16400	54
8	54660	98505	54742	18.26765	54687	16109	53
9	54950	98489	55033	18.17081	54978	15818	52
10	55241	98473	55325	18.07498	55269	15528	51
11	55531	98457	55617	17.98015	55560	15237	49
12	55822	98441	55909	17.88631	55851	14946	48
13	56112	98425	56201	17.79344	56141	14655	47
14	56402	98408	56492	17.70153	56432	14364	46
15	56693	98392	56784	17.61056	56723	14073	45
16	56983	98375	57076	17.52052	57014	13782	44
17	57274	98359	57368	17.43138	57305	13491	43
18	57564	98342	57660	17.34315	57596	13200	42
19	57854	98325	57952	17.25581	57887	12910	41
20	58145	98308	58243	17.16934	58178	12619	40
21	58435	98291	58535	17.08372	58469	12328	39
22	58726	98274	58827	16.99896	58759	12037	38
23	59016	98257	59119	16.91502	59050	11746	37
24	59306	98240	59411	16.83191	59341	11455	36
25	59597	98223	59703	16.74961	59632	11164	35
26	59887	98205	59995	16.66811	59923	10873	34
27	60178	98188	60287	16.58740	60214	10582	33
28	60468	98170	60579	16.50745	60505	10292	32
29	60758	98153	60871	16.42828	60796	10001	31
30	61049	98135	61163	16.34986	61087	09710	30
	0.0	**0.9**	**0.0**		**0.0**	**1.5**	
31	61339	98117	61455	16.27217	61377	09419	29
32	61629	98099	61747	16.19522	61668	09128	28
33	61920	98081	62039	16.11900	61959	08837	27
34	62210	98063	62331	16.04348	62250	08546	26
35	62500	98045	62623	15.96867	62541	08255	25
36	62791	98027	62915	15.89454	62832	07964	24
37	63081	98008	63207	15.82110	63123	07674	23
38	63371	97990	63499	15.74834	63414	07383	22
39	63661	97972	63791	15.67623	63705	07092	21
40	63952	97953	64083	15.60478	63995	06801	20
41	64242	97934	64375	15.53398	64286	06510	19
42	64532	97916	64667	15.46381	64577	06219	18
43	64823	97897	64959	15.39428	64868	05928	17
44	65113	97878	65251	15.32536	65159	05637	16
45	65403	97859	65543	15.25705	65450	05346	15
46	65693	97840	65836	15.18935	65741	05056	14
47	65984	97821	66128	15.12224	66032	04765	13
48	66274	97801	66420	15.05572	66323	04474	12
49	66564	97782	66712	14.98978	66613	04183	11
50	66854	97763	67004	14.92442	66904	03892	10
51	67145	97743	67297	14.85961	67195	03601	9
52	67435	97724	67589	14.79537	67486	03310	8
53	67725	97704	67881	14.73168	67777	03019	7
54	68015	97684	68173	14.66853	68068	02728	6
55	68306	97664	68465	14.60592	68359	02438	5
56	68596	97644	68758	14.54383	68650	02147	4
57	68886	97624	69050	14.48227	68941	01856	3
58	69176	97604	69342	14.42123	69231	01565	2
59	69466	97584	69635	14.36070	69522	01274	1
60	69757	97564	69927	14.30067	69813	00983	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **93°** = 5580′ **86°** = 5160′

4° = 240′ Sup. 175° = 10500′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.0	0.9	0.0		0.0	1.5	
0	69756	97564	69927	14.30067	69813	00983	60
1	70047	97544	70219	14.24113	70104	00692	59
2	70337	97523	70511	14.18209	70395	00401	58
3	70627	97503	70804	14.12354	70686	00110	57
						1.4	
4	70917	97482	71096	14.06546	70977	99820	56
5	71207	97461	71388	14.00786	71268	99529	55
6	71497	97441	71681	13.95072	71559	99238	54
7	71788	97420	71973	13.89405	71849	98947	53
8	72078	97399	72266	13.83783	72140	98656	52
9	72368	97378	72558	13.78206	72431	98365	51
10	72658	97357	72850	13.72674	72722	98074	50
11	72948	97336	73143	13.67186	73013	97783	49
12	73238	97314	73435	13.61741	73304	97493	48
13	73528	97293	73728	13.56339	73595	97202	47
14	73818	97272	74020	13.50980	73886	96911	46
15	74108	97250	74313	13.45662	74176	96620	45
16	74399	97229	74605	13.40387	74467	96329	44
17	74689	97207	74898	13.35152	74758	96038	43
18	74979	97185	75190	13.29957	75049	95747	42
19	75269	97163	75483	13.24803	75340	95456	41
20	75559	97141	75775	13.19688	75631	95165	40
21	75849	97119	76068	13.14613	75922	94875	39
22	76139	97097	76360	13.09576	76213	94584	38
23	76429	97075	76653	13.04577	76504	94293	37
24	76719	97053	76946	12.99616	76794	94002	36
25	77009	97030	77238	12.94692	77085	93711	35
26	77299	97008	77531	12.89806	77376	93420	34
27	77589	96985	77824	12.84956	77667	93129	33
28	77879	96963	78116	12.80142	77958	92838	32
29	78169	96940	78409	12.75363	78249	92547	31
30	78459	96917	78702	12.70621	78540	92257	30
	0.0	0.9	0.0		0.0	1.4	
31	78749	96894	78994	12.65913	78831	91966	29
32	79039	96871	79287	12.61239	79122	91675	28
33	79329	96848	79580	12.56600	79412	91384	27
34	79619	96825	79873	12.51994	79703	91093	26
35	79909	96802	80165	12.47422	79994	90802	25
36	80199	96779	80458	12.42883	80285	90511	24
37	80489	96755	80751	12.38377	80576	90220	23
38	80779	96732	81044	12.33903	80867	89929	22
39	81069	96708	81336	12.29461	81158	89639	21
40	81359	96685	81629	12.25051	81449	89348	20
41	81649	96661	81922	12.20672	81740	89057	19
42	81938	96637	82215	12.16324	82030	88766	18
43	82228	96613	82508	12.12006	82321	88475	17
44	82518	96589	82801	12.07719	82612	88184	16
45	82808	96565	83094	12.03462	82903	87893	15
46	83098	96541	83386	11.99235	83194	87602	14
47	83388	96517	83679	11.95037	83485	87311	13
48	83678	96493	83972	11.90868	83776	87021	12
49	83968	96468	84265	11.86728	84067	86730	11
50	84258	96444	84558	11.82617	84358	86439	10
51	84547	96419	84851	11.78533	84648	86148	9
52	84837	96395	85144	11.74478	84939	85857	8
53	85127	96370	85437	11.70450	85230	85566	7
54	85417	96345	85730	11.66449	85521	85275	6
55	85707	96320	86023	11.62476	85812	84984	5
56	85997	96295	86316	11.58529	86103	84693	4
57	86286	96270	86609	11.54609	86394	84403	3
58	86576	96242	86902	11.50715	86685	84112	2
59	86866	96220	87196	11.46847	86976	83821	1
60	87156	96195	87489	11.43005	87266	83530	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. 94° = 5640′ 85° = 5100′

5° = 300′ Sup. 174° = 10440′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.0	0.9	0.0		0.0	1.4	
0	87156	96195	87489	11.43005	87266	83530	60
1	87445	96169	87782	11.39189	87557	83239	59
2	87735	96144	88075	11.35397	87848	82948	58
3	88025	96118	88368	11.31630	88139	82657	57
4	88315	96093	88661	11.27889	88430	82366	56
5	88605	96067	88954	11.24171	88721	82075	55
6	88894	96041	89248	11.20478	89012	81785	54
7	89184	96015	89541	11.16809	89303	81494	53
8	89474	95989	89834	11.13164	89594	81203	52
9	89763	95963	90127	11.09542	89884	80912	51
10	90053	95937	90421	11.05943	90175	80621	50
11	90343	95911	90714	11.02367	90466	80330	49
12	90633	95884	91007	10.98815	90757	80039	48
13	90922	95858	91300	10.95285	91048	79748	47
14	91212	95831	91594	10.91778	91339	79457	46
15	91502	95805	91887	10.88292	91630	79167	45
16	91791	95778	92181	10.84829	91921	78876	44
17	92081	95751	92474	10.81387	92212	78585	43
18	92371	95725	92767	10.77967	92502	78294	42
19	92660	95698	93061	10.74569	92793	78003	41
20	92950	95671	93354	10.71191	93084	77712	40
21	93239	95644	93647	10.67835	93375	77421	39
22	93529	95616	93941	10.64499	93666	77130	38
23	93819	95589	94234	10.61184	93957	76839	37
24	94108	95562	94528	10.57890	94248	76549	36
25	94398	95534	94821	10.54615	94539	76258	35
26	94687	95507	95115	10.51361	94830	75967	34
27	94977	95479	95408	10.48126	95120	75676	33
28	95267	95452	95702	10.44911	95411	75385	32
29	95556	95424	95995	10.41716	95702	75094	31
30	95846	95396	96289	10.38540	95993	74803	30
	0.0	0.9	0.0		0.0	1.4	
31	96135	95368	96583	10.35383	96284	74512	29
32	96425	95340	96876	10.32245	96575	74221	28
33	96714	95312	97170	10.29126	96866	73931	27
34	97004	95284	97463	10.26025	97157	73640	26
35	97293	95256	97757	10.22943	97448	73349	25
36	97583	95227	98051	10.19879	97738	73058	24
37	97872	95199	98345	10.16833	98029	72767	23
38	98162	95170	98638	10.13805	98320	72476	22
39	98451	95142	98932	10.10795	98611	72185	21
40	98741	95113	99226	10.07803	98902	71894	20
41	99030	95084	99519	10.04828	99193	71603	19
42	99320	95055	99813	10.01871	99484	71313	18
			0.1				
43	99609	95027	00107	9.98931	99775	71022	17
					0.1		
44	99899	94998	00401	9.96007	00066	70731	16
	0.1						
45	00188	94968	00695	9.93101	00356	70440	15
46	00477	94939	00988	9.90211	00647	70149	14
47	00767	94910	01282	9.87338	00938	69858	13
48	01056	94881	01576	9.84482	01229	69567	12
49	01346	94851	01870	9.81641	01520	69276	11
50	01635	94822	02164	9.78817	01811	68985	10
51	01924	94792	02458	9.76009	02102	68695	9
52	02216	94762	02752	9.73217	02393	68404	8
53	02503	94733	03046	9.70441	02684	68113	7
54	02792	94703	03340	9.67680	02974	67822	6
55	03082	94673	03634	9.64935	03265	67531	5
56	03371	94643	03928	9.62205	03556	67240	4
57	03660	94613	04222	9.59490	03847	66949	3
58	03950	94582	04516	9.56791	04138	66658	2
59	04259	94552	04810	9.54106	04429	66367	1
60	04528	94522	05104	9.51436	04720	66077	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. 95° = 5700′ 84° = 5040′

6° = 360′ **Sup. 175°** = 10380′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.1	**0.9**	**0.1**		**0.1**	**1.4**	
0	0453	9452	0510	9.51436	0472	6608	60
1	0482	9449	0540	9.48781	0501	6579	59
2	0511	9446	0569	9.46141	0530	6549	58
3	0540	9443	0599	9.43515	0559	6520	57
4	0569	9440	0628	9.40904	0588	6491	56
5	0597	9437	0658	9.38307	0617	6462	55
6	0626	9434	0687	9.35724	0646	6433	54
7	0655	9431	0716	9.33155	0676	6404	53
8	0684	9428	0746	9.30599	0705	6375	52
9	0713	9424	0775	9.28058	0734	6346	51
10	0742	9421	0805	9.25530	0763	6317	50
11	0771	9418	0834	9.23016	0792	6288	49
12	0800	9415	0863	9.20516	0821	6259	48
13	0829	9412	0893	9.18028	0850	6229	47
14	0858	9409	0922	9.15554	0879	6200	46
15	0887	9406	0952	9.13093	0908	6171	45
16	0916	9402	0981	9.10646	0937	6142	44
17	0945	9399	1011	9.08211	0966	6113	43
18	0973	9396	1040	9.05789	0996	6084	42
19	1002	9393	1070	9.03379	1025	6055	41
20	1031	9390	1099	9.00983	1054	6026	40
21	1060	9386	1128	8.98598	1083	5997	39
22	1089	9383	1158	8.96227	1112	5968	38
23	1118	9380	1187	8.93867	1141	5939	37
24	1147	9377	1217	8.91520	1170	5909	36
25	1176	9374	1246	8.89185	1199	5880	35
26	1205	9370	1276	8.86862	1228	5851	34
27	1234	9367	1305	8.84551	1257	5822	33
28	1263	9364	1335	8.82252	1286	5793	32
29	1291	9360	1364	8.79964	1316	5764	31
30	1320	9357	1394	8.77689	1345	5735	30
	0.1	**0.9**	**0.1**		**0.1**	**1.4**	
31	1349	9354	1423	8.75425	1374	5706	29
32	1378	9351	1453	8.73172	1403	5677	28
33	1407	9347	1482	8.70931	1432	5648	27
34	1436	9344	1511	8.68701	1461	5619	26
35	1465	9341	1541	8.66482	1490	5589	25
36	1494	9337	1570	8.64275	1519	5560	24
37	1523	9334	1600	8.62078	1548	5531	23
38	1552	9331	1629	8.59893	1577	5502	22
39	1580	9327	1659	8.57718	1606	5473	21
40	1609	9324	1688	8.55555	1635	5444	20
41	1638	9320	1718	8.53402	1665	5415	19
42	1667	9317	1747	8.51259	1694	5386	18
43	1696	9314	1777	8.49128	1723	5357	17
44	1725	9310	1806	8.47007	1752	5328	16
45	1754	9307	1836	8.44896	1781	5299	15
46	1783	9303	1865	8.42795	1810	5270	14
47	1812	9300	1895	8.40705	1839	5240	13
48	1840	9297	1924	8.38625	1868	5211	12
49	1869	9293	1954	8.36555	1897	5182	11
50	1898	9290	1983	8.34496	1926	5153	10
51	1927	9286	2013	8.32446	1955	5124	9
52	1956	9283	2042	8.30406	1985	5095	8
53	1985	9279	2072	8.28376	2014	5066	7
54	2014	9276	2101	8.26355	2043	5037	6
55	2043	9272	2131	8.24345	2072	5008	5
56	2071	9269	2160	8.22345	2101	4979	4
57	2100	9265	2190	8.20352	2130	4950	3
58	2129	9262	2219	8.18370	2159	4920	2
59	2158	9258	2249	8.16398	2188	4891	1
60	2187	9255	2278	8.14435	2217	4862	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. 96° = 5760′ **83°** = 4980′

7° = 420′ **Sup. 172°** = 10320′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.1	**0.9**	**0.1**		**0.1**	**1.4**	
0	2187	9255	2278	8.14435	2217	4862	60
1	2216	9251	2308	8.12481	2246	4833	59
2	2245	9248	2338	8.10536	2275	4804	58
3	2274	9244	2367	8.08600	2305	4775	57
4	2302	9240	2397	8.06674	2334	4746	56
5	2331	9237	2426	8.04756	2363	4717	55
6	2360	9233	2456	8.02848	2392	4688	54
7	2389	9230	2485	8.00948	2421	4659	53
8	2418	9226	2515	7.99058	2450	4630	52
9	2447	9222	2544	7.97176	2479	4600	51
10	2476	9219	2574	7.95302	2508	4571	50
11	2504	9215	2603	7.93438	2537	4542	49
12	2533	9211	2633	7.91582	2566	4513	48
13	2562	9208	2662	7.89734	2595	4484	47
14	2591	9204	2692	7.87895	2624	4455	46
15	2620	9200	2722	7.86064	2654	4426	45
16	2649	9197	2751	7.84242	2683	4397	44
17	2678	9193	2781	7.82428	2712	4368	43
18	2706	9189	2810	7.80622	2741	4339	42
19	2735	9186	2840	7.78825	2770	4310	41
20	2764	9182	2869	7.77035	2799	4281	40
21	2793	9178	2899	7.75254	2828	4251	39
22	2822	9175	2929	7.73480	2857	4222	38
23	2851	9171	2958	7.71715	2886	4193	37
24	2880	9167	2988	7.69957	2915	4164	36
25	2908	9163	3017	7.68208	2944	4135	35
26	2937	9160	3047	7.66466	2974	4106	34
27	2966	9156	3076	7.64732	3003	4077	33
28	2995	9152	3106	7.63005	3032	4048	32
29	3024	9148	3136	7.61287	3061	4019	31
30	3053	9144	3165	7.59575	3090	3990	30
	0.1	**0.9**	**0.1**		**0.1**	**1.4**	
31	3081	9141	3195	7.57872	3119	3961	29
32	3110	9137	3224	7.56176	3148	3931	28
33	3139	9133	3254	7.54487	3177	3902	27
34	3168	9129	3284	7.52806	3206	3873	26
35	3197	9125	3313	7.51132	3235	3844	25
36	3226	9122	3343	7.49465	3264	3815	24
37	3254	9118	3372	7.47806	3294	3786	23
38	3283	9114	3402	7.46154	3323	3757	22
39	3312	9110	3432	7.44509	3352	3728	21
40	3341	9106	3461	7.42871	3381	3699	20
41	3370	9102	3491	7.41240	3410	3670	19
42	3399	9098	3521	7.39616	3439	3641	18
43	3427	9094	3550	7.37999	3468	3611	17
44	3456	9091	3580	7.36389	3497	3582	16
45	3485	9087	3609	7.34786	3526	3553	15
46	3514	9083	3639	7.33190	3555	3524	14
47	3543	9079	3669	7.31600	3584	3495	13
48	3572	9075	3698	7.30018	3614	3466	12
49	3600	9071	3728	7.28442	3643	3437	11
50	3629	9067	3758	7.26873	3672	3408	10
51	3658	9063	3787	7.25310	3701	3379	9
52	3687	9059	3817	7.23754	3730	3350	8
53	3716	9055	3847	7.22204	3759	3321	7
54	3744	9051	3876	7.20661	3788	3291	6
55	3773	9047	3906	7.19125	3817	3262	5
56	3802	9043	3935	7.17594	3846	3233	4
57	3831	9039	3965	7.16071	3875	3204	3
58	3860	9035	3995	7.14553	3904	3175	2
59	3889	9031	4024	7.13042	3933	3146	1
60	3917	9027	4054	7.11537	3963	3117	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. 97° = 5820′ **82°** = 4920′

8° = 480′ Sup. 171° = 10260′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.1	0.9	0.1		0.1	1.4	
0	3917	9027	4054	7.11537	3963	3117	60
1	3946	9023	4084	7.10038	3992	3088	59
2	3975	9019	4113	7.08546	4021	3059	58
3	4004	9015	4143	7.07059	4050	3030	57
4	4033	9011	4173	7.05579	4079	3001	56
5	4061	9006	4202	7.04105	4108	2972	55
6	4090	9002	4232	7.02637	4137	2942	54
7	4119	8998	4262	7.01174	4166	2913	53
8	4148	8994	4291	6.99718	4195	2884	52
9	4177	8990	4321	6.98268	4224	2855	51
10	4205	8986	4351	6.96823	4253	2826	50
11	4234	8982	4381	6.95385	4283	2797	49
12	4263	8978	4410	6.93952	4312	2768	48
13	4292	8973	4440	6.92525	4341	2739	47
14	4320	8969	4470	6.91104	4370	2710	46
15	4349	8965	4499	6.89688	4399	2681	45
16	4378	8961	4529	6.88278	4428	2652	44
17	4407	8957	4559	6.86874	4457	2622	43
18	4436	8953	4588	6.85475	4486	2593	42
19	4464	8948	4618	6.84082	4515	2564	41
20	4493	8944	4648	6.82694	4544	2535	40
21	4522	8940	4678	6.81312	4573	2506	39
22	4551	8936	4707	6.79936	4603	2477	38
23	4580	8931	4737	6.78564	4632	2448	37
24	4608	8927	4767	6.77199	4661	2419	36
25	4637	8923	4796	6.75838	4690	2390	35
26	4666	8919	4826	6.74483	4719	2361	34
27	4695	8914	4856	6.73133	4748	2332	33
28	4723	8910	4886	6.71789	4777	2302	32
29	4752	8906	4915	6.70450	4806	2273	31
30	4781	8902	4945	6.69116	4835	2244	30
	0.1	0.9	0.1		0.1	1.4	
31	4810	8897	4975	6.67787	4864	2215	29
32	4838	8893	5005	6.66463	4893	2186	28
33	4867	8889	5034	6.65144	4923	2157	27
34	4896	8884	5064	6.63831	4952	2128	26
35	4925	8880	5094	6.62523	4981	2099	25
36	4954	8876	5124	6.61219	5010	2070	24
37	4982	8871	5153	6.59921	5039	2041	23
38	5011	8867	5183	6.58627	5068	2012	22
39	5040	8863	5213	6.57339	5097	1982	21
40	5069	8858	5243	6.56055	5126	1953	20
41	5097	8854	5272	6.54777	5155	1924	19
42	5126	8849	5302	6.53503	5184	1895	18
43	5155	8845	5332	6.52234	5213	1866	17
44	5184	8841	5362	6.50970	5242	1837	16
45	5212	8836	5391	6.49710	5272	1808	15
46	5241	8832	5421	6.48456	5301	1779	14
47	5270	8827	5451	6.47206	5330	1750	13
48	5299	8823	5481	6.45961	5359	1721	12
49	5327	8818	5511	6.44720	5388	1692	11
50	5356	8814	5540	6.43484	5417	1663	10
51	5385	8809	5570	6.42253	5446	1633	9
52	5414	8805	5600	6.41026	5475	1604	8
53	5442	8800	5630	6.39804	5504	1575	7
54	5471	8796	5660	6.38587	5533	1546	6
55	5500	8791	5689	6.37374	5562	1517	5
56	5529	8787	5719	6.36165	5592	1488	4
57	5557	8782	5749	6.34961	5621	1459	3
58	5586	8778	5779	6.33761	5650	1430	2
59	5615	8773	5809	6.32566	5679	1401	1
60	5643	8769	5838	6.31375	5708	1372	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. 98° = 5880′ 81° = 4860′

9° = 540′ Sup. 170° = 10200′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.1	0.9	0.1		0.1	1.4	
0	5643	8769	5838	6.31375	5708	1372	60
1	5672	8764	5868	6.30189	5737	1343	59
2	5701	8760	5898	6.29007	5766	1313	58
3	5730	8755	5928	6.27829	5795	1284	57
4	5758	8751	5958	6.26655	5824	1255	56
5	5787	8746	5988	6.25486	5853	1226	55
6	5816	8741	6017	6.24321	5882	1197	54
7	5845	8737	6047	6.23160	5912	1168	53
8	5873	8732	6077	6.22003	5941	1139	52
9	5902	8728	6107	6.20851	5970	1110	51
10	5931	8723	6137	6.19703	5999	1081	50
11	5959	8718	6167	6.18559	6028	1052	49
12	5988	8714	6196	6.17419	6057	1023	48
13	6017	8709	6226	6.16283	6086	0993	47
14	6046	8704	6256	6.15151	6115	0964	46
15	6074	8700	6286	6.14023	6144	0935	45
16	6103	8695	6316	6.12899	6173	0906	44
17	6132	8690	6346	6.11779	6202	0877	43
18	6160	8686	6376	6.10664	6232	0848	42
19	6189	8681	6406	6.09552	6261	0819	41
20	6218	8676	6435	6.08444	6290	0790	40
21	6246	8671	6465	6.07340	6319	0761	39
22	6275	8667	6495	6.06240	6348	0732	38
23	6304	8662	6525	6.05143	6377	0703	37
24	6333	8657	6555	6.04051	6406	0673	36
25	6361	8652	6585	6.02962	6435	0644	35
26	6390	8648	6615	6.01878	6464	0615	34
27	6419	8643	6645	6.00797	6493	0586	33
28	6447	8638	6674	5.99720	6522	0557	32
29	6476	8633	6704	5.98646	6551	0528	31
30	6505	8629	6734	5.97576	6581	0499	30
	0.1	0.9	0.1		0.1	1.4	
31	6533	8624	6764	5.96510	6610	0470	29
32	6562	8619	6794	5.95448	6639	0441	28
33	6591	8614	6824	5.94390	6668	0412	27
34	6620	8609	6854	5.93335	6697	0383	26
35	6648	8604	6884	5.92283	6726	0354	25
36	6677	8600	6914	5.91235	6755	0324	24
37	6706	8595	6944	5.90191	6784	0295	23
38	6734	8590	6974	5.89151	6813	0266	22
39	6763	8585	7004	5.88114	6842	0237	21
40	6792	8580	7033	5.87080	6871	0208	20
41	6820	8575	7063	5.86051	6901	0179	19
42	6849	8570	7093	5.85024	6930	0150	18
43	6878	8565	7123	5.84001	6959	0121	17
44	6906	8561	7153	5.82982	6988	0092	16
45	6935	8556	7183	5'81966	7017	0063	15
46	6964	8551	7213	5.80953	7046	0034	14
47	6992	8546	7243	5.79944	7075	0004	13
						1.3	
48	7021	8541	7273	5.78938	7104	9975	12
49	7050	8536	7303	5.77936	7133	9946	11
50	7078	8531	7333	5.76937	7162	9917	10
51	7107	8526	7363	5.75941	7191	9888	9
52	7136	8521	7393	5.74949	7221	9859	8
53	7164	8516	7423	5.73960	7250	9830	7
54	7193	8511	7453	5.72974	7279	9801	6
55	7222	8506	7483	5.71992	7308	9772	5
56	7250	8501	7513	5.71013	7337	9743	4
57	7279	8496	7543	5.70037	7366	9714	3
58	7308	8491	7573	5.69064	7395	9684	2
59	7336	8486	7603	5.68094	7424	9655	1
60	7365	8481	7633	5.67128	7453	9626	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. 99° = 5940′ 80° = 4800′

10° = 600′ Sup. **169°** = 10140′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.1	0.9	0.1		0.1	1.3	
0	7365	8481	7633	5.67128	7453	9626	60
1	7393	8476	7663	5.66165	7482	9597	59
2	7422	8471	7693	5.65205	7511	9568	58
3	7451	8466	7723	5.64248	7541	9539	57
4	7479	8461	7753	5.63295	7570	9510	56
5	7508	8455	7783	5.62344	7599	9481	55
6	7537	8450	7813	5.61397	7628	9452	54
7	7565	8445	7843	5.60452	7657	9423	53
8	7594	8440	7873	5.59511	7686	9394	52
9	7623	8435	7903	5.58573	7715	9364	51
10	7651	8430	7933	5.57638	7744	9335	50
11	7680	8425	7963	5.56706	7773	9306	49
12	7708	8420	7993	5.55777	7802	9277	48
13	7737	8414	8023	5.54851	7831	9248	47
14	7766	8409	8053	5.53927	7860	9219	46
15	7794	8404	8083	5.53007	7890	9190	45
16	7823	8399	8113	5.52090	7919	9161	44
17	7852	8394	8143	5.51176	7948	9132	43
18	7880	8389	8173	5.50264	7977	9103	42
19	7909	8383	8203	5.49356	8006	9074	41
20	7937	8378	8233	5.48451	8035	9045	40
21	7966	8373	8263	5.47548	8064	9015	39
22	7995	8368	8293	5.46648	8093	8986	38
23	8023	8362	8323	5.45751	8122	8957	37
24	8052	8357	8353	5.44857	8151	8928	36
25	8081	8352	8383	5.43966	8180	8899	35
26	8109	8347	8414	5.43077	8210	8870	34
27	8138	8341	8444	5.42192	8239	8841	33
28	8166	8336	8474	5.41309	8268	8812	32
29	8195	8331	8504	5.40429	8297	8783	31
30	8224	8325	8534	5.39552	8326	8754	30
31	8252	8320	8564	5.38677	8355	8725	29
32	8281	8315	8594	5.37805	8384	8695	28
	0.1	0.9	0.1		0.1	1.3	
33	8309	8310	8624	5.36936	8413	8666	27
34	8338	8304	8654	5.36070	8442	8637	26
35	8367	8299	8684	5.35206	8471	8608	25
36	8395	8294	8714	5.34345	8500	8579	24
37	8424	8288	8745	5.33487	8530	8550	23
38	8452	8283	8775	5.32631	8559	8521	22
39	8481	8277	8805	5.31778	8588	8492	21
40	8509	8272	8835	5.30928	8617	8463	20
41	8538	8267	8865	5.30080	8646	8434	19
42	8567	8261	8895	5.29235	8675	8405	18
43	8595	8256	8925	5.28393	8704	8375	17
44	8624	8250	8955	5.27553	8733	8346	16
45	8652	8245	8986	5.26715	8762	8317	15
46	8681	8240	9016	5.25880	8791	8288	14
47	8710	8234	9046	5.25048	8820	8259	13
48	8738	8229	9076	5.24218	8850	8230	12
49	8767	8223	9106	5.23391	8879	8201	11
50	8795	8218	9136	5.22566	8908	8172	10
51	8824	8212	9166	5.21744	8937	8143	9
52	8852	8207	9197	5.20925	8966	8114	8
53	8881	8201	9227	5.20107	8995	8085	7
54	8910	8196	9257	5.19293	9024	8055	6
55	8938	8190	9287	5.18480	9053	8026	5
56	8967	8185	9317	5.17671	9082	7997	4
57	8995	8179	9347	5.16863	9111	7968	3
58	9024	8174	9378	5.16058	9140	7939	2
59	9052	8168	9408	5.15256	9169	7910	1
60	9081	8163	9438	5.14455	9199	7881	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **100°** = 6000′ **79°** = 4740′

11° = 660′ Sup. **168°** = 10080′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.1	0.9	0.1		0.1	1.3	
0	9081	8163	9438	5.14455	9199	7881	60
1	9109	8157	9468	5.13658	9228	7852	59
2	9138	8152	9498	5.12862	9257	7823	58
3	9167	8146	9529	5.12069	9286	7794	57
4	9195	8140	9559	5.11279	9315	7765	56
5	9224	8135	9589	5.10490	9344	7736	55
6	9252	8129	9619	5.09704	9373	7706	54
7	9281	8124	9649	5.08921	9402	7677	53
8	9309	8118	9680	5.08139	9431	7648	52
9	9338	8112	9710	5.07360	9460	7619	51
10	9366	8107	9740	5.06584	9489	7590	50
11	9395	8101	9770	5.05809	9519	7561	49
12	9423	8096	9801	5.05037	9548	7532	48
13	9452	8090	9831	5.04267	9577	7503	47
14	9480	8084	9861	5.03499	9606	7474	46
15	9509	8079	9891	5.02734	9635	7445	45
16	9538	8073	9921	5.01971	9664	7416	44
17	9566	8067	9952	5.01210	9693	7386	43
18	9595	8061	9982	5.00451	9722	7357	42
			0.2				
19	9623	8056	0012	4.99695	9751	7328	41
20	9652	8050	0042	4.98940	9780	7299	40
21	9680	8044	0073	4.98188	9809	7270	39
22	9709	8039	0103	4.97438	9839	7241	38
23	9737	8033	0133	4.96690	9868	7212	37
24	9766	8027	0164	4.95945	9897	7183	36
25	9794	8021	0194	4.95201	9926	7154	35
26	9823	8016	0224	4.94460	9955	7125	34
27	9851	8010	0254	4.93721	9984	7096	33
					0.2		
28	9880	8004	0285	4.92984	0013	7066	32
29	9908	7998	0315	4.92249	0042	7037	31
30	9937	7992	0345	4.91516	0071	7008	30
31	9965	7987	0376	4.90785	0100	6979	29
32	9994	7981	0406	4.90056	0129	6950	28
	0.2	0.9	0.2		0.2	1.3	
33	0022	7975	0436	4.89330	0159	6921	27
34	0053	7969	0466	4.88605	0188	6892	26
35	0079	7963	0497	4.87882	0217	6863	25
36	0108	7958	0527	4.87162	0246	6834	24
37	0136	7952	0557	4.86444	0275	6805	23
38	0165	7946	0588	4.85727	0304	6776	22
39	0193	7940	0618	4.85013	0333	6746	21
40	0222	7934	0648	4.84300	0362	6717	20
41	0250	7928	0679	4.83590	0391	6688	19
42	0279	7922	0709	4.82882	0420	6659	18
43	0307	7916	0739	4.82175	0449	6630	17
44	0336	7910	0770	4.81471	0478	6601	16
45	0364	7905	0800	4.80769	0508	6572	15
46	0393	7899	0830	4.80068	0537	6543	14
47	0421	7893	0861	4.79370	0566	6514	13
48	0450	7887	0891	4.78673	0595	6485	12
49	0478	7881	0921	4.77978	0624	6456	11
50	0507	7875	0952	4.77286	0653	6427	10
51	0535	7869	0982	4.76595	0682	6397	9
52	0563	7863	1013	4.75906	0711	6368	8
53	0592	7857	1043	4.75219	0740	6339	7
54	0620	7851	1073	4.74534	0769	6310	6
55	0649	7845	1104	4.73851	0798	6281	5
56	0677	7839	1134	4.73170	0828	6252	4
57	0706	7833	1164	4.72490	0857	6223	3
58	0734	7827	1195	4.71813	0886	6194	2
59	0763	7821	1225	4.71137	0915	6165	1
60	0791	7815	1256	4.70463	0944	6136	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **101°** = 6060′ **78°** = 4680

12° = 720′ Sup. **167°** = 10020′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.2	0.9	0.2		0.2	1.3	
0	0791	7815	1256	4.70463	0944	6136	60
1	0820	7809	1286	4.69791	0973	6107	59
2	0848	7803	1316	4.69121	1003	6077	58
3	0877	7797	1347	4.68452	1031	6048	57
4	0905	7790	1377	4.67786	1060	6019	56
5	0933	7784	1408	4.67421	1089	5990	55
6	0962	7778	1438	4.66458	1118	5961	54
7	0990	7772	1469	4.65797	1148	5932	53
8	1019	7766	1499	4.65138	1177	5903	52
9	1047	7760	1529	4.64480	1206	5874	51
10	1076	7754	1560	4.63825	1235	5845	50
11	1104	7748	1590	4.63171	1264	5816	49
12	1132	7742	1621	4.62518	1293	5787	48
13	1161	7735	1651	4.61868	1322	5758	47
14	1189	7729	1682	4.61219	1351	5728	46
15	1218	7723	1712	4.60572	1380	5699	45
16	1246	7717	1743	4.59927	1409	5670	44
17	1275	7711	1773	4.59283	1438	5641	43
18	1303	7705	1804	4.58641	1468	5612	42
19	1331	7698	1834	4.58001	1497	5583	41
20	1360	7692	1864	4.57363	1526	5554	40
21	1388	7686	1895	4.56726	1555	5525	39
22	1417	7680	1925	4.56091	1584	5496	38
23	1445	7673	1956	4.55458	1613	5467	37
24	1474	7667	1986	4.54826	1642	5437	36
25	1502	7661	2017	4.54196	1671	5408	35
26	1530	7655	2047	4.53568	1700	5379	34
27	1559	7648	2078	4.52941	1729	5350	33
28	1587	7642	2108	4.52316	1758	5321	32
29	1616	7636	2139	4.51693	1787	5292	31
30	1644	7630	2169	4.51071	1817	5263	30
	0.2	0.9	0.2		0.2	1.3	
31	1672	7623	2200	4.50451	1846	5234	29
32	1701	7617	2230	4.49832	1875	5205	28
33	1729	7611	2261	4.49215	1904	5176	27
34	1758	7604	2292	4.48600	1933	5147	26
35	1786	7598	2322	4.47986	9962	5118	25
36	1814	7592	2353	4.47374	1991	5088	24
37	1843	7585	2383	4.46764	2020	5059	23
38	1871	7579	2414	4.46155	2049	5030	22
39	1899	7573	2444	4.45548	2078	5001	21
40	1928	7566	2475	4.44942	2107	4972	20
41	1956	7560	2505	4.44338	2137	4943	19
42	1985	7553	2536	4.43735	2166	4914	18
43	2013	7547	2567	4.43134	2195	4885	17
44	2041	7541	2597	4.42534	2224	4856	16
45	2070	7534	2628	4.41936	2253	4827	15
46	2098	7528	2658	4.41340	2282	4798	14
47	2126	7521	2689	4.40745	2311	4768	13
48	2155	7515	2719	4.40152	2340	4739	12
49	2183	7508	2750	4.39560	2369	4710	11
50	2212	7502	2781	4.38969	2398	4681	10
51	2240	7490	2811	4.38381	2427	4652	9
52	2268	7489	2842	4.37793	2457	4623	8
53	2297	7483	2872	4.37207	2486	4594	7
54	2325	7476	2903	4.36623	2515	4565	6
55	2353	7470	2934	4.36040	2544	4536	5
56	2382	7463	2964	4.35459	2573	4507	4
57	2410	7457	2995	4.34879	2602	4478	3
58	2438	7450	3026	4.34300	2631	4448	2
59	2467	7444	3056	4.33723	2660	4419	1
60	2495	7437	3087	4.33148	2689	4390	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **102°** = 6120′ **77°** = 4620′

13° = 780′ Sup. **166°** = 9960′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.2	0.9	0.2		0.2	1.3	
0	2495	7437	3087	4.33148	2689	4390	60
1	2523	7430	3117	4.32573	2718	4361	59
2	2552	7424	3148	4.32001	2747	4332	58
3	2580	7417	3179	4.31430	2776	4303	57
4	2608	7411	3209	4.30860	2806	4274	56
5	2637	7404	3240	4.30291	2835	4245	55
6	2665	7398	3271	4.29724	2864	4216	54
7	2693	7391	3301	4.29159	2893	4187	53
8	2722	7384	3332	4.28595	2922	4158	52
9	2750	7378	3363	4.28032	2951	4129	51
10	2778	7371	3393	4.27471	2980	4099	50
11	2807	7365	3424	4.26911	3009	4070	49
12	2835	7358	3455	4.26352	3038	4041	48
13	2863	7351	3485	4.25795	3067	4012	47
14	2892	7345	3516	4.25239	3096	3983	46
15	2920	7338	3547	4.24685	3126	3954	45
16	2948	7331	3578	4.24132	3155	3925	44
17	2977	7325	3608	4.23580	3184	3896	43
18	3005	7318	3639	4.23030	3213	3867	42
19	3033	7311	3670	4.22481	3242	3838	41
20	3062	7304	3700	4.21933	3271	3809	40
21	3090	7298	3731	4.21387	3300	3779	39
22	3118	7291	3762	4.20842	3329	3750	38
23	3146	7284	3793	4.20298	3358	3721	37
24	3175	7278	3823	4.19756	3387	3692	36
25	3203	7271	3854	4.19215	3416	3663	35
26	3231	7264	3885	4.18675	3446	3634	34
27	3260	7257	3916	4.18137	3475	3605	33
28	3288	7251	3946	4.17600	3504	3576	32
29	3316	7244	3977	4.17064	3533	3547	31
30	3345	7237	4008	4.16530	3562	3518	30
	0.2	0.9	0.2		0.2	1.3	
31	3373	7230	4039	4.15997	3591	3489	29
32	3401	7223	4069	4.15465	3620	3459	28
33	3429	7217	4100	4.14934	3649	3430	27
34	3458	7210	4131	4.14405	3678	3401	26
35	3486	7203	4162	4.13877	3707	3372	25
36	3514	7196	4193	4.13350	3736	3343	24
37	3542	7189	4223	4.12825	3766	3314	23
38	3571	7182	4254	4.12301	3795	3285	22
39	3599	7176	4285	4.11778	3824	3256	21
40	3627	7169	4316	4.11256	3853	3227	20
41	3656	7162	4347	4.10736	3882	3198	19
42	3684	7155	4377	4.10216	3911	3169	18
43	3712	7148	4408	4.09699	3940	3139	17
44	3740	7141	4439	4.09182	3969	3110	16
45	3769	7134	4470	4.08666	3998	3081	15
46	3797	7127	4501	4.08152	4027	3052	14
47	3825	7120	4532	4.07639	4056	3023	13
48	3853	7113	4562	4.07127	4085	2994	12
49	3882	7106	4593	4.06616	4115	2965	11
50	3910	7100	4624	4.06107	4144	2936	10
51	3938	7093	4655	4.05599	4173	2907	9
52	3966	7086	4686	4.05093	4202	2878	8
53	3995	7079	4717	4.04586	4231	2849	7
54	4023	7072	4747	4.04081	4260	2820	6
55	4051	7065	4778	4.03578	4289	2790	5
56	4079	7058	4809	4.03076	4318	2761	4
57	4108	7051	8440	4.02574	4347	2732	3
58	4136	7044	4871	4.02074	4376	2703	2
59	4164	7037	4902	4.01576	4405	2674	1
60	4192	7030	4933	4.01078	4435	2645	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **103°** = 6180′ **76°** = 4560′

14° = 840′ Sup. 165° = 9900′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.2	0.9	0.2		0.2	1.3	
0	4192	7030	4933	4.01078	4435	2645	60
1	4220	7023	4964	4.00582	4464	2616	59
2	4249	7015	4995	4.00086	4493	2587	58
3	4277	7008	5026	3.99592	4522	2558	57
4	4305	7001	5056	3.99099	4551	2529	56
5	4333	6994	5087	3.98607	4580	2500	55
6	4361	6987	5118	3.98117	4609	2470	54
7	4390	6980	5149	3.97627	4638	2441	53
8	4418	6973	5180	3.97139	4667	2412	52
9	4446	6966	5211	3.96651	4696	2383	51
10	4474	6959	5242	3.96165	4725	2354	50
11	4503	6952	5273	3.95680	4755	2325	49
12	4531	6945	5304	3.95196	4784	2296	48
13	4559	6937	5335	3.94713	4813	2267	47
14	4587	6930	5366	3.94232	4842	2238	46
15	4615	6923	5397	3.93751	4871	2209	45
16	4644	6916	5428	3.93271	4900	2180	44
17	4672	6909	5459	3.92793	4929	2150	43
18	4700	6902	5490	3.92316	4958	2121	42
19	4728	6894	5521	3.91839	4987	2092	41
20	4756	6887	5552	3.91364	5016	2063	40
21	4784	6880	5583	3.90890	5045	2034	39
22	4813	6873	5614	3.90417	5075	2005	38
23	4841	6866	5645	3.89945	5104	1976	37
24	4869	6859	5676	3.89474	5133	1947	36
25	4897	6851	5707	3.89004	5162	1918	35
26	4925	6844	5738	3.88536	5191	1889	34
27	4953	6837	5769	3.88068	5220	1860	33
28	4982	6829	5800	3.87601	5249	1830	32
29	5010	6822	5831	3.87136	5278	1801	31
30	5038	6815	5862	3.86671	5307	1772	30
	0.2	0.9	0.2		0.2	1.3	
31	5066	6807	5893	3.86208	5336	1743	29
32	5094	6800	5924	3.85745	5365	1714	28
33	5122	6793	5955	3.85284	5394	1685	27
34	5151	6786	5986	3.84824	5424	1656	26
35	5179	6778	6017	3.84364	5453	1627	25
36	5207	6771	6048	3.83906	5482	1598	24
37	5235	6764	6079	3.83449	5511	1569	23
38	5263	6756	6110	3.82992	5540	1540	22
39	5291	6749	6141	3.82537	5569	1511	21
40	5320	6742	6172	3.82083	5598	1481	20
41	5348	6734	6203	3.81630	5627	1452	19
42	5376	6727	6235	3.81177	5656	1423	18
43	5404	6719	6266	3.80726	5685	1394	17
44	5432	6712	6297	3.80276	5714	1365	16
45	5460	6705	6328	3.79827	5744	1336	15
46	5488	6697	6359	3.79378	5773	1307	14
47	5516	6690	6390	3.78931	5802	1278	13
48	5545	6682	6421	3.78485	5831	1249	12
49	5573	6675	6452	3.78040	5860	1220	11
50	5601	6667	6483	3.77595	5889	1191	10
51	5629	6660	6515	3.77152	5918	1161	9
52	5657	6653	6546	3.76709	5947	1132	8
53	5685	6645	6577	3.76268	5976	1103	7
54	5713	6638	6608	3.75828	6005	1074	6
55	5741	6630	6639	3.75388	6034	1045	5
56	5769	6623	6670	3.74950	6064	1016	4
57	5798	6615	6701	3.74512	6093	0987	3
58	5826	6608	6733	3.74075	6122	0958	2
59	5854	6600	6764	3.73640	6151	0929	1
60	5882	6593	6795	3.73205	6180	0900	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. 104° = 6240′ 75° = 4500′

15° = 900′ Sup. 164° = 9840′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.2	0.9	0.2		0.2	1.3	
0	5882	6593	6795	3.73205	6180	0900	60
1	5910	6585	6826	3.72771	6209	0871	59
2	5938	6578	6857	3.72338	6238	0841	58
3	5966	6570	6888	3.71907	6267	0812	57
4	5994	6562	6920	3.71476	6296	0783	56
5	6022	6555	6951	3.71046	6325	0754	55
6	6050	6547	6982	3.70616	6354	0725	54
7	6079	6540	7013	3.70188	6384	0696	53
8	6107	6532	7044	3.69761	6413	0667	52
9	6135	6524	7076	3.69335	6442	0638	51
10	6163	6517	7107	3.68909	6471	0609	50
11	6191	6509	7138	3.68485	6500	0580	49
12	6219	6502	7169	3.68061	6529	0551	48
13	6247	6494	7201	3.67638	6558	0521	47
14	6275	6486	7232	3.67217	6587	0492	46
15	6303	6479	7263	3.66796	6616	0463	45
16	6331	6471	7294	3.66376	6645	0434	44
17	6359	6463	7326	3.65957	6674	0405	43
18	6387	6456	7357	3.65538	6703	0376	42
19	6415	6448	7388	3.65121	6733	0347	41
20	6443	6440	7419	3.64705	6762	0318	40
21	6471	6433	7451	3.64289	6791	0289	39
22	6500	6425	7482	3.63874	6820	0260	38
23	6528	6417	7513	3.63461	6849	0231	37
24	6556	6410	7545	3.63048	6878	0202	36
25	6584	6402	7576	3.62636	6907	0172	35
26	6612	6394	7607	3.62224	6936	0143	34
27	6640	6386	7638	3.61814	6965	0114	33
28	6668	6379	7670	3.61405	6994	0085	32
29	6696	6371	7701	3.60996	7023	0056	31
30	6724	6363	7732	3.60588	7053	0027	30
	0.2	0.9	0.2		0.2	1.2	
31	6752	6355	7764	3.60181	7082	9998	29
32	6780	6347	7795	3.59775	7111	9969	28
33	6808	6340	7826	3.59370	7140	9940	27
34	6836	6332	7858	3.58966	7169	9911	26
35	6864	6324	7889	3.58562	7198	9882	25
36	6892	6316	7920	3.58160	7227	9852	24
37	6920	6308	7952	3.57758	7256	9823	23
38	6948	6301	7983	3.57357	7285	9794	22
39	6976	6293	8015	3.56957	7314	9765	21
40	7004	6285	8046	3.56557	7343	9736	20
41	7032	6277	8077	3.56159	7373	9707	19
42	7060	6269	8109	3.55761	7402	9678	18
43	7088	6261	8140	3.55364	7431	9649	17
44	7116	6253	8172	3.54968	7460	9620	16
45	7144	6246	8203	3.54573	7489	9591	15
46	7172	6238	8234	3.54179	7518	9562	14
47	7200	6230	8266	3.53785	7547	9532	13
48	7228	6222	8297	3.53393	7576	9503	12
49	7256	6214	8329	3.53001	7605	9474	11
50	7284	6206	8360	3,52609	7634	9445	10
51	7312	6198	8391	3.52219	7663	9416	9
52	7340	6190	8423	3.51829	7693	9387	8
53	7368	6182	8454	3.51441	7722	9358	7
54	7396	6174	8486	3.51053	7751	9329	6
55	7424	6166	8517	3.50666	7780	9300	5
56	7452	6158	8549	3.50279	7809	9271	4
57	7480	6150	8580	3.49894	7838	9242	3
58	7508	6142	8612	3.49509	7867	9212	2
59	7536	6134	8643	3.49125	7896	9183	1
60	7564	6126	8675	3.48741	7925	9154	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. 105° = 6300′ 74° = 4440′

16° = 960′ Sup. **163°** = 9780′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.2	**0.9**	**0.2**		**0.2**	**1.2**	
0	7564	6126	8675	3.48741	7925	9154	60
1	7592	6118	8706	3.48359	7954	9125	59
2	7620	6110	8738	3.47977	7983	9096	58
3	7648	6102	8769	3.47595	8012	9067	57
4	7676	6094	8800	3.47216	8042	9038	56
5	7704	6086	8832	3.46837	8071	9009	55
6	7731	6078	8864	3.46458	8100	8980	54
7	7759	6070	8895	3.46080	8129	8951	53
8	7787	6062	8927	3.45703	8158	8922	52
9	7815	6054	8958	3.45327	8187	8893	51
10	7843	6046	8990	3.44951	8216	8863	50
11	7871	6037	9021	3.44576	8245	8884	49
12	7899	6029	9053	3.44202	8274	8805	48
13	7927	6021	9084	3.43829	8303	8776	47
14	7955	6013	9116	3.43456	8332	8747	46
15	7983	6005	9147	3.43084	8362	8718	45
16	8011	5997	9179	3.42713	8391	8689	44
17	8039	5989	9210	3.42343	8420	8660	43
18	8067	5981	9242	3.41973	8449	8631	42
19	8095	5972	9274	3.41604	8478	8602	41
20	8123	5964	9305	3.41236	8507	8573	40
21	8150	5956	9337	3.40869	8536	8543	39
22	8178	5948	9368	3.40502	8565	8514	38
23	8206	5940	9400	3.40136	8594	8485	37
24	8234	5931	9432	3.39771	8623	8456	36
25	8262	5923	9463	3.39406	8652	8427	35
26	8290	5915	9495	3.39042	8682	8398	34
27	8318	5907	9526	3.38679	8711	8369	33
28	8346	5898	9558	3.38317	8740	8340	32
29	8374	5890	9590	3.37955	8769	8311	31
30	8402	5882	9621	3.37594	8798	8282	30
	0.2	**0.9**	**0.2**		**0.2**	**1.2**	
31	8429	5874	9653	3.37234	8827	8253	29
32	8457	5865	9685	3.36875	8856	8223	28
33	8485	5857	9716	3.36516	8885	8194	27
34	8513	5849	9748	3.36158	8914	8165	26
35	8541	5841	9780	3.35800	8943	8136	25
36	8569	5832	9811	3.35443	8972	8107	24
37	8597	5824	9843	3.35087	9002	8078	23
38	8625	5816	9875	3.34732	9031	8049	22
39	8652	5807	9906	3.34377	9060	8020	21
40	8680	5799	9938	3.34023	9089	7991	20
41	8708	5791	9970	3.33670	9118	7962	19
			0.3				
42	8736	5782	0001	3.33317	9147	7933	18
43	8764	5774	0033	3.32965	9176	7903	17
44	8792	5766	0065	3.32614	9205	7874	16
45	8820	5757	0097	3.32264	9234	7845	15
46	8847	5749	0128	3.31914	9263	7816	14
47	8875	5740	0160	3.31565	9292	7787	13
48	8903	5732	0192	3.31216	9321	7758	12
49	8931	5724	0224	3.30868	9351	7729	11
50	8959	5715	0255	3.30521	9380	7700	10
51	8987	5707	0287	3.30174	9409	7671	9
52	9015	5698	0319	3.29829	9438	7642	8
53	9042	5690	0351	3.29483	9467	7613	7
54	9070	5681	0382	3.29139	9496	7584	6
55	9098	5673	0414	3.28795	9525	7554	5
56	9126	5664	0446	3.28452	9554	7525	4
57	9154	5656	0478	3.28109	9583	7496	3
58	9182	5647	0509	3.27767	9612	7467	2
59	9209	5639	0541	3.27426	9641	7438	1
60	9237	5630	0573	3.27085	9671	7409	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **106°** = 6360′ **73°** = 4380′

17° = 1020′ Sup. **162°** = 9720′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.2	**0.9**	**0.3**		**0.2**	**1.2**	
0	9237	5630	0537	3.27085	9671	7409	60
1	9265	5622	0605	3.26745	9700	7380	59
2	9293	5613	0637	3.26406	9729	7351	58
3	9321	5605	0669	3.26067	9758	7322	57
4	9348	5596	0700	3.25729	9787	7293	56
5	9376	5588	0732	3.25392	9816	7264	55
6	9404	5579	0764	3.25055	9845	7234	54
7	9432	5571	0796	3.24719	9874	7205	53
8	9460	5562	0828	3.24383	9903	7176	52
9	9487	5554	0860	3.24048	9932	7147	51
10	9515	5545	0891	3.23714	9561	7118	50
11	9543	5536	0923	3.23381	9991	7089	49
					0.3		
12	9571	5528	0955	3.23048	0020	7060	48
13	9599	5519	0987	3.22715	0049	7021	47
14	9626	5511	1019	3.22384	0078	6992	46
15	9654	5502	1051	3.22053	0107	6963	45
16	9682	5493	1083	3.21722	0136	6934	44
17	9710	5485	1115	3.21392	0165	6914	43
18	9737	5476	1147	3.21063	0194	6885	42
19	9765	5467	1178	3.20734	0223	6856	41
20	9793	5459	1210	3.20406	0252	6827	40
21	9821	5450	1242	3.20079	0281	6798	39
22	9849	5441	1274	3.19752	0311	6769	38
23	9876	5433	1306	3.19426	0340	6740	37
24	9904	5424	1338	3.19100	0369	6711	36
25	9932	5415	1370	3.18775	0398	6682	35
26	9960	5407	1402	3.18451	0427	6653	34
27	9987	5398	1434	3.18127	0456	6624	33
28	0015	5389	1466	3.17804	0485	6594	32
29	0043	5380	1498	3.17481	0514	6565	31
30	0071	5372	1530	3.17159	0543	6536	30
	0.3	**0.9**	**0.3**		**0.3**	**1.2**	
31	0098	5363	1562	3.16838	0572	6507	29
32	0126	5354	1594	3.16517	0601	6478	28
33	0154	5345	1626	3.16197	0630	6449	27
34	0182	5337	1658	3.15877	0660	6420	26
35	0209	5328	1690	3.15558	0689	6391	25
36	0237	5319	1722	3.15240	0718	6362	24
37	0265	5310	1754	3.14922	0747	6333	23
38	0292	5301	1786	3.14605	0776	6304	22
39	0320	5293	1818	3.14288	0805	6275	21
40	0348	5284	1850	3.13972	0834	6245	20
41	0376	5275	1882	3.13656	0863	6216	19
42	0403	5266	1914	3.13341	0892	6187	18
43	0431	5257	1946	3.13027	0921	6158	17
44	0459	5248	1978	3.12713	0950	6129	16
45	0486	5240	2010	3.12400	0980	6100	15
46	0514	5231	2042	3.12087	1009	6071	14
47	0542	5222	2074	3.11775	1038	6042	13
48	0570	5213	2106	3.11464	1067	6013	12
49	0597	5204	2139	3.11153	1096	5984	11
50	0625	5195	2171	3.10842	1125	5955	10
51	0653	5186	2203	3.10532	1154	5925	9
52	0680	5177	2235	3.10223	1183	5896	8
53	0708	5168	2267	3.09914	1212	5867	7
54	0736	5159	2299	3.09606	1241	5838	6
55	0763	5150	2331	3.09298	1270	5809	5
56	0791	5142	2363	3.08991	1300	5780	4
57	0819	5133	2396	3.08685	1329	5751	3
58	0846	5124	2428	3.08379	1358	5722	2
59	0874	5115	2460	3.08073	1387	5693	1
60	0902	5106	2492	3.07768	1416	5664	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **107°** = 6420′ **72°** = 4320′

18° = 1080′ Sup. 161° = 9660′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.3	0.9	0.3		0.3	1.2	
0	0902	5106	2492	3.07768	1416	5664	60
1	0930	5097	2524	3.07464	1445	5635	59
2	0957	5088	2556	3.07160	1474	5605	58
3	0985	5079	2588	3.06857	1503	5576	57
4	1012	5070	2621	3.06554	1532	5547	56
5	1040	5061	2653	3.06252	1561	5518	55
6	1068	5052	2685	3.05950	1590	5489	54
7	1095	5043	2717	3.05649	1619	5460	53
8	1123	5033	2749	3.05349	1649	5431	52
9	1151	5024	2782	3.05049	1678	5402	51
10	1178	5015	2814	3.04749	1707	5373	50
11	1206	5006	2846	3.04450	1736	5344	49
12	1233	4997	2878	3.04152	1765	5315	48
13	1261	4988	2911	3.03854	1794	5286	47
14	1289	4979	2943	3.03556	1823	5256	46
15	1316	4970	2975	3.03260	1852	5227	45
16	1344	4961	3007	3.02963	1881	5198	44
17	1372	4952	3040	3.02667	1910	5169	43
18	1399	4943	3072	3.02372	1939	5140	42
19	1427	4933	3104	3.02077	1969	5111	41
20	1454	4924	3136	3.01783	1998	5082	40
21	1482	4915	3169	3.01489	2027	5053	39
22	1510	4906	3201	3.01196	2056	5024	38
23	1537	4897	3233	3.00903	2085	4995	37
24	1565	4888	3266	3.00611	2114	4966	36
25	1592	4878	3298	3.00319	2143	4936	35
26	1620	4869	3330	3.00028	2172	4907	34
27	1648	4860	3363	2.99738	2201	4878	33
28	1675	4851	3395	2.99447	2230	4849	32
29	1703	4842	3427	2.99158	2259	4820	31
30	1730	4832	3460	2.98869	2289	4791	30
	0.3	0.9	0.3		0.3	1.2	
31	1758	4823	3491	2,98580	2318	4762	29
32	1786	4814	3524	2,98292	2347	4733	28
33	1813	4805	3557	2,98004	2376	4704	27
34	1841	4795	3589	2.97717	2405	4675	26
35	1868	4786	3621	2.97430	2434	4646	25
36	1896	4777	3654	2.97144	2463	4616	24
37	1923	4768	3686	2.96858	2492	4587	23
38	1951	4758	3718	2.96573	2521	4558	22
39	1979	4749	3751	2.96288	2550	4529	21
40	2006	4740	3783	2.96004	2579	4500	20
41	2034	4730	3816	2.95720	2609	4471	19
42	2061	4721	3848	2.95437	2638	4442	18
43	2089	4712	3881	2.95155	2667	4413	17
44	2116	4702	3913	2.94872	2696	4384	16
45	2144	4693	3945	2.94590	2725	4355	15
46	2171	4684	3978	2.24309	2754	4326	14
47	2199	4674	4010	2.94028	2783	4296	13
48	2227	4665	4043	2.93748	2812	4267	12
49	2254	4656	4075	2.93468	2841	4238	11
50	2282	4646	4108	2.93189	2870	4209	10
51	2309	4637	4140	2.92910	2899	4180	9
52	2337	4627	4173	2.92632	2928	4151	8
53	2364	4618	4205	2.92354	2958	4122	7
54	2392	4609	4238	2.92076	2987	4093	6
55	2419	4599	4270	2.91799	3016	4064	5
56	2447	4590	4303	2.91523	3045	4035	4
57	2474	4580	4335	2.91246	3074	4006	3
58	2502	4571	4368	2.90971	3103	3977	2
59	2529	4561	4400	2.90696	3132	3947	1
60	2557	4552	4433	2.90421	3161	3918	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

71° = 4260′

19° = 1140′ Sup. 160° = 9600′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.3	0.9	0.3		0.3	1.2	
0	2557	4552	4433	2.90421	3161	3918	60
1	2584	4542	4465	2.90147	3190	3889	59
2	2612	4533	4498	2.89873	3219	3860	58
3	2639	4523	4530	2.89600	3248	3831	57
4	2667	4514	4563	2.89327	3278	3802	56
5	2694	4504	4596	2.89055	3307	3773	55
6	2722	4495	4628	2.88783	3336	3744	54
7	2749	4485	4661	2.88511	3365	3715	53
8	2777	4476	4693	2.88240	3394	3686	52
9	2804	4466	4726	2.87970	3423	3657	51
10	2832	4457	4758	2.87700	3452	3627	50
11	2859	4447	4791	2.87430	3481	3598	49
12	2887	4438	4824	2.87161	3510	3569	48
13	2914	4428	4856	2.86892	3539	3540	47
14	2942	4418	4889	2.86624	3568	3511	46
15	2969	4409	4922	2.86356	3598	3482	45
16	2997	4399	4954	2.86089	3627	3453	44
17	3024	4390	4987	2.85822	3656	3424	43
18	3051	4380	5019	2.85555	3685	3395	42
19	3079	4370	5052	2.85289	3714	3366	41
20	3106	4361	5085	2.85023	3743	3337	40
21	3134	4351	5117	2.84758	3772	3307	39
22	3161	4342	5150	2.84494	3801	3278	38
23	3189	4332	5183	2.84229	3830	3249	37
24	3216	4322	5216	2.83965	3859	3220	36
25	3244	4313	5248	2.83702	3888	3191	35
26	3271	4303	5281	2.83439	3918	3162	34
27	3298	4293	5314	2.83176	3947	3133	33
28	3326	4284	5346	2.82914	3976	3104	32
29	3353	4274	5379	2.82653	4005	3075	31
30	3381	4264	5412	2.82391	4034	3046	30
	0.3	0.9	0.3		0.3	1.2	
31	3408	4254	5445	2.82130	4063	3017	29
32	3436	4245	5477	2.81870	4092	2987	28
33	3463	4235	5510	2.81610	4121	2958	27
34	3490	4225	5543	2.81350	4150	2929	26
35	3518	4215	5576	2.81091	4179	2900	25
36	3545	4206	5608	2.80833	4208	2871	24
37	3573	4196	5641	2.80574	4237	2842	23
38	3600	4186	5674	2.80316	4267	2813	22
39	3627	4176	5707	2.80059	4296	2784	21
40	3655	4167	5740	2.79802	4325	2755	20
41	3682	4157	5772	2.79545	4354	2726	19
42	3710	4147	5805	2.79289	4383	2697	18
43	3737	4137	5838	2.79033	4412	2668	17
44	3764	4127	5871	2.78778	4441	2638	16
45	3792	4118	5904	2.78523	4470	2609	15
46	3819	4108	5937	2.78269	4499	2580	14
47	3846	4098	5969	2.78014	4528	2551	13
48	3874	4088	6002	2.77761	4557	2522	12
49	3901	4078	6035	2.77507	4587	2493	11
50	3929	4068	6068	2.77254	4616	2464	10
51	3956	4058	6101	2.77002	4645	2435	9
52	3983	4049	6134	2.76750	4674	2406	8
53	4011	4039	6167	2.76498	4703	2377	7
54	4038	4029	6199	2.76247	4732	2348	6
55	4065	4019	6232	2.75996	4761	2318	5
56	4093	4009	6265	2.75746	4790	2289	4
57	4120	3999	6298	2.75496	4819	2260	3
58	4147	3989	6331	2.75246	4848	2231	2
59	4175	3979	6364	2.74997	4877	2202	1
60	4202	3969	6397	2.74748	4907	2173	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. 109° = 6540′ 70° = 4200′

20° = 1200′ Sup. **159°** = 9540′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.3	0.9	0.3		0.3	1.2	
0	4202	3969	6397	2.74748	4907	2173	60
1	4229	3959	6430	2.74499	4936	2144	59
2	4257	3949	6463	2.74251	4965	2115	58
3	4284	3939	6496	2.74004	4994	2086	57
4	4311	3929	6529	2.73756	5023	2057	56
5	4339	3919	6562	2.73509	5052	2028	55
6	4366	3909	6595	2.73263	5081	1998	54
7	4393	3899	6628	2.73017	5110	1969	53
8	4421	3889	6661	2.72771	5139	1940	52
9	4448	3879	6694	2.72526	5168	1911	51
10	4475	3869	6727	2.72281	5197	1882	50
11	4503	3859	6760	2.72036	5227	1853	49
12	4530	3849	6793	2.71792	5256	1824	48
13	4557	3839	6826	2.71548	5285	1795	47
14	4584	3829	6859	2.71305	5314	1766	46
15	4612	3819	6892	2.71062	5343	1737	45
16	4639	3809	6925	2.70819	5372	1708	44
17	4666	3799	6958	2.70577	5401	1678	43
18	4694	3789	6991	2.70335	5430	1649	42
19	4721	3779	7024	2.70094	5459	1620	41
20	4748	3769	7057	2.69853	5488	1591	40
21	4775	3759	7090	2.69612	5517	1562	39
22	4803	3748	7123	2.69371	5546	1533	38
23	4830	3738	7157	2.69131	5576	1504	37
24	4857	3728	7190	2.68892	5605	1475	36
25	4884	3718	7223	2.68653	5634	1446	35
26	4912	3708	7256	2.68414	5663	1417	34
27	4939	3698	7289	2.68175	5692	1388	33
28	4966	3688	7322	2.67937	5721	1359	32
29	4993	3677	7355	2.67700	5750	1329	31
30	5021	3667	7388	2.67462	5779	1300	30
	0.3	0.9	0.3		0.3	1.2	
31	5048	3657	7422	2.67225	5808	1271	29
32	5075	3647	7455	2.66989	5837	1242	28
33	5102	3637	7488	2.66752	5866	1213	27
34	5130	3626	7521	2.66516	5896	1184	26
35	5157	3616	7554	2.66281	5925	1155	25
36	5184	3606	7588	2.66046	5954	1126	24
37	5211	3596	7621	2.65811	5983	1097	23
38	5239	3585	7654	2.65576	6012	1068	22
39	5266	3575	7687	2.65342	6041	1039	21
40	5293	3565	7720	2.65109	6070	1009	20
41	5320	3555	7754	2.64875	6099	0980	19
42	5347	3544	7787	2.64642	6128	0951	18
43	5375	3534	7820	2.64410	6157	0922	17
44	5402	3524	7853	2.64177	6186	0893	16
45	5429	3514	7887	2.63945	6216	0864	15
46	5456	3503	7920	2.63714	6245	0835	14
47	5483	3493	7953	2.63483	6274	0806	13
48	5511	3483	7986	2.63252	6303	0777	12
49	5538	3472	8020	2.63021	6332	0748	11
50	5565	3462	8053	2.62791	6361	0719	10
51	5592	3452	8086	2.62561	6390	0689	9
52	5619	3441	8120	2.62332	6419	0660	8
53	5647	3431	8153	2.62103	6448	0631	7
54	5674	3420	8186	2.61874	6477	0602	6
55	5701	3410	8220	2.61646	6506	0573	5
56	5728	3400	8253	2.61418	6536	0544	4
57	5755	3389	8286	2.61190	6565	0515	3
58	5782	3379	8320	2.60963	6594	0486	2
59	5810	3368	8353	2.60736	6623	0457	1
60	5837	3358	8386	2.60509	6652	0428	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **110°** = 6600′ **69°** = 4140′

21° = 1260 Sup. **158°** = 9480.

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.3	0.9	0.3		0.3	1.2	
0	5837	3358	8386	2.60509	6652	0428	60
1	5864	3348	8420	2.60283	6681	0399	59
2	5891	3337	8453	2.60057	6710	0369	58
3	5918	3327	8487	2.59831	6739	0340	57
4	5945	3316	8520	2.59606	6768	0311	56
5	5973	3306	8553	2.59381	6797	0282	55
6	6000	3295	8587	2.59156	6826	0253	54
7	6027	3285	8620	2.58932	6855	0224	53
8	6054	3274	8654	2.58708	6885	0195	52
9	6081	3264	8687	2.58484	6914	0166	51
10	6108	3253	8721	2.58261	6943	0137	50
11	6135	3243	8754	2.58038	6972	0108	49
12	6162	3232	8787	2.57815	7001	0079	48
13	6190	3222	8821	2.57593	7030	0050	47
14	6217	3211	8854	2.57371	7059	0020	46
						1.1	
15	6244	3201	8888	2.57150	7088	9991	45
16	6271	3190	8921	2.56928	7117	9962	44
17	6298	3180	8955	2.56707	7146	9933	43
18	6325	3169	8988	2.56487	7175	9904	42
19	6352	3159	9022	2.56266	7205	9875	41
20	6379	3148	9055	2.56046	7234	9846	40
21	6406	3137	9089	2.55827	7263	9817	39
22	6433	3127	9122	2.55608	7292	9788	38
23	6461	3116	9156	2.55389	7321	9759	37
24	6488	3106	9190	2.55170	7350	9730	36
25	6515	3095	9223	2.54952	7379	9700	35
26	6542	3084	9257	2.54734	7408	9671	34
27	6569	3074	9290	2.54516	7437	9642	33
28	6596	3063	9324	2.54299	7466	9613	32
29	6623	3052	9357	2.54082	7495	9584	31
30	6650	3042	9391	2.53865	7525	9555	30
	0.3	0.9	0.3		0.3	1.1	
31	6677	3031	9425	2.53648	7554	9526	29
32	6704	3020	9458	2.53432	7583	9497	28
33	6731	3010	9492	2.53217	7612	9468	27
34	6758	2999	9526	2.53001	7641	9439	26
35	6785	2988	9559	2.52786	7670	9410	25
36	6812	2978	9593	2.52571	7699	9380	24
37	6839	2967	9626	2.52357	7728	9351	23
38	6867	2956	9660	2.52142	7757	9322	22
39	6894	2945	9694	2.51929	7786	9293	21
40	6921	2935	9727	2.51715	7815	9264	20
41	6948	2924	9761	2.51502	7845	9235	19
42	6975	2913	9795	2.51289	7874	9206	18
43	7002	2903	9829	2.51076	7903	9177	17
44	7029	2892	9862	2.50864	7932	9148	16
45	7056	2881	9896	2.50652	7961	9119	15
46	7083	2870	9930	2.50440	7990	9090	14
47	7110	2859	9963	2.50229	8019	9060	13
48	7137	2849	9997	2.50018	8048	9031	12
			0.4				
49	7164	2838	0031	2.49807	8077	9002	11
50	7191	2827	0065	2.49597	8106	8973	10
51	7218	2816	0098	2.49386	8135	8944	9
52	7245	2805	0132	2.49177	8164	8915	8
53	7272	2794	0166	2.48967	8194	8886	7
54	7299	2784	0200	2.48758	8223	8857	6
55	7326	2773	0234	2.48549	8252	8828	5
56	7353	2762	0267	2.48340	8281	8799	4
57	7380	2751	0301	2.48132	8310	8770	3
58	7407	2740	0335	2.47924	8339	8741	2
59	7434	2729	0369	2.47716	8368	8711	1
60	7461	2718	0403	2.47509	8397	8682	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **111°** = 6660′ **68°** = 4080′

22° = 1320′ Sup. **157°** = 9420′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.3	**0.9**	**0.4**		**0.3**	**1.1**	
0	7461	2718	0403	2.47509	8397	8682	60
1	7488	2707	0436	2.47302	8426	8653	59
2	7515	2697	0470	2.47095	8455	8624	58
3	7542	2686	0504	2.46888	8484	8595	57
4	7569	2675	0538	2.46682	8514	8566	56
5	7595	2664	0572	2.46476	8543	8537	55
6	7622	2653	0606	2.46270	8572	8508	54
7	7649	2642	0640	2.46065	8601	8479	53
8	7676	2631	0674	2.45860	8630	8450	52
9	7703	2620	0707	2.45655	8659	8421	51
10	7730	2609	0741	2.45451	8688	8391	50
11	7757	2598	0775	2.45246	8717	8362	49
12	7784	2587	0809	2.45043	8746	8333	48
13	7811	2576	0843	2.44839	8775	8304	47
14	7838	2565	0877	2.44636	8804	8275	46
15	7865	2554	0911	2.44433	8834	8246	45
16	7892	2543	0945	2.44230	8863	8217	44
17	7919	2532	0979	2.44027	8892	8188	43
18	7946	2521	1013	2.43825	8921	8159	42
19	7973	2510	1047	2.43623	8950	8130	41
20	7999	2499	1081	2.43422	8979	8101	40
21	8026	2488	1115	2.43220	9008	8071	39
22	8053	2477	1149	2.43019	9037	8042	38
23	8080	2466	1183	2.42819	9066	8013	37
24	8107	2455	1217	2.42618	9095	7984	36
25	8134	2444	1251	2.42418	9124	7955	35
26	8161	2432	1285	2.42218	9154	7926	34
27	8188	2421	1319	2.42019	9183	7897	33
28	8215	2410	1353	2.41819	9212	7868	32
29	8241	2399	1387	2.41620	9241	7839	31
30	8268	2388	1421	2.41422	9270	7810	30
	0.3	**0.9**	**0.4**		**0.3**	**1.1**	
31	8295	2377	1455	2.41223	9299	7781	29
32	8322	2366	1490	2.41025	9328	7751	28
33	8349	2355	1524	2.40827	9357	7722	27
34	8376	2343	1558	2.40629	9386	7693	26
35	8403	2332	1592	2.40432	9415	7664	25
36	8430	2321	1626	2.40235	9444	7635	24
37	8456	2310	1660	2.40038	9473	7606	23
38	8483	2299	1694	2.39841	9503	7577	22
39	8510	2287	1728	2.39645	9532	7548	21
40	8537	2276	1763	2.39449	9561	7519	20
41	8564	2265	1797	2.39253	9590	7490	19
42	8591	2254	1831	2.39058	9619	7461	18
43	8617	2243	1865	2.38863	9648	7432	17
44	8644	2231	1899	2.38668	9677	7402	16
45	8671	2220	1933	2.38473	9706	7373	15
46	8698	2209	1968	2.38279	9735	7344	14
47	8725	2198	2002	2.38084	9764	7315	13
48	8752	2186	2036	2.37891	9793	7286	12
49	8778	2175	2070	2.37697	9823	7257	11
50	8805	2164	2105	2.37504	9852	7228	10
51	8832	2152	2139	2.37311	9881	7199	9
52	8859	2141	2173	2.37118	9910	7170	8
53	8886	2130	2207	2.36925	9939	7141	7
54	8912	2119	2242	2.36733	9968	7112	6
55	8939	2107	2276	2.36541	9997	7082	5
					0.4		
56	8966	2096	2310	2.36349	0026	7053	4
57	8993	2085	2345	2.36158	0055	7024	3
58	9020	2073	2379	2.35967	0084	6995	2
59	9046	2062	2413	2.35776	0113	6966	1
60	9073	2050	2447	2.35585	0143	6937	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **112°** = 6720′ **67°** = 4020′

23° = 1380′ Sup. **156°** = 9360′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.3	**0.9**	**0.4**		**0.4**	**1.1**	
0	9073	2050	2447	2.35585	0143	6937	60
1	9100	2039	2482	2.35395	0172	6908	59
2	9127	2028	2516	2.35205	0201	6879	58
3	9153	2016	2551	2.35015	0230	6850	57
4	9180	2005	2585	2.34825	0259	6821	56
5	9207	1994	2619	2.34636	0288	6792	55
6	9234	1982	2654	2.34447	0317	6762	54
7	9260	1971	2688	2.34258	0346	6733	53
8	9287	1959	2722	2.34069	0375	6704	52
9	9314	1948	2757	2.33881	0404	6675	51
10	9341	1936	2791	2.33693	0433	6646	50
11	9368	1925	2826	2.33505	0463	6617	49
12	9394	1914	2860	2.33317	0492	6588	48
13	9421	1902	2894	2.33130	0521	6559	47
14	9448	1891	2929	2.32943	0550	6530	46
15	9474	1879	2963	2.32756	0579	6501	45
16	9501	1868	2998	2.32570	0608	6472	44
17	9528	1856	3032	2.32383	0637	6442	43
18	9555	1845	3067	2.32197	0666	6413	42
19	9581	1833	3101	2.32012	0695	6384	41
20	9608	1822	3136	2.31826	0724	6355	40
21	9635	1810	3170	2.31641	0753	6326	39
22	9661	1799	3205	2.31456	0782	6297	38
23	9688	1787	3239	2.31271	0812	6268	37
24	9715	1775	3274	2.31086	0841	6239	36
25	9741	1764	3308	2.30902	0870	6210	35
26	9768	1752	3343	2.30718	0899	6181	34
27	9795	1741	3378	2.30534	0928	6152	33
28	9822	1729	3412	2.30351	0957	6123	32
29	9848	1718	3447	2.30167	0986	6093	31
30	9875	1706	3481	2.29984	1015	6064	30
	0.3	**0.9**	**0.4**		**0.4**	**1.1**	
31	9902	1694	3516	2.29801	1044	6035	29
32	9928	1683	3550	2.29619	1073	6006	28
33	9955	1671	3585	2.29437	1102	5977	27
34	9982	1660	3620	2.29254	1132	5948	26
	0.4						
35	0008	1648	3654	2.29073	1161	5919	25
36	0035	1636	3689	2.28891	1190	5890	24
37	0062	1625	3724	2.28710	1219	5861	23
38	0088	1613	3758	2.28528	1248	5832	22
39	0115	1601	3793	2.28348	1277	5803	21
40	0141	1590	3828	2.28167	1306	5773	20
41	0168	1578	3862	2.27987	1335	5744	19
42	0195	1566	3897	2.27806	1364	5715	18
43	0221	1555	3932	2.27626	1393	5686	17
44	0248	1543	3966	2.27447	1422	5657	16
45	0275	1531	4001	2.27267	1452	5628	15
46	0301	1519	4036	2.27088	1481	5599	14
47	0328	1508	4071	2.26909	1510	5570	13
48	0355	1496	4105	2.26730	1539	5541	12
49	0381	1484	4140	2.26552	1568	5512	11
50	0408	1472	4175	2.26374	1597	5483	10
51	0434	1461	4210	2.26196	1626	5453	9
52	0461	1449	4244	2.26018	1655	5424	8
53	0488	1437	4279	2.25840	1684	5395	7
54	0514	1425	4314	2.25663	1713	5366	6
55	0541	1414	4349	2.25486	1742	5337	5
56	0567	1402	4384	2.25309	1771	5308	4
57	0594	1390	4418	2.25132	1801	5279	3
58	0621	1378	4453	2.24956	1830	5250	2
59	0647	1366	4488	2.24780	1859	5221	1
60	0674	1355	4523	2.24604	1888	5192	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **113°** = 6780′ **66°** = 3960′

24° = 1440′ Sup. **155°** = 9300′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.4	**0.9**	**0.4**		**0.4**	**1.1**	
0	0674	1355	4523	2.24604	1888	5192	60
1	0700	1343	4558	2.24428	1917	5163	59
2	0727	1331	4593	2.24252	1946	5134	58
3	0753	1319	4627	2.24077	1975	5104	57
4	0780	1307	4662	2.23902	2004	5075	56
5	0806	1295	4697	2.23727	2033	5046	55
6	0833	1283	4732	2.23553	2062	5017	54
7	0860	1272	4767	2.23378	2091	4988	53
8	0886	1260	4802	2.23204	2121	4959	52
9	0913	1248	4837	2.23030	2150	4930	51
10	0939	1236	4872	2.22857	2179	4901	50
11	0966	1224	4907	2.22683	2208	4872	49
12	0992	1212	4942	2.22510	2237	4843	48
13	1019	1200	4977	2.22337	2266	4814	47
14	1045	1188	5012	2.22164	2295	4784	46
15	1072	1176	5047	2.21992	2324	4755	45
16	1098	1164	5082	2.21819	2353	4726	44
17	1125	1152	5117	2.21647	2382	4697	43
18	1151	1140	5152	2.21475	2411	4668	42
19	1178	1128	5187	2.21304	2441	4639	41
20	1204	1116	5222	2.21132	2470	4610	40
21	1231	1104	5257	2.20961	2499	4581	39
22	1257	1092	5292	2.20790	2528	4552	38
23	1284	1080	5327	2.20619	2557	4523	37
24	1310	1068	5362	2.20449	2586	4494	36
25	1337	1056	5397	2.20278	2615	4464	35
26	1363	1044	5432	2.20108	2644	4435	34
27	1390	1032	5467	2.19938	2673	4406	33
28	1416	1020	5502	2.19769	2702	4377	32
29	1443	1008	5538	2.19599	2731	3448	31
30	1469	0996	5573	2.19430	2761	4319	30
	0.4	**0.9**	**0.4**		**0.4**	**1.1**	
31	1496	0984	5608	2.19261	2790	4290	29
32	1522	0972	5643	2.19092	2819	4261	28
33	1549	0960	5678	2.18923	2848	4232	27
34	1575	0948	5713	2.18755	2877	4203	26
35	1602	0936	5748	2.18587	2906	4174	25
36	1628	0924	5784	2.18419	2935	4144	24
37	1655	0911	5819	2.18251	2964	4115	23
38	1681	0899	5854	2.18084	2993	4086	22
39	1707	0887	5889	2.17916	3022	4057	21
40	1734	0875	5924	2.17749	3051	4028	20
41	1760	0863	5960	2.17582	3080	3999	19
42	1787	0851	5995	2.17416	3110	3970	18
43	1813	0839	6030	2.17249	3139	3941	17
44	1840	0826	6065	2.17083	3168	3912	16
45	1866	0814	6101	2.16917	3197	3883	15
46	1892	0802	6136	2.16751	3226	3854	14
47	1919	0790	6171	2.16585	3255	3825	13
48	1945	0778	6206	2.16420	3284	3795	12
49	1972	0766	6242	2.16255	3313	3766	11
50	1998	0753	6277	2.16090	3342	3737	10
51	2024	0741	6312	2.15925	3371	3708	9
52	2051	0729	6348	2.15760	3400	3679	8
53	2077	0717	6383	2.15596	3430	3650	7
54	2104	0704	6418	2.15432	3459	3621	6
55	2130	0692	6454	2.15268	3488	3592	5
56	2156	0680	6489	2.15104	3517	3563	4
57	2183	0668	6525	2.14940	3546	3534	3
58	2209	0655	6560	2.14777	3575	3505	2
59	2235	0643	6595	2.14614	3604	3475	1
60	2262	0631	6631	2.14451	3633	3446	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **114°** = 6840′ **65°** = 3900′

25° = 1500′ Sup. **154°** = 9240′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.4	**0.9**	**0.4**		**0.4**	**1.1**	
0	2262	0631	6631	2.14451	3633	3446	60
1	2288	0618	6666	2.14288	3662	3417	59
2	2315	0606	6702	2.14125	3691	3388	58
3	2341	0594	6737	2.13963	3720	3359	57
4	2367	0582	6773	2.13801	3750	3330	56
5	2394	0569	6808	2.13639	3779	3301	55
6	2420	0557	6843	2.13477	3808	3272	54
7	2446	0545	6879	2.13316	3837	3243	53
8	2473	0532	6914	2.13154	3866	3214	52
9	2499	0520	6950	2.12993	3895	3185	51
10	2525	0507	6985	2.12832	3924	3155	50
11	2552	0495	7021	2.12671	3953	3126	49
12	2578	0483	7056	2.12511	3982	3097	48
13	2604	0470	7092	2.12350	4011	3068	47
14	2631	0458	7128	2.12190	4040	3039	46
15	2657	0446	7163	2.12030	4070	3010	45
16	2683	0433	7199	2.11871	4099	2981	44
17	2709	0421	7234	2.11711	4128	2952	43
18	2736	0408	7270	2.11552	4157	2923	42
19	2762	0396	7305	2.11392	4186	2894	41
20	2788	0383	7341	2.11233	4215	2865	40
21	2815	0371	7377	2.11075	4244	2835	39
22	2844	0358	7412	2.10916	4273	2806	38
23	2867	0346	7448	2.10758	4302	2777	37
24	2894	0334	7483	2.10600	4331	2748	36
25	2920	0321	7519	2.10441	4360	2719	35
26	2946	0309	7555	2.10284	4389	2690	34
27	2972	0296	7590	2.10126	4419	2661	33
28	2999	0284	7626	2.09969	4448	2632	32
29	3025	0271	7662	2.09811	4477	2603	31
30	3051	0259	7698	2.09654	4506	2574	30
	0.4	**0.9**	**0.4**		**0.4**	**1.1**	
31	3077	0246	7733	2.09498	4535	2545	29
32	3104	0233	7769	2.09341	4564	2516	28
33	3130	0221	7805	2.09184	4593	2486	27
34	3156	0208	7840	2.09028	4622	2457	26
35	3182	0196	7876	2.08872	4651	2428	25
36	3209	0183	7912	2.08716	4680	2399	24
37	3235	0171	7948	2.08560	4709	2370	23
38	3261	0158	7984	2.08405	4739	2341	22
39	3287	0146	8019	2.08250	4768	2312	21
40	3313	0133	8055	2.08094	4797	2283	20
41	3340	0120	8091	2.07939	4826	2254	19
42	3366	0108	8127	2.07785	4855	2225	18
43	3392	0095	8163	2.07630	4884	2196	17
44	3418	0082	8198	2.07476	4913	2166	16
45	3445	0070	8234	2.07321	4942	2137	15
46	3471	0057	8270	2.07167	4971	2108	14
47	3497	0045	8306	2.07014	5000	2079	13
48	3523	0032	8342	2.06860	5029	2050	12
49	3549	0019	8378	2.06706	5059	2021	11
50	3575	0007	8414	2.06553	5088	1992	10
		0.8					
51	3602	9994	8450	2.06400	5117	1963	9
52	3628	9981	8486	2.06247	5146	1934	8
53	3654	9968	8521	2.06094	5175	1905	7
54	3680	9956	8557	2.05942	5204	1876	6
55	3706	9943	8593	2.05790	5233	1846	5
56	3732	9930	8629	2.05637	5262	1817	4
57	3759	9918	8665	2.05485	5291	1788	3
58	3785	9905	8701	2.05333	5320	1759	2
59	3811	9892	8737	2.05182	5349	1730	1
60	3837	9879	8773	2.05030	5379	1701	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **115°** = 6900′ **64°** = 3840′

26° = 1560′ Sup. **153°** = 9180′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.4	**0.8**	**0.4**		**0.4**	**1.1**	
0	3837	9879	8773	2.05030	5379	1701	60
1	3863	9867	8809	2.04879	5408	1672	59
2	3889	9854	8845	2.04728	5437	1643	58
3	3916	9841	8881	2.04577	5466	1614	57
4	3942	9828	8917	2.04426	5495	1585	56
5	3968	9816	8953	2.04276	5524	1556	55
6	3994	9803	8989	2.04125	5553	1526	54
7	4020	9790	9026	2.03975	5582	1497	53
8	4046	9777	9062	2.03825	5611	1468	52
9	4072	9764	9098	2.03675	5640	1439	51
10	4098	9752	9134	2.03526	5669	1410	50
11	4124	9739	9170	2.03376	5698	1381	49
12	4151	9726	9206	2.03227	5728	1352	48
13	4177	9713	9242	2.03078	5757	1323	47
14	4203	9700	9278	2.02929	5786	1294	46
15	4229	9687	9315	2.02780	5815	1265	45
16	4255	9674	9351	2.02631	5844	1236	44
17	4281	9662	9387	2.02483	5873	1207	43
18	4307	9649	9423	2.02335	5902	1177	42
19	4333	9636	9459	2.02187	5931	1148	41
20	4359	9623	9495	2.02039	5960	1119	40
21	4385	9610	9532	2.01891	5989	1090	39
22	4411	9597	9568	2.01743	6018	1061	38
23	4437	9584	9604	2.01596	6048	1032	37
24	4464	9571	9640	2.01449	6077	1003	36
25	4490	9558	9677	2.01302	6106	0974	35
26	4516	9545	9713	2.01155	6135	0945	34
27	4542	9532	9749	2.01008	6164	0916	33
28	4568	9519	9786	2.00862	6193	0887	32
29	4594	9506	9822	2.00715	6222	0857	31
30	4620	9493	9858	2.00569	6251	0828	30
	0.4	**0.8**	**0.4**		**0.4**	**1.1**	
31	4646	9480	9894	2.00423	6280	0799	29
32	4672	9467	9931	2.00277	6309	0770	28
33	4698	9454	9967	2.00131	6338	0741	27
			0.5				
34	4724	9441	0004	1.99986	6368	0712	26
35	4750	9428	0040	1.99841	6397	0683	25
36	4776	9415	0076	1.99695	6426	0654	24
37	4802	9402	0113	1.99550	6455	0625	23
38	4828	9389	0149	1.99406	6484	0596	22
39	4854	9376	0185	1.99261	6513	0567	21
40	4880	9363	0222	1.99116	6542	0537	20
41	4906	9350	0258	1.98972	6571	0508	19
42	4932	9337	0295	1.98828	6600	0479	18
43	4958	9324	0331	1.98684	6629	0450	17
44	4984	9311	0368	1.98540	6658	0421	16
45	5010	9298	0404	1.98396	6688	0392	15
46	5036	9285	0441	1.98253	6717	0363	14
47	5062	9272	0477	1.98110	6746	0334	13
48	5088	9259	0514	1.97966	6775	0305	12
49	5114	9245	0550	1.97823	6804	0276	11
50	5140	9232	0587	1.97680	6833	0247	10
51	5166	9219	0623	1.97538	6862	0217	9
52	5192	9206	0660	1.97395	6891	0188	8
53	5218	9193	0696	1.97253	6920	0159	7
54	5243	9180	0733	1.97111	6949	0130	6
55	5269	9167	0769	1.96969	6978	0101	5
56	5295	9153	0806	1.96827	7007	0072	4
57	5321	9140	0843	1.96685	7037	0043	3
58	5347	9127	0879	1.96544	7066	0014	2
						1.0	
59	5373	9114	0916	1.96402	7095	9985	1
60	5399	9101	0953	1.96261	7124	9956	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. 116° = 6960′ **63°** = 3780′

27° = 1620′ **Sup. 152°** = 9120′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.4	**0.8**	**0.5**		**0.4**	**1.0**	
0	5399	9101	0953	1.96261	7124	9956	60
1	5425	9087	0989	1.96120	7153	9927	59
2	5451	9074	1026	1.95979	7182	9898	58
3	5477	9061	1063	1.95838	7211	9868	57
4	5503	9048	1099	1.95698	7240	9839	56
5	5529	9035	1136	1.95557	7269	9810	55
6	5554	9021	1173	1.95417	7298	9781	54
7	5580	9008	1209	1.95277	7327	9752	53
8	5606	8995	1246	1.95137	7357	9723	52
9	5632	8981	1283	1.94997	7386	9694	51
10	5658	8968	1319	1.94858	7415	9665	50
11	5684	8955	1356	1.94718	7444	9636	49
12	5710	8942	1393	1.94579	7473	9607	48
13	5736	8928	1430	1.94440	7502	9578	47
14	5762	8915	1467	1.94301	7531	9548	46
15	5787	8902	1503	1.94162	7560	9519	45
16	5813	8888	1540	1.94023	7589	9490	44
17	5839	8875	1577	1.93885	7618	9461	43
18	5865	8862	1614	1.93746	7647	9432	42
19	5891	8848	1651	1.93608	7677	9403	41
20	5917	8835	1688	1.93470	7706	9374	40
21	5942	8822	1724	1.93332	7735	9345	39
22	5968	8808	1761	1.93195	7764	9316	38
23	5994	8795	1798	1.93057	7793	9287	37
24	6020	8782	1835	1.92920	7822	9258	36
25	6046	8768	1872	1.92782	7851	9228	35
26	6072	8755	1909	1.92645	7880	9199	34
27	6097	8741	1946	1.92508	7909	9170	33
28	6123	8728	1983	1.92371	7938	9141	32
27	6149	8715	2020	1.92235	7967	9112	31
30	6175	8701	2057	1.92098	7997	9083	30
	0.4	**0.8**	**0.5**		**0.4**	**1.0**	
31	6201	8688	2094	1.91962	8026	9054	29
32	6226	8674	2131	1.91826	8055	9025	28
33	6252	8661	2168	1.91690	8084	8996	27
34	6278	8647	2205	1.91554	8113	8967	26
35	6304	8634	2242	1.91418	8142	8938	25
36	6330	8620	2279	1.91282	8171	8908	24
37	6355	8607	2316	1.91147	8200	8879	23
38	6381	8593	2353	1.91012	8229	8850	22
39	6407	8580	2390	1.90876	8258	8821	21
40	6433	8566	2427	1.90741	8287	8792	20
41	6458	8553	2464	1.90607	8316	8763	19
42	6484	8539	2501	1.90472	8346	8734	18
43	6510	8526	2538	1.90337	8375	8705	17
44	6536	8512	2575	1.90203	8404	8676	16
45	6561	8499	2613	1.90069	8433	8647	15
46	6587	8485	2650	1.89935	8462	8618	14
47	6613	8472	2687	1.89801	8491	8589	13
48	6639	8458	2724	1.89667	8520	8559	12
49	6664	8445	2761	1.89533	8549	8530	11
50	6690	8431	2798	1.89400	8578	8501	10
51	6716	8417	2836	1.89266	8607	8472	9
52	6742	8404	2873	1.89133	8636	8443	8
53	6767	8390	2910	1.89000	8666	8414	7
54	6793	8377	2947	1.88867	8695	8385	6
55	6819	8363	2985	1.88734	8724	8356	5
56	6844	8349	3022	1.88602	8753	8327	4
57	6870	8336	3059	1.88469	8782	8298	3
58	6896	8322	3096	1.88337	8811	8269	2
59	6921	8308	3134	1.88205	8840	8239	1
60	6947	8295	3171	1.88073	8869	8210	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. 117° = 7020′ **62°** = 3720′

28° = 1680′ Sup. **151°** = 9060′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.4	**0.8**	**0.5**		**0.4**	**1.0**	
0	6947	8295	3171	1.88073	8869	8210	60
1	6973	8281	3208	1.87941	8898	8181	59
2	6999	8267	3246	1.87809	8927	8152	58
3	7024	8254	3283	1.87677	8956	8123	57
4	7050	8240	3320	1.87546	8986	8094	56
5	7076	8226	3358	1.87415	9015	8065	55
6	7101	8213	3395	1.87283	9044	8036	54
7	7127	8199	3432	1.87152	9073	8007	53
8	7152	8185	3470	1.87021	9102	7978	52
9	7178	8172	3507	1.86891	9131	7949	51
10	7204	8158	3545	1.86760	9160	7919	50
11	7229	8144	3582	1.86630	9189	7890	49
12	7255	8130	3620	1.86499	9218	7861	48
13	7281	8117	3657	1.86369	9247	7832	47
14	7306	8103	3694	1.86239	9276	7803	46
15	7332	8089	3732	1.86109	9306	7774	45
16	7358	8075	3769	1.85979	9335	7745	44
17	7383	8062	3807	1.85850	9364	7716	43
18	7409	8048	3844	1.85720	9393	7687	42
19	7434	8034	3882	1.85591	9422	7658	41
20	7460	8020	3920	1.85462	9451	7629	40
21	7486	8006	3957	1.85333	9480	7599	39
22	7511	7993	3995	1.85204	9509	7570	38
23	7537	7979	4032	1.85075	9538	7541	37
24	7562	7965	4070	1.84946	9567	7512	36
25	7588	7951	4107	1.84818	9596	7483	35
26	7614	7937	4145	1.84689	9625	7454	34
27	7639	7923	4183	1.84561	9655	7425	33
28	7665	7909	4220	1.84433	9684	7396	32
29	7690	7896	4258	1.84305	9713	7367	31
30	7716	7882	4296	1.84177	9742	7338	30
	0.4	**0.8**	**0.5**		**0.4**	**1.0**	
31	7741	7868	4333	1.84049	9771	7309	29
32	7767	7854	4371	1.83922	9800	7280	28
33	7793	7840	4409	1.83794	9829	7250	27
34	7818	7826	4446	1.83667	9858	7221	26
35	7844	7812	4484	1.83540	9887	7192	25
36	7869	7798	4522	1.83413	9916	7163	24
37	7895	7784	4560	1.83286	9945	7134	23
38	7920	7770	4597	1.83159	9975	7105	22
					0.5		
39	7946	7756	4635	1.83033	0004	7076	21
40	7971	7743	4673	1.82906	0033	7047	20
41	7997	7729	4711	1.82780	0062	7018	19
42	8022	7715	4748	1.82654	0091	6989	18
43	8048	7701	4786	1.82528	0120	6960	17
44	8073	7687	4824	1.82402	0149	6930	16
45	8099	7673	4862	1.82276	0178	6901	15
46	8124	7659	4900	1.82150	0207	6872	14
47	8150	7645	4938	1.82025	0236	6843	13
48	8175	7631	4975	1.81899	0265	6814	12
49	8201	7617	5013	1.81774	0295	6785	11
50	8226	7603	5051	1.81649	0324	6756	10
51	8252	7589	5089	1.81524	0353	6727	9
52	8277	7575	5127	1.81399	0382	6698	8
53	8303	7560	5165	1.81274	0411	6669	7
54	8328	7546	5203	1.81150	0440	6640	6
55	8354	7532	5241	1.81025	0469	6610	5
56	8379	7518	5279	1.80901	0498	6581	4
57	8405	7504	5317	1.80777	0527	6552	3
58	8430	7490	5355	1.80653	0556	6523	2
59	8456	7476	5393	1.80529	0585	6494	1
60	8481	7462	5431	1.80405	0614	6465	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **118°** = 7080′ **61°** = 3660′

29° = 1740′ Sup. **150°** = 9000′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.4	**0.8**	**0.5**		**0.5**	**1.0**	
0	8481	7462	5431	1.80405	0614	6465	60
1	8506	7448	5469	1.80281	0644	6436	59
2	8532	7434	5507	1.80158	0673	6407	58
3	8557	7420	5545	1.80034	0702	6378	57
4	8583	7405	5583	1.79911	0731	6349	56
5	8608	7391	5621	1.79788	0760	6320	55
6	8634	7377	5659	1.79665	0789	6291	54
7	8659	7363	5697	1.79542	0818	6261	53
8	8684	7349	5736	1.79419	0847	6232	52
9	8710	7335	5774	1.79296	0876	6203	51
10	8735	7321	5812	1.79174	0905	6174	50
11	8761	7306	5850	1.79051	0934	6145	49
12	8786	7292	5888	1.78929	0964	6116	48
13	8811	7278	5926	1.78807	0993	6087	47
14	8837	7264	5964	1.78685	1022	6058	46
15	8862	7250	6003	1.78563	1051	6029	45
16	8887	7235	6041	1.78441	1080	6000	44
17	8913	7221	6079	1.78319	1109	5971	43
18	8938	7207	6117	1.78198	1138	5941	42
19	8964	7193	6156	1.78077	1167	5912	41
20	8989	7178	6194	1.77955	1196	5883	40
21	9014	7164	6232	1.77834	1225	5854	39
22	9040	7150	6270	1.77713	1254	5825	38
23	9065	7136	6309	1.77592	1284	5796	37
24	9090	7121	6347	1.77471	1313	5767	36
25	9116	7107	6385	1.77351	1342	5738	35
26	9141	7093	6424	1.77230	1371	5709	34
27	9166	7079	6462	1.77110	1400	5680	33
28	9192	7064	6500	1.76990	1429	5651	32
29	9217	7050	6539	1.76869	1458	5621	31
30	9242	7036	6577	1.76749	1487	5592	30
	0.4	**0.8**	**0.5**		**0.5**	**1.0**	
31	9268	7021	6616	1.76629	1516	5563	29
32	9293	7007	6654	1.76510	1545	5534	28
33	9318	6993	6693	1.76390	1574	5505	27
34	9344	6978	6731	1.76271	1604	5476	26
35	9369	6964	6770	1.76151	1633	5447	25
36	9394	6949	6808	1.76032	1662	5418	24
37	9419	6935	6846	1.75913	1691	5389	23
38	9445	6921	6885	1.75794	1720	5360	22
39	9470	6906	6923	1.75675	1749	5331	21
40	9495	6892	6962	1.75556	1778	5301	20
41	9521	6878	7000	1.75437	1807	5272	19
42	9546	6863	7039	1.75319	1836	5243	18
43	9571	6849	7078	1.75200	1865	5214	17
44	9596	6834	7116	1.75082	1894	5185	16
45	9622	6820	7155	1.74964	1923	5156	15
46	9647	6805	7193	1.74846	1953	5127	14
47	9672	6791	7232	1.74728	1982	5098	13
48	9697	6777	7271	1.74610	2011	5069	12
49	9723	6762	7309	1.74492	2040	5040	11
50	9748	6748	7348	1.74375	2069	5011	10
51	9773	6733	7386	1.74257	2098	4982	9
52	9798	6719	7425	1.74140	2127	4952	8
53	9824	6704	7464	1.74022	2156	4923	7
54	9849	6690	7503	1.73905	2185	4894	6
55	9874	6675	7541	1.73788	2214	4865	5
56	9899	6661	7580	1.73671	2243	4836	4
57	9924	6646	7619	1.73555	2273	4807	3
58	9950	6632	7657	1.73438	2302	4778	2
59	9975	6617	7696	1.73321	2331	4749	1
	0.5						
60	0000	6603	7735	1.73205	2360	4720	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **119°** = 7140′ **60°** = 3600′

30° = 1800′ Sup. **149°** = 8940′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.5	0.8	0.5		0.5	1.0	
0	0000	6603	7735	1.73205	2360	4720	60
1	0025	6588	7774	1.73089	2389	4691	59
2	0050	6573	7813	1.72973	2418	4662	58
3	0076	6559	7851	1.72857	2447	4632	57
4	0101	6544	7890	1.72741	2476	4603	56
5	0126	6530	7929	1.72625	2505	4574	55
6	0151	6515	7968	1.72509	2534	4545	54
7	0176	6501	8007	1.72393	2563	4516	53
8	0201	6486	8046	1.72278	2593	4487	52
9	0227	6471	8085	1.72163	2622	4458	51
10	0252	6457	8124	1.72047	2651	4429	50
11	0277	6442	8162	1.71932	2680	4400	49
12	0302	6427	8201	1.71817	2709	4371	48
13	0327	6413	8240	1.71702	2738	4342	47
14	0352	6398	8279	1.71588	2767	4312	46
15	0377	6384	8318	1.71473	2796	4283	45
16	0403	6369	8357	1.71358	2825	4254	44
17	0428	6354	8396	1.71244	2854	4225	43
18	0453	6340	8435	1.71129	2883	4196	42
19	0478	6325	8474	1.71015	2913	4167	41
20	0503	6310	8513	1.70901	2942	4138	40
21	0528	6295	8552	1.70787	2971	4109	39
22	0553	6281	8591	1.70673	3000	4080	38
23	0578	6266	8631	1.70560	3029	4051	37
24	0603	6251	8670	1.70446	3058	4022	36
25	0628	6237	8709	1.70332	3087	3992	35
26	0654	6222	8748	1.70219	3116	3963	34
27	0679	6207	8787	1.70106	3145	3934	33
28	0704	6192	8826	1.69992	3174	3905	32
29	0729	6178	8865	1.69879	3203	3876	31
30	0754	6163	8904	1.69766	3232	3847	30
	0.5	0.8	0.5		0.5	1.0	
31	0779	6148	8944	1.69653	3262	3818	29
32	0804	6133	8983	1.69541	3291	3789	28
33	0829	6119	9022	1.69428	3320	3760	27
34	0854	6104	9061	1.69315	3349	3731	26
35	0879	6089	9101	1.69203	3378	3702	25
36	0904	6074	9140	1.69091	3407	3673	24
37	0929	6059	9179	1.68979	3436	3643	23
38	0954	6045	9218	1.68866	3465	3614	22
39	0979	6030	9258	1.68754	3494	3585	21
40	1004	6015	9297	1.68643	3523	3556	20
41	1029	6000	9336	1.68531	3552	3527	19
42	1054	5985	9376	1.68419	3582	3498	18
43	1079	5970	9415	1.68308	3611	3469	17
44	1104	5956	9454	1.68196	3640	3440	16
45	1129	5941	9494	1.68085	3669	3411	15
46	1154	5926	9533	1.67974	3698	3382	14
47	1179	5911	9573	1.67863	3727	3353	13
48	1204	5896	9612	1.67752	3756	3323	12
49	1229	5881	9651	1.67641	3785	3294	11
50	1254	5866	9691	1.67530	3814	3265	10
51	1279	5851	9730	1.67419	3843	3236	9
52	1304	5836	9770	1.67309	3872	3207	8
53	1329	5821	9809	1.67198	3902	3178	7
54	1354	5806	9849	1.67088	3931	3149	6
55	1379	5792	9888	1.66978	3960	3120	5
56	1404	5777	9928	1.66867	3989	3091	4
57	1429	5762	9967	1.66757	4018	3062	3
			0.6				
58	1454	5747	0007	1.66647	4047	3033	2
59	1479	5732	0046	1.66538	4076	3003	1
60	1504	5717	0086	1.66428	4105	2974	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **120°** = 7200′ **59°** = 3540′

31° = 1860′ Sup. **148°** = 8880′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.5	0.8	0.6		0.5	1.0	
0	1504	5717	0086	1.66428	4105	2974	60
1	1529	5702	0126	1.66318	4134	2945	59
2	1554	5687	0165	1.66209	4163	2916	58
3	1579	5672	0205	1.66099	4192	2887	57
4	1604	5657	0245	1.65990	4222	2858	56
5	1628	5642	0284	1.65881	4251	2829	55
6	1653	5627	0324	1.65772	4280	2800	54
7	1678	5612	0364	1.65663	4309	2771	53
8	1703	5597	0403	1.65554	4338	2742	52
9	1728	5582	0443	1.65445	4367	2713	51
10	1753	5567	0483	1.65337	4396	2683	50
11	1778	5551	0522	1.65228	4425	2654	49
12	1803	5536	0562	1.65120	4454	2625	48
13	1828	5521	0602	1.65011	4483	2596	47
14	1852	5506	0642	1.64903	4512	2567	46
15	1877	5491	0681	1.64795	4541	2538	45
16	1902	5476	0721	1.64687	4571	2509	44
17	1927	5461	0761	1.64579	4600	2480	43
18	1952	5446	0801	1.64471	4629	2451	42
19	1977	5431	0841	1.64363	4658	2422	41
20	2002	5416	0881	1.64256	4687	2393	40
21	2026	5400	0921	1.64148	4716	2364	39
22	2051	5385	0960	1.64041	4745	2334	38
23	2076	5370	1000	1.63934	4774	2305	37
24	2101	5355	1040	1.63826	4803	2276	36
25	2126	5340	1080	1.63719	4832	2247	35
26	2151	5325	1120	1.63612	4861	2218	34
27	2175	5310	1160	1.63505	4891	2189	33
28	2200	5294	1200	1.63398	4920	2160	32
29	2225	5279	1240	1.63292	4949	2131	31
30	2250	5264	1280	1.63185	4978	2102	30
	0.5	0.8	0.6		0.5	1.0	
31	2275	5229	1320	1.63079	5007	2073	29
32	2299	5234	1360	1.62972	5036	2044	28
33	2324	5218	1400	1 62866	5065	2014	27
34	2349	5203	1440	1.62760	5094	1985	26
35	2374	5188	1480	1.62654	5123	1956	25
36	2399	5173	1520	1.62548	5152	1927	24
37	2423	5157	1561	1.62442	5181	1898	23
38	2448	5142	1601	1.62336	5211	1869	22
39	2473	5127	1641	1.62230	5240	1840	21
40	2498	5112	1681	1.62125	5269	1811	20
41	2522	5096	1721	1.62019	5298	1782	19
42	2547	5081	1761	1.61914	5327	1753	18
43	2572	5066	1801	1.61808	5356	1724	17
44	2597	5051	1842	1.61703	5385	1694	16
45	2621	5035	1882	1.61598	5414	1665	15
46	2646	5020	1922	1.61493	5443	1636	14
47	2671	5005	1962	1.61388	5472	1607	13
48	2696	4989	2003	1.61283	5501	1578	12
49	2720	4974	2043	1.61179	5531	1549	11
50	2745	4959	2083	1.61074	5560	1520	10
51	2770	4943	2124	1.60970	5589	1491	9
52	2794	4928	2164	1.60865	5618	1462	8
53	2819	4913	2204	1.60761	5647	1433	7
54	2844	4897	2245	1.60657	5676	1404	6
55	2869	4882	2285	1.60553	5705	1374	5
56	2893	4866	2325	1.60449	5734	1345	4
57	2918	4851	2366	1.60345	5763	1316	3
58	2943	4836	2406	1.60241	5792	1287	2
59	2967	4820	2446	1.60137	5821	1258	1
60	2992	4805	2487	1.60033	5850	1229	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **121°** = 7260′ **58°** = 3480′

32° = 1920′ Sup. **147°** = 8820′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.5	0.8	0.6		0.5	1.0	
0	2992	4805	2487	1.60033	5850	1229	60
1	3017	4789	2527	1.59930	5880	1200	59
2	3041	4774	2568	1.59826	5909	1171	58
3	3066	4759	2608	1.59723	5938	1142	57
4	3091	4743	2649	1.59620	5967	1113	56
5	3115	4728	2689	1.59517	5996	1084	55
6	3140	4712	2730	1.59414	6025	1055	54
7	3164	4697	2770	1.59311	6054	1025	53
8	3189	4681	2811	1.59208	6083	0996	52
9	3214	4666	2852	1.59105	6112	0967	51
10	3238	4650	2892	1.59002	6141	0938	50
11	3263	4635	2933	1.58900	6170	0909	49
12	3288	4619	2973	1.58797	6200	0880	48
13	3312	4604	3014	1.58695	6229	0851	47
14	3337	4588	3055	1.58593	6258	0822	46
15	3361	4573	3095	1.58490	6287	0793	45
16	3386	4557	3136	1.58388	6316	0764	44
17	3411	4542	3177	1.58286	6345	0735	43
18	3435	4526	3217	1.58184	6374	0705	42
19	3460	4511	3258	1.58083	6403	0676	41
20	3484	4495	3299	1.57981	6432	0647	40
21	3509	4480	3340	1.57879	6461	0618	39
22	3534	4464	3380	1.57778	6490	0589	38
23	3558	4448	3421	1.57676	6520	0560	37
24	3583	4433	3462	1.57575	6549	0531	36
25	3607	4417	3503	1.57474	6578	0502	35
26	3632	4402	3544	1.57372	6607	0473	34
27	3656	4386	3584	1.57271	6636	0444	33
28	3681	4370	3625	1.57170	6665	0415	32
29	3705	4355	3666	1.57069	6694	0385	31
30	3730	4339	3707	1.56969	6723	0356	30
	0.5	0.8	0.6		0.5	1.0	
31	3754	4324	3748	1.56868	6752	0327	29
32	3779	4308	3789	1.56767	6781	0298	28
33	3804	4292	3830	1.56667	6810	0269	27
34	3828	4277	3871	1.56566	6840	0240	26
35	3854	4261	3912	1.56466	6869	0211	25
36	3877	4245	3953	1.56366	6898	0182	24
37	3902	4230	3994	1.56265	6927	0153	23
38	3926	4214	4035	1.56165	6956	0124	22
39	3951	4198	4076	1.56065	6985	0095	21
40	3975	4182	4117	1.55966	7014	0065	20
41	4000	4167	4158	1.55866	7043	0036	19
42	4024	4151	4199	1.55766	7072	0007	18
						0.9	
43	4049	4135	4240	1.55666	7101	9978	17
44	4073	4120	4281	1.55567	7130	9949	16
45	4097	4104	4322	1.55467	7159	9920	15
46	4122	4088	4363	1.55368	7189	9891	14
47	4146	4072	4404	1.55269	7218	9862	13
48	4171	4057	4446	1.55170	7247	9833	12
49	4195	4041	4487	1.55071	7276	9804	11
50	4220	4025	4528	1.54972	7305	9775	10
51	4244	4009	4569	1.54873	7334	9746	9
52	4269	3994	4610	1.54774	7363	9716	8
53	4293	3978	4652	1.54675	7392	9687	7
54	4317	3962	4693	1.54576	7421	9658	6
55	4342	3946	4734	1.54478	7450	9629	5
56	4366	3930	4775	1.54379	7479	9600	4
57	4391	3915	4817	1.54281	7509	9571	3
58	4415	3899	4858	1.54183	7538	9542	2
59	4439	3883	4899	1.54085	7567	9513	1
60	4464	3867	4941	1.53986	7596	9484	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **122°** = 7320′ **57°** = 3420′

33° = 1980′ Sup. **146°** = 8760′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.5	0.8	0.6		0.5	0.9	
0	4464	3867	4941	1.53986	7596	9484	60
1	4488	3851	4982	1.53888	7625	9455	59
2	4513	3835	5023	1.53791	7654	9426	58
3	4537	3819	5065	1.53693	7683	9396	57
4	4561	3804	5106	1.53595	7712	9367	56
5	4586	3788	5148	1.53497	7741	9338	55
6	4610	3772	5189	1.53400	7770	9309	54
7	4635	3756	5231	1.53302	7799	9280	53
8	4659	3740	5272	1.53205	7829	9251	52
9	4683	3724	5314	1.53107	7858	9222	51
10	4708	3708	5355	1.53010	7887	9193	50
11	4732	3692	5397	1.52913	7916	9164	49
12	4756	3676	5438	1.52816	7945	9135	48
13	4781	3660	5480	1.52719	7974	9106	47
14	4805	3645	5521	1.52622	8003	9076	46
15	4829	3629	5563	1.52525	8032	9047	45
16	4854	3613	5604	1.52429	8061	9018	44
17	4878	3597	5646	1.52332	8090	8989	43
18	4902	3581	5688	1.52235	8119	8960	42
19	4927	3565	5729	1.52139	8149	8931	41
20	4951	3549	5771	1.52043	8178	8902	40
21	4975	3533	5813	1.51946	8207	8873	39
22	4999	3517	5854	1.51850	8236	8844	38
23	5024	3501	5896	1.51754	8265	8815	37
24	5048	3485	5938	1.51658	8294	8786	36
25	5072	3469	5980	1.51562	8323	8756	35
26	5097	3453	6021	1.51466	8352	8727	34
27	5121	3437	6063	1.51370	8381	8698	33
28	5145	3421	6105	1.51275	8410	8669	32
29	5169	3405	6147	1.51179	8439	8640	31
30	5194	3389	6189	1.51084	8468	8611	30
	0.5	0.8	0.6		0.5	0.9	
31	5218	3373	6230	1.50988	8498	8582	29
32	5242	3356	6272	1.50893	8527	8553	28
33	5266	3340	6314	1.50797	8556	8524	27
34	5291	3324	6356	1.50702	8585	8495	26
35	5315	3308	6398	1.50607	8614	8466	25
36	5339	3292	6440	1.50512	8643	8437	24
37	5363	3276	6482	1.50417	8672	8407	23
38	5388	3260	6524	1.50322	8701	8378	22
39	5412	3244	6566	1.50228	8730	8349	21
40	5436	3228	6608	1.50133	8759	8320	20
41	5460	3212	6650	1.50038	8788	8291	19
42	5484	3195	6692	1.49944	8818	8262	18
43	5509	3179	6734	1.49849	8847	8233	17
44	5533	3163	6776	1.49755	8876	8204	16
45	5557	3147	6818	1.49661	8906	8175	15
46	5581	3131	6860	1.49566	8934	8146	14
47	5605	3115	6902	1.49472	8963	8117	13
48	5630	3098	6944	1.49378	8992	8087	12
49	5654	3082	6986	1.49284	9021	8058	11
50	5678	3066	7028	1.49190	9050	8029	10
51	5702	3050	7071	1.49097	9079	8000	9
52	5726	3034	7113	1.49003	9108	7971	8
53	5750	3017	7155	1.48909	9138	7942	7
54	5775	3001	7197	1.48816	9167	7913	6
55	5799	2985	7239	1.48722	9196	7884	5
56	5823	2969	7282	1.48629	9225	7855	4
57	5847	2953	7324	1.48536	9254	7826	3
58	5871	2936	7366	1.48442	9283	7797	2
59	5895	2920	7409	1.48349	9312	7767	1
60	5919	2904	7451	1.48256	9341	7738	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **123°** = 7380′ **56°** = 3360′

34° = 2040′ Sup. **145°** = 8700′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.5	0.8	0.6		0.5	0.9	
0	5919	2904	7451	1.48256	9341	7738	60
1	5943	2887	7493	1.48163	9370	7709	59
2	5968	2871	7536	1.48070	9399	7680	58
3	5992	2855	7578	1.47977	9428	7651	57
4	6016	2839	7620	1.47885	9458	7622	56
5	6040	2822	7663	1.47792	9487	7593	55
6	6064	2806	7705	1.47699	9516	7564	54
7	6088	2790	7748	1.47607	9545	7535	53
8	6112	2773	7790	1.47514	9574	7506	52
9	6136	2757	7832	1.47422	9603	7477	51
10	6160	2741	7875	1.47330	9632	7448	50
11	6184	2724	7917	1.47238	9661	7418	49
12	6208	2708	7960	1.47146	9690	7389	48
13	6232	2692	8002	1.47054	9719	7360	47
14	6256	2675	8045	1.46962	9748	7331	46
15	6280	2659	8088	1.46870	9777	7302	45
16	6305	2643	8130	1.46778	9807	7273	44
17	6329	2626	8173	1.46686	9836	7244	43
18	6353	2610	8215	1.46595	9865	7215	42
19	6377	2593	8258	1.46503	9894	7186	41
20	6401	2577	8301	1.46411	9923	7157	40
21	6425	2561	8343	1.46320	9952	7128	39
22	6449	2544	8386	1.46229	9981	7098	38
					0.6		
23	6473	2528	8429	1.46137	0010	7069	37
24	6497	2511	8471	1.46046	0039	7040	36
25	6521	2495	8514	1.45955	0068	7011	35
26	6545	2478	8557	1.45864	0097	6982	34
27	6569	2462	8599	1.45773	0127	6953	33
28	6593	2446	8642	1.45682	0156	6924	32
29	6617	2429	8685	1.45592	0185	6895	31
30	6641	2413	8728	1.45501	0214	6866	30
	0.5	0.8	0.6		0.6	0.9	
31	6665	2396	8771	1.45410	0243	6837	29
32	6689	2380	8814	1.45320	0272	6808	28
33	6713	2363	8857	1.45229	0301	6778	27
34	6736	2347	8900	1.45139	0330	6749	26
35	6760	2330	8942	1.45048	0359	6720	25
36	6784	2314	8985	1.44958	0388	6691	24
37	6808	2297	9028	1.44868	0417	6662	23
38	6832	2281	9071	1.44778	0447	6633	22
39	6856	2264	9114	1.44688	0476	6604	21
40	6889	2248	9157	1.44598	0505	6575	20
41	6904	2231	9200	1.44508	0534	6546	19
42	6928	2214	9243	1.44418	0563	6517	18
43	6952	2198	9286	1.44329	0592	6488	17
44	6976	2181	9329	1.44239	0621	6458	16
45	7000	2165	9372	1.44149	0650	6429	15
46	7024	2148	9416	1.44060	0679	6400	14
47	7047	2132	9459	1.43970	0708	6371	13
48	7071	2115	9502	1.43881	0737	6342	12
49	7095	2098	9545	1.43792	0766	6313	11
50	7119	2082	9588	1.43703	0796	6284	10
51	7143	2065	9631	1.43614	0825	6255	9
52	7167	2048	9675	1.43525	0854	6226	8
53	7191	2032	9718	1.43436	0883	6197	7
54	7215	2015	9761	1.43347	0912	6168	6
55	7238	1999	9804	1.43258	0941	6138	5
56	7262	1982	9847	1.43169	0970	6109	4
57	7286	1965	9891	1.43080	0999	6080	3
58	7310	1949	9934	1.42992	1028	6051	2
59	7334	1932	9977	1.42903	1057	6022	1
			0.7				
60	7358	1915	0021	1.42815	1086	5993	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **124°** = 7440′ **55°** = 3300′

35° = 2100′ Sup. **144°** = 8640′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.5	0.8	0.7		0.6	0.9	
0	7358	1915	0021	1.42815	1086	5993	60
1	7381	1899	0064	1.42726	1116	5964	59
2	7405	1882	0107	1.42638	1145	5935	58
3	7429	1865	0151	1.42550	1174	5906	57
4	7453	1848	0194	1.42462	1203	5877	56
5	7477	1832	0238	1.42374	1232	5848	55
6	7501	1815	0281	1.42286	1261	5819	54
7	7524	1798	0325	1.42198	1290	5789	53
8	7548	1781	0368	1.42110	1319	5760	52
9	7572	1765	0412	1.42022	1348	5731	51
10	7596	1748	0455	1.41934	1377	5702	50
11	7619	1731	0499	1.41847	1406	5673	49
12	7643	1714	0542	1.41759	1436	5644	48
13	7667	1698	0586	1.41672	1465	5615	47
14	7691	1681	0629	1.41584	1494	5586	46
15	7715	1664	0673	1.41497	1523	5557	45
16	7738	1647	0717	1.41409	1552	5528	44
17	7762	1631	0760	1.41322	1581	5499	43
18	7786	1614	0804	1.41235	1610	5469	42
19	7809	1597	0848	1.41148	1639	5440	41
20	7833	1580	0891	1.41061	1668	5411	40
21	7857	1563	0935	1.40974	1697	5382	39
22	7881	1546	0979	1.40887	1726	5353	38
23	7904	1530	1023	1.40800	1756	5324	37
24	7928	1513	1066	1.40714	1785	5295	36
25	7952	1496	1110	1.40627	1814	5266	35
26	7976	1479	1154	1.40540	1843	5237	34
27	7999	1462	1198	1.40454	1872	5208	33
28	8023	1445	1242	1.40367	1901	5179	32
29	8047	1428	1285	1.40281	1930	5149	31
30	8070	1412	1329	1.40195	1959	5120	30
	0.5	0.8	0.7		0.6	0.9	
31	8094	1395	1373	1.40109	1988	5091	29
32	8118	1378	1417	1.40022	2017	5062	28
33	8141	1361	1461	1.39936	2046	5033	27
34	8165	1344	1505	1.39850	2075	5004	26
35	8189	1327	1549	1.39764	2105	4975	25
36	8212	1310	1593	1.39679	2134	4946	24
37	8236	1293	1637	1.39593	2163	4917	23
38	8260	1276	1681	1.39507	2192	4888	22
39	8283	1259	1725	1.39421	2221	4859	21
40	8307	1242	1769	1.39336	2250	4830	20
41	8330	1225	1813	1.39250	2279	4800	19
42	8354	1208	1857	1.39165	2308	4771	18
43	8378	1191	1901	1.39079	2337	4742	17
44	8401	1174	1946	1.38994	2366	4713	16
45	8425	1157	1990	1.38909	2395	4684	15
46	8449	1140	2034	1.38824	2425	4655	14
47	8472	1123	2078	1.38738	2454	4626	13
48	8496	1106	2122	1.38653	2483	4597	12
49	8519	1089	2166	1.38568	2512	4568	11
50	8543	1072	2211	1.38484	2541	4539	10
51	8567	1055	2255	1.38399	2570	4510	9
52	8590	1038	2299	1.38314	2599	4480	8
53	8614	1021	2344	1.38229	2628	4451	7
54	8637	1004	2388	1.38145	2657	4422	6
55	8661	0987	2432	1.38060	2686	4393	5
56	8684	0970	2477	1.37976	2715	4364	4
57	8708	0953	2521	1.37891	2745	4335	3
58	8731	0936	2565	1.37807	2774	4306	2
59	8755	0919	2610	1.37722	2803	4277	1
60	8779	0902	2654	1.37638	2832	4248	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **125°** = 7500′ **54°** = 3240′

36° = 2160′ Sup. **143°** = 8580′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.5	**0.8**	**0.7**		**0.6**	**0.9**	
0	8779	0902	2654	1.37638	2832	4248	60
1	8802	0885	2699	1.37554	2861	4219	59
2	8826	0867	2743	1.37470	2890	4190	58
3	8849	0850	2788	1.37386	2919	4160	57
4	8873	0833	2832	1.37302	2948	4131	56
5	8896	0816	2877	1.37218	2977	4102	55
6	8920	0799	2921	1.37134	3006	4073	54
7	8943	0782	2966	1.37050	3035	4044	53
8	8967	0765	3010	1.36967	3065	4015	52
9	8990	0748	3055	1.36883	3094	3986	51
10	9014	0730	3100	1.36800	3123	3957	50
11	9037	0713	3144	1.36716	3152	3928	49
12	9061	0696	3189	1.36633	3181	3899	48
13	9084	0679	3234	1.36549	3210	3870	47
14	9108	0662	3278	1.36466	3239	3840	46
15	9131	0644	3323	1.36383	3268	3811	45
16	9154	0627	3368	1.36300	3297	3782	44
17	9178	0610	3413	1.36217	3326	3753	43
18	9201	0593	3457	1.36133	3355	3724	42
19	9225	0576	3502	1.36051	3384	3695	41
20	9248	0558	3547	1.35968	3414	3666	40
21	9272	0541	3592	1.35885	3443	3637	39
22	9295	0524	3637	1.35802	3472	3608	38
23	9318	0507	3681	1.35719	3501	3579	37
24	9342	0489	3726	1.35637	3530	3550	36
25	9365	0472	3771	1.35554	3559	3521	35
26	9389	0455	3816	1.35472	3588	3491	34
27	9412	0438	3861	1.35389	3617	3462	33
28	9435	0420	3906	1.35307	3646	3433	32
29	9459	0403	3951	1.35224	3675	3404	31
30	9482	0386	3996	1.35142	3704	3375	30
	0.5	**0.8**	**0.7**		**0.6**	**0.9**	
31	9506	0368	4041	1.35060	3734	3346	29
32	9529	0351	4086	1.34978	3763	3317	28
33	9552	0334	4131	1.34896	3792	3288	27
34	9576	0316	4176	1.34814	3821	3259	26
35	9599	0299	4221	1.34732	3850	3230	25
36	9622	0282	4267	1.34650	3879	3201	24
37	9646	0264	4312	1.34568	3908	3171	23
38	9669	0247	4357	1.34487	3937	3142	22
39	9693	0230	4402	1.34405	3966	3113	21
40	9716	0212	4447	1.34323	3995	3084	20
41	9739	0195	4492	1.34242	4024	3055	19
42	9763	0178	4538	1.34160	4054	3026	18
43	9786	0160	4583	1.34079	4083	2997	17
44	9809	0143	4628	1.33998	4112	2968	16
45	9832	0125	4674	1.33916	4141	2939	15
46	9856	0108	4719	1.33835	4170	2910	14
47	9879	0091	4764	1.33754	4199	2881	13
48	9902	0073	4810	1.33673	4228	2851	12
49	9926	0056	4855	1.33592	4257	2822	11
50	9949	0038	4900	1.33511	4286	2793	10
51	9972	0021	4946	1.33430	4315	2764	9
52	9995	0003	4991	1.33349	4344	2735	8
	0.6	**0.7**					
53	0019	9986	5037	1.33268	4374	2706	7
54	0042	9968	5082	1.33187	4403	2677	6
55	0065	9951	5128	1.33107	4432	2648	5
56	0089	9934	5173	1.33026	4461	2619	4
57	0112	9916	5219	1.32050	4490	2590	3
58	0135	9899	5264	1.32865	4519	2561	2
59	0158	9881	5310	1.32785	4548	2531	1
60	0181	9864	5355	1.32704	4577	2502	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **126°** = 7560′ **53°** = 3180′

37° = 2220′ Sup. **142°** = 8520′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.6	**0.7**	**0.7**		**0.6**	**0.9**	
0	0181	9864	5355	1.32704	4577	2502	60
1	0205	9846	5401	1.32624	4606	2473	59
2	0228	9829	5447	1.32544	4635	2444	58
3	0251	9811	5492	1.32464	4664	2415	57
4	0274	9793	5538	1.32384	4693	2386	56
5	0298	9776	5584	1.32304	4723	2357	55
6	0321	9758	5629	1.32224	4752	2328	54
7	0344	9741	5675	1.32144	4781	2299	53
8	0367	9723	5721	1.32064	4810	2270	52
9	0390	9706	5767	1.31984	4839	2241	51
10	0414	9688	5812	1.31904	4868	2212	50
11	0437	9671	5858	1.31825	4897	2182	49
12	0460	9653	5904	1.31745	4926	2153	48
13	0483	9635	5950	1.31666	4955	2124	47
14	0506	9618	5996	1.31586	4984	2095	46
15	0529	9600	6042	1.31507	5013	2066	45
16	0553	9583	6088	1.31427	5043	2037	44
17	0576	9565	6134	1.31348	5072	2008	43
18	0599	9547	6180	1.31269	5101	1979	42
19	0622	9530	6226	1.31190	5130	1950	41
20	0645	9512	6272	1.31110	5159	1921	40
21	0668	9494	6318	1.31031	5188	1892	39
22	0691	9477	6364	1.30952	5217	1862	38
23	0714	9459	6410	1.30873	5246	1833	37
24	0738	9441	6456	1.30795	5275	1804	36
25	0761	9424	6502	1.30716	5304	1775	35
26	0784	9406	6548	1.30637	5333	1746	34
27	0807	9388	6594	1.30558	5363	1717	33
28	0830	9371	6640	1.30480	5392	1688	32
29	0853	9353	6686	1.30401	5421	1659	31
30	0876	9335	6733	1.30323	5450	1630	30
	0.6	**0.7**	**0.7**		**0.6**	**0.9**	
31	0899	9318	6779	1.30244	5479	1601	29
32	0922	9300	6825	1.30166	5508	1572	28
33	0945	9282	6871	1.30087	5537	1542	27
34	0968	9264	6918	1.30009	5566	1513	26
35	0991	9247	6964	1.29931	5595	1484	25
36	1015	9229	7010	1.29853	5624	1455	24
37	1038	9211	7057	1.29775	5653	1426	23
38	1061	9193	7103	1.29696	5683	1397	22
39	1084	9176	7149	1.29618	5712	1368	21
40	1107	9158	7196	1.29541	5741	1339	20
41	1130	9140	7242	1.29463	5770	1310	19
42	1153	9122	7289	1.29385	5799	1281	18
43	1176	9105	7335	1.29307	5828	1252	17
44	1199	9087	7382	1.29229	5857	1222	16
45	1222	9069	7428	1.29152	5886	1193	15
46	1245	9051	7475	1.29074	5915	1164	14
47	1268	9033	7521	1.28997	5944	1135	13
48	1291	9015	7568	1.28919	5973	1106	12
49	1314	8998	7615	1.28842	6002	1077	11
50	1337	8980	7661	1.28764	6032	1048	10
51	1360	8962	7708	1.28687	6061	1019	9
52	1383	8944	7754	1.28610	6090	0990	8
53	1406	8926	7801	1.28533	6119	0961	7
54	1429	8908	7848	1.28456	6148	0932	6
55	1451	8891	7895	1.28379	6177	0903	5
56	1474	8873	7941	1.28302	6206	0873	4
57	1497	8855	7988	1.28225	6235	0844	3
58	1520	8837	8035	1.28148	6264	0815	2
59	1543	8819	8082	1.28071	6293	0786	1
60	1566	8801	8129	1.27994	6322	0757	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **127°** = 7620′ **52°** = 3120

38° = 2280′ Sup. **141°** = 8460′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.6	**0.7**	**0.7**		**0.6**	**0.9**	
0	1566	8801	8129	1.27994	6322	0757	60
1	1589	8783	8175	1.27917	6352	0728	59
2	1612	8765	8222	1.27841	6381	0699	58
3	1635	8747	8269	1.27764	6410	0670	57
4	1658	8729	8316	1.27688	6439	0641	56
5	1681	8711	8363	1.27611	6468	0612	55
6	1704	8693	8410	1.27535	6497	0583	54
7	1726	8676	8457	1.27458	6526	0553	53
8	1749	8658	8504	1.27382	6555	0524	52
9	1772	8640	8551	1.27306	6584	0495	51
10	1795	8622	8599	1.27230	6613	0466	50
11	1818	8604	8645	1.27153	6642	0437	49
12	1841	8586	8692	1.27077	6672	0408	48
13	1864	8568	8739	1.27001	6701	0379	47
14	1887	8550	8786	1.26925	6730	0350	46
15	1909	8532	8834	1.26849	6759	0321	15
16	1932	8514	8881	1.26774	6788	0292	44
17	1955	8496	8928	1.26698	6817	0263	43
18	1978	8478	8975	1.26622	6846	0233	42
19	2001	8460	9022	1.26546	6875	0204	41
20	2024	8442	9070	1.26471	6904	0175	40
21	2046	8424	9117	1.26395	6933	0146	39
22	2069	8405	9164	1.26319	6962	0117	38
23	2092	8387	9212	1.26244	6992	0088	37
24	2115	8369	9259	1.26169	7021	0059	36
25	2138	8351	9306	1.26093	7050	0030	35
26	2160	8333	9354	1.26018	7079	0001	34
						0.8	
27	2183	8315	9401	1.25943	7108	9972	33
28	2206	8297	9449	1.25867	7137	9943	32
29	2229	8279	9496	1.25792	7166	9913	31
30	2251	8261	9544	1.25717	7195	9884	30
	0.6	**0.7**	**0.7**		**0.6**	**0.8**	
31	2274	8243	9591	1.25642	7224	9855	29
32	2297	8225	9639	1.25567	7253	9826	28
33	2320	8206	9686	1.25492	7282	9797	27
34	2342	8188	9734	1.25417	7311	9768	26
35	2365	8170	9781	1.25343	7341	9739	25
36	2388	8152	9829	1.25268	7370	9710	24
37	2411	8134	9877	1.25193	7399	9681	23
38	2433	8116	9924	1.25118	7428	9652	22
39	2456	8098	9972	1.25044	7457	9623	21
			0.8				
40	2479	8079	0020	1.24969	7486	9594	20
41	2502	8061	0067	1.24895	7515	9564	19
42	2524	8043	0115	1.24820	7544	9535	18
43	2547	8025	0163	1.24746	7573	9506	17
44	2570	8007	0211	1.24672	7602	9477	16
45	2592	7988	0258	1.24597	7631	9448	15
46	2615	7970	0306	1.24523	7661	9419	14
47	2638	7952	0354	1.24449	7690	9390	13
48	2660	7934	0402	1.24375	7719	9361	12
49	2683	7916	0450	1.24301	7748	9332	11
50	2706	7897	0498	1.24227	7777	9303	10
51	2728	7879	0546	1.24153	7806	9274	9
52	2751	7861	0594	1.24079	7835	9244	8
53	2774	7843	0642	1.24005	7864	9215	7
54	2796	7824	0690	1.23931	7893	9186	6
55	2819	7806	0738	1.23858	7922	9157	5
56	2842	7788	0786	1.23784	7951	9128	4
57	2864	7769	0834	1.23710	7981	9099	3
58	2887	7751	0882	1.23637	8010	9070	2
59	2909	7733	0930	1.23563	8039	9041	1
60	2932	7715	0978	1.23490	8068	9012	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **128°** = 7680′ **51°** = 3060′

39° = 2340′ Sup. **140°** = 8400′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.6	**0.7**	**0.8**		**0.6**	**0.8**	
0	2932	7715	0978	1.23490	8068	9012	60
1	2955	7696	1027	1.23416	8097	8983	59
2	2977	7678	1075	1.23343	8126	8954	58
3	3000	7660	1123	1.23270	8155	8924	57
4	3022	7641	1171	1.23196	8184	8895	56
5	3045	7623	1220	1.23123	8213	8866	55
6	3068	7605	1268	1.23050	8242	8837	54
7	3090	7586	1316	1.22977	8271	8808	53
8	3113	7568	1364	1.22904	8301	8779	52
9	3135	7550	1413	1.22831	8330	8750	51
10	3158	7531	1461	1.22758	8359	8721	50
11	3180	7513	1510	1.22685	8388	8692	49
12	3203	7494	1558	1.22612	8417	8663	48
13	3225	7476	1606	1.22539	8446	8634	47
14	3248	7458	1655	1.22467	8475	8604	46
15	3271	7439	1703	1.22394	8504	8575	45
16	3293	7421	1752	1.22321	8533	8546	44
17	3316	7402	1800	1.22249	8562	8517	43
18	3338	7384	1849	1.22176	8591	8488	42
19	3361	7366	1898	1.22104	8620	8459	41
20	3383	7347	1946	1.22031	8650	8430	40
21	3406	7329	1995	1.21959	8679	8401	39
22	3428	7310	2044	1.21886	8708	8372	38
23	3451	7292	2092	1.21814	8737	8343	37
24	3473	7273	2141	1.21742	8766	8314	36
25	3496	7255	2190	1.21670	8795	8285	35
26	3518	7236	2238	1.21598	8824	8255	34
27	3540	7218	2287	1.21526	8853	8226	33
26	3563	7199	2336	1.21454	8882	8197	32
29	3585	7181	2385	1.21382	8911	8168	31
30	3608	7162	2434	1.21310	8940	8139	30
	0.6	**0.7**	**0.8**		**0.6**	**0.8**	
31	3630	7144	2483	1.21238	8970	8110	29
32	3653	7125	2531	1.21166	8999	8081	28
33	3675	7107	2580	1.21094	9028	8052	27
34	3698	7088	2629	1.21023	9057	8023	26
35	3720	7070	2678	1.20951	9086	7994	25
36	3742	7051	2727	1.20879	9115	7965	24
37	3765	7033	2776	1.20808	9144	7935	23
38	3787	7014	2825	1.20736	9173	7906	22
39	3810	6996	2874	1.20665	9202	7877	21
40	3832	6977	2923	1.20593	9231	7848	20
41	3854	6959	2972	1.20522	9260	7819	19
42	3877	6940	3022	1.20451	9290	7790	18
43	3899	6921	3071	1.20379	9319	7761	17
44	3922	6903	3120	1.20308	9348	7732	16
45	3944	6884	3169	1.20237	9377	7703	15
46	3966	6866	3218	1.20166	9406	7674	14
47	3989	6847	3268	1.20095	9435	7645	13
48	4011	6828	3317	1.20024	9464	7615	12
49	4033	6810	3366	1.19953	9493	7586	11
50	4056	6791	3415	1.19882	9522	7557	10
51	4078	6772	3465	1.19811	9551	7528	9
52	4100	6754	3514	1.19740	9580	7499	8
53	4123	6735	3564	1.19669	9609	7470	7
52	4145	6717	3613	1.19599	9639	7441	6
55	4167	6698	3662	1.19528	9668	7412	5
56	4190	6679	3712	1.19457	9697	7383	4
57	4212	6661	3761	1.19387	9726	7354	3
58	4234	6642	3811	1.19316	9755	7325	2
59	4256	6623	3860	1.19246	9784	7296	1
60	4279	6604	3910	1.19175	9813	7266	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **129** = 7740′ **50°** = 3000′

40° = 2400′ Sup. 139° = 8340′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.6	0.7	0.8		0.6	0.8	
0	4279	6604	3910	1.19175	9813	7266	60
1	4301	6586	3960	1.19105	9842	7237	59
2	4323	6567	4009	1.19035	9871	7208	58
3	4346	6548	4059	1.18964	9900	7179	57
4	4368	6530	4108	1.18894	9929	7150	56
5	4390	6511	4158	1.18824	9959	7121	55
6	4412	6492	4208	1.18754	9988	7092	54
					0.7		
7	4435	6473	4258	1.18684	0017	7063	53
8	4457	6455	4307	1.18614	0046	7034	52
9	4479	6436	4357	1.18544	0075	7005	51
10	4501	6417	4407	1.18474	0104	6976	50
11	4524	6398	4457	1.18404	0133	6946	49
12	4546	6380	4507	1.18334	0162	6917	48
13	4568	6361	4556	1.18264	0191	6888	47
14	4590	6342	4606	1.18194	0220	6859	46
15	4612	6323	4656	1.18125	0249	6830	45
16	4635	6304	4706	1.18055	0279	6801	44
17	4657	6286	4756	1.17986	0308	6772	43
18	4679	6267	4806	1.17916	0337	6743	42
19	4701	6248	4856	1.17846	0366	6714	41
20	4723	6229	4906	1.17777	0395	6685	40
21	4746	6210	4956	1.17708	0424	6656	39
22	4768	6192	5006	1.17638	0453	6626	38
23	4790	6173	5057	1.17569	0482	6597	37
24	4812	6154	5107	1.17500	0511	6568	36
25	4834	6135	5157	1.17430	0540	6539	35
26	4856	6116	5207	1.17361	0569	6510	34
27	4878	6097	5257	1.17292	0599	6481	33
28	4901	6078	5307	1.17223	0628	6452	32
29	4923	6059	5358	1.17154	0657	6423	31
30	4945	6041	5408	1.17085	0686	6394	30
	0.6	0.7	0.8		0.7	0.8	
31	4967	6022	5458	1.17016	0715	6365	29
32	4989	6003	5509	1.16947	0744	6336	28
33	5011	5984	5559	1.16878	0773	6306	27
34	5033	5965	5609	1.16809	0802	6277	26
35	5055	5946	5660	1.16741	0831	6248	25
36	5077	5927	5710	1.16672	0860	6219	24
37	5099	5908	5761	1.16603	0889	6190	23
38	5122	5889	5811	1.16535	0918	6161	22
39	5144	5870	5862	1.16466	0948	6132	21
40	5166	5851	5912	1.16398	0977	6103	20
41	5188	5832	5963	1.16329	1006	6074	19
42	5210	5813	6014	1.16261	1035	6045	18
43	5232	5794	6064	1.16192	1064	6016	17
44	5254	5775	6115	1.16124	1093	5987	16
45	5276	5756	6166	1.16056	1122	5957	15
46	5298	5738	6216	1.15987	1151	5928	14
47	5320	5719	6267	1.15919	1180	5899	13
48	5342	5699	6318	1.15851	1209	5870	12
49	5364	5680	6368	1.15783	1238	5841	11
50	5386	5661	6419	1.15715	1268	5812	10
51	5408	5642	6470	1.15647	1297	5783	9
52	5430	5623	6521	1.15579	1326	5754	8
53	5452	5604	6572	1.15511	1355	5725	7
54	5474	5585	6623	1.15443	1384	5696	6
55	5496	5566	6674	1.15375	1413	5667	5
56	5518	5547	6725	1.15308	1442	5637	4
57	5540	5528	6776	1.15240	1471	5608	3
58	5562	5509	6827	1.15172	1500	5579	2
59	5584	5490	6878	1.15104	1529	5550	1
60	5606	5471	6929	1.15037	1558	5521	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. 130° = 7800′ 49° = 2940′

41° = 2460′ Sup. 138° = 8280′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.6	0.7	0.8		0.7	0.8	
0	5606	5471	6929	1.15037	1558	5521	60
1	5628	5452	6980	1.14970	1588	5492	59
2	5650	5433	7031	1.14902	1617	5463	58
3	5672	5414	7082	1.14834	1646	5434	57
4	5694	5395	7133	1.14767	1675	5405	56
5	5716	5375	7184	1.14699	1704	5376	55
6	5738	5356	7236	1.14632	1733	5347	54
7	5759	5337	7287	1.14565	1762	5317	53
8	5781	5318	7338	1.14498	1791	5288	52
9	5803	5299	7389	1.14430	1820	5259	51
10	5825	5280	7441	1.14363	1849	5230	50
11	5847	5261	7492	1.14296	1878	5201	49
12	5869	5241	7543	1.14229	1908	5172	48
13	5891	5222	7595	1.14162	1937	5143	47
14	5913	5203	7646	1.14095	1966	5114	46
15	5935	5184	7698	1.14028	1995	5085	45
16	5956	5165	7749	1.13961	2024	5056	44
17	5978	5146	7801	1.13894	2053	5027	43
18	6000	5126	7852	1.13828	2082	4997	42
19	6022	5107	7904	1.13761	2111	4968	41
20	6044	5088	7955	1.13694	2140	4939	40
21	6066	5069	8007	1.13627	2169	4910	39
22	6088	5050	8059	1.13561	2198	4881	38
23	6109	5030	8110	1.13494	2227	4852	37
24	6131	5011	8162	1.13428	2257	4823	36
25	6153	4992	8214	1.13361	2286	4794	35
26	6175	4973	8265	1.13295	2315	4765	34
27	6197	4953	8317	1.13228	2344	4736	33
28	6218	4934	8369	1.13162	2373	4707	32
29	6240	4915	8421	1.13096	2402	4678	31
30	6262	4896	8473	1.13029	2431	4648	30
	0.6	0.7	0.8		0.7	0.8	
31	6284	4876	8524	1.12963	2460	4619	29
32	6306	4857	8576	1.12897	2489	4590	28
33	6327	4838	8628	1.12831	2518	4561	27
34	6349	4818	8680	1.12765	2547	4532	26
35	6371	4799	8732	1.12699	2577	4503	25
36	6393	4780	8784	1.12633	2606	4474	24
37	6414	4760	8836	1.12567	2635	4445	23
38	6436	4741	8888	1.12501	2664	4416	22
39	6458	4722	8940	1.12435	2693	4387	21
40	6480	4703	8992	1.12369	2722	4358	20
41	6501	4683	9045	1.12303	2751	4328	19
42	6523	4664	9097	1.12238	2780	4299	18
43	6545	4644	9149	1.12172	2809	4270	17
44	6566	4625	9201	1.12106	2838	4241	16
45	6588	4606	9253	1.12041	2867	4212	15
46	6610	4586	9306	1.11975	2897	4183	14
47	6632	4567	9358	1.11909	2926	4154	13
48	6653	4548	9410	1.11844	2955	4125	12
49	6675	4528	9463	1.11778	2984	4096	11
50	6697	4509	9515	1.11714	3013	4067	10
51	6718	4490	9567	1.11648	3042	4038	9
52	6740	4470	9620	1.11582	3071	4008	8
53	6762	4451	9672	1.11517	3100	3979	7
54	6783	4431	9725	1.11452	3129	3950	6
55	6805	4412	9777	1.11387	3158	3921	5
56	6827	4392	9830	1.11321	3187	3892	4
57	6848	4373	9883	1.11256	3217	3863	3
58	6870	4353	9935	1.11191	3246	3834	2
59	6891	4334	9988	1.11126	3275	3805	1
			0.9				
60	6913	4314	0040	1.11061	3304	3776	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. 131° = 7860′ 48° = 2880′

42° = 2520′ Sup. **137°** = 8220′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.6	0.7	0.9		0.7	0.8	
0	6913	4314	0040	1.11061	3304	3776	60
1	6935	4295	0093	1.10996	3333	3747	59
2	6956	4276	0146	1.10931	3362	3718	58
3	6978	4256	0199	1.10867	3391	3688	57
4	6999	4237	0251	1.10802	3420	3659	56
5	7021	4217	0304	1.10737	3449	3630	55
6	7043	4198	0357	1.10672	3478	3601	54
7	7064	4178	0410	1.10607	3507	3572	53
8	7086	4159	0463	1.10543	3536	3543	52
9	7107	4139	0516	1.10478	3566	3514	51
10	7129	4120	0569	1.10414	3595	3485	50
11	7151	4100	0621	1.10349	3624	3456	49
12	7172	4080	0674	1.10285	3653	3427	48
13	7194	4061	0727	1.10220	3682	3398	47
14	7215	4041	0781	1.10156	3711	3369	46
15	7237	4022	0834	1.10091	3740	3339	45
16	7258	4002	0887	1.10027	3769	3310	44
17	7280	3983	0940	1.09963	3798	3281	43
18	7301	3963	0993	1.09899	3827	3252	42
19	7323	3944	1046	1.09834	3856	3223	41
20	7344	3924	1099	1.09770	3886	3194	40
21	7366	3904	1153	1.09706	3915	3165	39
22	7387	3885	1206	1.09642	3944	3136	38
23	7409	3865	1259	1.09578	3973	3107	37
24	7430	3846	1313	1.09514	4002	3078	36
25	7452	3826	1366	1.09450	4031	3049	35
26	7473	3806	1419	1.09386	4060	3019	34
27	7495	3787	1473	1.09322	4089	2990	33
28	7516	3767	1526	1.09258	4118	2961	32
29	7538	3747	1580	1.09195	4147	2932	31
30	7559	3728	1633	1.09131	4176	2903	30
	0.6	0.7	0.9		0.7	0.8	
31	7580	3708	1687	1.09067	4206	2874	29
32	7602	3688	1740	1.09003	4235	2845	28
33	7623	3669	1794	1.08940	4264	2816	27
34	7645	3649	1847	1.08876	4293	2787	26
35	7666	3629	1901	1.08813	4322	2758	25
36	7688	3610	1955	1.08749	4351	2729	24
37	7709	3590	2008	1.08686	4380	2699	23
38	7730	3570	2062	1.08622	4409	2670	22
39	7752	3551	2116	1.08559	4438	2641	21
40	7773	3531	2170	1.08496	4467	2612	20
41	7795	3511	2223	1.08432	4496	2583	19
42	7816	3491	2277	1.08369	4526	2554	18
43	7837	3472	2331	1.08306	4555	2525	17
44	7859	3452	2385	1.08243	4584	2496	16
45	7880	3432	2439	1.08179	4613	2467	15
46	7901	3412	2493	1.08116	4642	2438	14
47	7923	3393	2547	1.08053	4671	2409	13
48	7944	3373	2601	1.07990	4700	2379	12
49	7965	3353	2655	1.07927	4729	2350	11
50	7987	3333	2709	1.07864	4758	2321	10
51	8008	3314	2763	1.07801	4787	2292	9
52	8029	3294	2817	1.07738	4816	2263	8
53	8051	3274	2872	1.07676	4845	2234	7
54	8072	3254	2926	1.07613	4875	2205	6
55	8093	3234	2980	1.07550	4904	2176	5
56	8115	3215	3034	1.07487	4933	2147	4
57	8136	3195	3088	1.07425	4962	2118	3
58	8157	3175	3143	1.07362	4991	2089	2
59	8179	3155	3197	1.07299	5020	2060	1
60	8200	3135	3252	1.07237	5049	2030	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **132°** = 7920′ **47°** = 2820′

43° = 2580′ Sup. **136°** = 8160′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.6	0.7	0.9		0.7	0.8	
0	8200	3135	3252	1.07237	5049	2030	60
1	8221	3116	3306	1.07174	5078	2001	59
2	8242	3096	3360	1.07112	5107	1972	58
3	8264	3076	3415	1.07049	5136	1943	57
4	8285	3056	3469	1.06987	5165	1914	56
5	8306	3036	3524	1.06925	5195	1885	55
6	8327	3016	3578	1.06862	5224	1856	54
7	8349	2996	3633	1.06800	5253	1827	53
8	8370	2976	3688	1.06738	5282	1798	52
9	8391	2957	3742	1.06676	5311	1769	51
10	8412	2937	3797	1.06613	5340	1740	50
11	8433	2917	3852	1.06551	5369	1710	49
12	8455	2997	3906	1.06489	5398	1681	48
13	8476	2877	3961	1.06427	5427	1652	47
14	8497	2857	4016	1.06365	5456	1623	46
15	8518	2837	4071	1.06303	5485	1594	45
16	8539	2817	4125	1.06241	5515	1565	44
17	8561	2797	4180	1.06179	5544	1536	43
18	8582	2777	4235	1.06117	5573	1507	42
19	8603	2757	4290	1.06056	5602	1478	41
20	8624	2737	4345	1.05994	5631	1449	40
21	8645	2717	4400	1.05932	5660	1420	39
22	8666	2697	4455	1.05870	5689	1390	38
23	8688	2677	4510	1.05809	5718	1361	37
24	8709	2657	4565	1.05747	5747	1332	36
25	8730	2637	4620	1.05685	5776	1303	35
26	8751	2617	4676	1.05624	5805	1274	34
27	8772	2597	4731	1.05562	5835	1245	33
28	8793	2577	4786	1.05501	5864	1216	32
29	8814	2557	4841	1.05439	5893	1187	31
30	8835	2537	4896	1.05378	5922	1158	30
	0.6	0.7	0.9		0.7	0.8	
31	8857	2517	4952	1.05317	5951	1129	29
32	8878	2497	5007	1.05255	5980	1100	28
33	8899	2477	5062	1.05194	6009	1070	27
34	8920	2457	5118	1.05133	6038	1041	26
35	8941	2437	5173	1.05072	6067	1012	25
36	8962	2417	5229	1.05010	6096	0983	24
37	8983	2397	5284	1.04949	6125	0954	23
38	9004	2377	5340	1.04888	6154	0925	22
39	9025	2357	5395	1.04827	6184	0896	21
40	9046	2337	5451	1.04766	6213	0867	20
41	9067	2317	5506	1.04705	6242	0838	19
42	9088	2297	5562	1.04644	6271	0809	18
43	9109	2277	5618	1.04583	6300	0780	17
44	9130	2257	5673	1.04522	6329	0751	16
45	9151	2236	5729	1.04461	6358	0721	15
46	9172	2216	5785	1.04401	6387	0692	14
47	9193	2196	5841	1.04340	6416	0663	13
48	9214	2176	5897	1.04279	6445	0634	12
49	9235	2156	5952	1.04218	6474	0605	11
50	9256	2136	6008	1.04158	6504	0576	10
51	9277	2116	6064	1.04097	6533	0547	9
52	9298	2095	6120	1.04036	6562	0518	8
53	9319	2075	6176	1.03976	6591	0489	7
54	9340	2055	6232	1.03915	6620	0460	6
55	9361	2035	6288	1.03855	6649	0431	5
56	9382	2015	6344	1.03794	6678	0401	4
57	9403	1995	6400	1.03734	6707	0372	3
58	9424	1974	6457	1.03674	6736	0343	2
59	9445	1954	6513	1.03613	6765	0314	1
60	9466	1934	6569	1.03553	6794	0285	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **133°** = 7980′ **46°** = 2760′

44° = 2640′ Sup. **135°** = 8100′

′	SIN.	COS.	TAN.	COT.	ARC.	COM. OF ARC.	
	0.6	**0.7**	**0.9**		**0.7**	**0.8**	
0	9466	1934	6569	1.03553	6794	0285	60
1	9487	1914	6625	1.03493	6824	0256	59
2	9508	1894	6681	1.03431	6853	0227	58
3	9529	1873	6738	1.03372	6882	0198	57
4	9549	1853	6794	1.03312	6911	0169	56
5	9570	1833	6850	1.03252	6940	0140	55
6	9591	1813	6907	1.03192	6969	0111	54
7	9612	1792	6963	1.03132	6998	0081	53
8	9633	1772	7020	1.03072	7027	0052	52
9	9654	1752	7076	1.03012	7056	0023	51
						0.7	
10	9675	1732	7133	1.02952	7085	9994	50
11	9696	1711	7189	1.02892	7114	9965	49
12	9717	1691	7246	1.02832	7144	9936	48
13	9737	1671	7302	1.02772	7173	9907	47
14	9758	1650	7359	1.02713	7202	9878	46
15	9779	1630	7416	1.02653	7231	9849	45
16	9800	1610	7472	1.02593	7260	9820	44
17	9821	1590	7529	1.02533	7289	9791	43
18	9842	1569	7586	1.02474	7318	9761	42
19	9862	1549	7643	1.02414	7347	9732	41
20	9883	1529	7700	1.02355	7376	9703	40
21	9904	1508	7756	1.02295	7405	9674	39
22	9925	1488	7813	1.02236	7434	9645	38
23	9946	1468	7870	1.02176	7463	9616	37
24	9966	1447	7927	1.02117	7493	9587	36
25	9987	1427	7984	1.02057	7522	9558	35
	0.7						
26	0008	1407	8041	1.01998	7551	9529	34
27	0029	1386	8098	1.01939	7580	9500	33
28	0049	1366	8155	1.01879	7609	9471	32
29	0070	1345	8213	1.01820	7638	9442	31
30	0091	1325	8270	1.01761	7667	9412	30
	0.7	**0.7**	**0.9**		**0.7**	**0.7**	
31	0112	1305	8327	1.01702	7696	9383	29
32	0132	1284	8384	1.01642	7725	9354	28
33	0153	1264	8441	1.01583	7754	9325	27
34	0174	1243	8499	1.01524	7783	9296	26
35	0195	1223	8556	1.01465	7813	9267	25
36	0215	1203	8613	1.01406	7842	9238	24
37	0236	1182	8671	1.01347	7871	9209	23
38	0257	1162	8728	1.01288	7900	9180	22
39	0277	1141	8786	1.01229	7929	9151	21
40	0298	1121	8843	1.01170	7958	9122	20
41	0319	1100	8901	1.01112	7987	9092	19
42	0339	1080	8958	1.01053	8016	9063	18
43	0360	1059	9016	1.00994	8045	9034	17
44	0381	1039	9073	1.00935	8074	9005	16
45	0401	1019	9131	1.00876	8103	8976	15
46	0422	0998	9189	1.00818	8133	8947	14
47	0443	0978	9247	1.00759	8162	8918	13
48	0463	0957	9304	1.00701	8191	8889	12
49	0484	0937	9362	1.00642	8220	8860	11
50	0505	0916	9420	1.00583	8249	8831	10
51	0525	0896	9478	1.00525	8278	8802	9
52	0546	0875	9536	1.00467	8307	8772	8
53	0567	0855	9594	1.00408	8336	8743	7
54	0587	0834	9652	1.00350	8365	8714	6
55	0608	0813	9710	1.00291	8394	8685	5
56	0628	0793	9768	1.00233	8423	8656	4
57	0649	0772	9826	1.00175	8452	8627	3
58	0670	0752	9884	1.00116	8482	8598	2
59	0690	0731	9942	1.00058	8511	8569	1
			1.0				
60	0711	0711	0000	1.00000	8540	8540	0
	COS.	SIN.	COT.	TAN.	COM. OF ARC.	ARC.	′

Sup. **134°** = 8040′ **45°** = 2700′

APPLICATION OF THE EQUATION OF THE THIRD DEGREE AND THE TRIGONOMETRIC SOLUTION OF THE IRREDUCIBLE CASE

1072. Continuing from the point where we left off in (592) from the general equation

$$x^3 + px + q = 0,$$

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}. \qquad (592)$$

CASE 1. *One real and two imaginary roots.*

If the quantity

$$\frac{q^2}{4} + \frac{p^3}{27} > 0,$$

the equation has only one real root of a sign opposite to that of its last term q, and two imaginary roots. Designating the values of the cubic radicals by A and B, the three roots of the equation are:

$$x_1 = A + B \text{ (real)}, \qquad (2)$$

$$\left.\begin{aligned} x_2 &= A\alpha + B\alpha^2 \\ x_3 &= A\alpha^2 + B\alpha \end{aligned}\right\} \text{(imaginary)}. \qquad (3)$$

α is one of the two imaginary cube roots of one, that is,

$$\alpha = \frac{-1 \pm \sqrt{-3}}{2}.$$

EXAMPLE 1. Calculate the radius and altitude of a cylinder inscribed in a sphere, such that the area of its lateral surface is equal to the area of the two zones of one base, which are determined by the cylinder.

Solution. Let R be the radius of the sphere, x the radius of the cylinder, and $2y$ its altitude. Then the lateral surface of the cylinder equals $4\pi xy$ and the surface of each zone $2\pi R(R - y)$, and the equation of the problem is

$$4\pi xy = 4\pi R(R - y),$$

or

$$xy = R(R - y). \qquad (1)$$

The following relation exists between the three quantities R, x, and y:

$$R^2 = x^2 + y^2, \qquad (1022)$$

and

$$x = \sqrt{R^2 - y^2}. \qquad (2)$$

Dividing (1) by (2),

$$y = \frac{R(R-y)}{\sqrt{R^2-y^2}}.$$

Then, $$y^2 = \frac{R^2(R-y)^2}{R^2-y^2} = \frac{R^2(R-y)}{R+y}.$$

Transposing,

$$y^3 + Ry^2 + R^2y - R^3 = 0.$$

Taking $R = 1$, this equation becomes:

$$y^3 + y^2 + y - 1 = 0. \tag{3}$$

The term y^2 may be eliminated by substituting,*

$$y = u - \frac{1}{3}. \tag{4}$$

After the substitution the equation (3) becomes:

$$u^3 + \frac{2u}{3} - \frac{34}{27} = 0. \tag{5}$$

Finally, to eliminate the denominators, write $u = \frac{z}{3}$ in equation (5), which then becomes:

$$z^3 + 6z - 34 = 0. \tag{6}$$

The equations (4) and (6) give:

$$y = \frac{z}{3} - \frac{1}{3} = \frac{z-1}{3}.$$

It remains now to solve equation (6), which, according to the equation of the third degree, gives:

$$z = \sqrt[3]{17 + \sqrt{297}} + \sqrt[3]{17 - \sqrt{297}}.$$

Here the radical of the second degree is real, the equation has one real root and two imaginary ones; it is the first case, as explained above.

* Let the general equation of the third degree be:

$$x^3 + Ax^2 + Bx + C = 0. \tag{1}$$

Write $$x = y + h;$$

then equation (1) becomes:

$$y^3 + y^2(3h + A) + y(3h^2 + 2Ah + B) + h^3 + Ah^2 + Bh + C = 0.$$

The quantity h being indeterminate, we may write,

$$3h + A = 0, \text{ from which } h = -\frac{A}{3}.$$

Substituting this value of h in all the terms of the preceding equation, we get:

$$y^3 + py + q = 0.$$

Solving, $$z = 2.631,$$

and $$y = \frac{z - 1}{3} = 0.5436.$$

The altitude of the cylinder is then

$$2\,y = 1.0872.$$

The equation (2) will give the radius of the cylinder,

$$x = \sqrt{R^2 - y^2} = \sqrt{1 - \overline{0.5436}^2} = 0.8451.$$

The other two roots of the equation (6) are imaginary; they are given by the equations (2) and (3) (see CASE 1, page 445).

REMARK. If the radius of the sphere were R, the preceding solution would give the radius of the cylinder as:

$$x = 0.8451\,R,$$

and the altitude as:

$$2\,y = 1.0872\,R.$$

CASE 2. *Three real roots of which two are equal.*

If the quantity

$$\frac{q^2}{4} + \frac{p^3}{27} = 0,$$

the equation has two equal roots of the same sign as the independent term q, and one root of sign opposite to that of q.

The roots are,

$$x_1 = x_2 = \frac{-3\,q}{2\,p} \text{ (equal roots)},$$

$$x_3 = \frac{3\,q}{p} \text{ (single root)}.$$

REMARK. The absolute value of the last root is double that of the two equal ones.

EXAMPLE. The equation

$$x^3 - 3\,x + 2 = 0,$$

gives the following values:

$$x_1 = x_2 = \frac{-3\,q}{2\,p} = \frac{-3 \times 2}{-2 \times 3} = +1,$$

$$x_3 = \frac{3\,q}{p} = \frac{6}{-3} = -2.$$

CASE 3. *The irreducible case. Three real roots.*

If the quantity

$$\frac{q^2}{4} + \frac{p^3}{27} < 0,$$

the equation has three real roots; but the value of x is composed of the sum of two imaginary quantities, which are calculated by trigonometric formulas, as will be shown below.

The trigonometric solution of the irreducible case of the equation of the third degree.

The equation of the 3d degree being reduced to the form

$$x^3 + px + q = 0, \tag{1}$$

the general value of x is (592),

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

If the sum $\frac{q^2}{4} + \frac{p^3}{27} < 0$, the value of x appears under the form of the sum of two imaginary quantities.

Writing

$$-\frac{q}{2} = \rho \cos \phi \text{ and } \frac{q^2}{4} + \frac{p^3}{27} = -\rho^2 \sin^2 \phi,$$

we have $$\rho = \sqrt{-\frac{p^3}{27}} \text{ and } \cos \phi = \frac{-q}{2\rho}.$$

Then the values of the three roots are:

$$x_1 = 2\sqrt[3]{\rho} \cos \frac{\phi}{3},$$

$$x_2 = -2\sqrt[3]{\rho} \cos\left(60^\circ - \frac{\phi}{3}\right),$$

$$x_3 = +2\sqrt[3]{} \cos\left(120^\circ - \frac{\phi}{3}\right).$$

REMARK. If the last two roots are equal, we have:

$$\phi = 0^\circ.$$

NOTE. If the $\cos \phi = \frac{-q}{2\rho}$ is negative, the angle ϕ' is found which is a supplement of ϕ and has the same cosine with the sign $+$. This angle ϕ' should replace ϕ in the values of the three roots.

EXAMPLE 1. Solve the equation:

$$x^3 + 5x + 1 = 0.$$

Comparing with the general form,

$$x^3 + px + q = 0,$$

we have:
$$\frac{q^2}{4} + \frac{p^3}{27} = \frac{1}{4} - \frac{125}{27} < 0.$$

Thus, the example reduces to the irreducible case, and the formulas given above are to be applied.

$$\rho = \sqrt{\frac{-p^3}{27}} = \sqrt{\frac{125}{27}} \text{ and } \cos\phi = \frac{-q}{2\rho} = \frac{-1}{2\rho}.$$

CALCULATION OF ρ

$$\begin{aligned} \log 125 &= 2.0969100 \\ c^t \log 27 &= 8.5686362 \\ &\quad -10 \\ \hline \log\rho &= \tfrac{1}{2}(0.6655462) \end{aligned}$$

or $\log\rho = 0.3327731$

CALCULATION OF ϕ

$$\begin{aligned} \log 1 &= 0.0000000 \\ c^t \log 2 &= 9.6989700 \\ c^t \log\rho &= 9.6672269 \\ &\quad -20.0000000 \\ \hline \log\cos\phi &= \bar{1}.3661969 \\ \phi &= 76^\circ\, 33'\, 53'' \end{aligned}$$

The value of $\cos\phi$ being negative, ϕ must be replaced by its supplement ϕ', that is,

$$\phi' = 103^\circ\, 26'\, 7'';$$

then
$$\frac{\phi'}{3} = 34^\circ\, 28'\, 42.3'',$$

$$60^\circ - \frac{\phi'}{3} = 25^\circ\, 31'\, 17.7'',$$

$$120^\circ - \frac{\phi'}{3} = 85^\circ\, 31'\, 17.7''.$$

Calculation of the three roots.

CALCULATION OF x_1

$$\begin{aligned} \log 2 &= 0.3010300 \\ \log\sqrt[3]{\rho} &= 0.1109243 \\ \log\cos\frac{\phi'}{3} &= \bar{1}.9161061 \\ \hline \log x_1 &= 0.3280604 \end{aligned}$$

from which
$$x_1 = +\,2.128$$

CALCULATION OF x_2

$$\begin{aligned}
\log 2 &= 0.3010300\\
\log \sqrt[3]{\rho} &= 0.1109245\\
\log \cos\left(60^\circ - \frac{\phi'}{3}\right) &= \bar{1}.9554101\\
\hline
\log(-x_2) &= 0.3673646
\end{aligned}$$

from which $x_2 = -2.330$

CALCULATION OF x_3

$$\begin{aligned}
\log 2 &= 0.3010300\\
\log \sqrt[3]{\rho} &= 0.1109245\\
\log \cos\left(120^\circ - \frac{\phi'}{3}\right) &= \bar{2}.8925602\\
\hline
\log x_3 &= \bar{1}.3045147\\
x_3 &= 0.2016
\end{aligned}$$

NOTE. The calculations being so laborious, it is quite necessary to prove that the roots are correct by substituting their values in the given equation

$$x^3 - 5x + 1 = 0,$$

or in

$$x^3 - 5x + 1 = y,$$

and making sure that two consecutive values which differ by $\frac{1}{1000}$, for example, give two values preceded by unlike signs for the sum y of the terms of the equation.

Proof of $x_1 = 2.128$:

for $x_1 = 2.128$ $y = -0.0036$
for $x_1 = 2.129$ $y = +0.00499$

Proof of $x_2 = -2.330$:

for $x_2 = -2.331$ $y = -0.0010$
for $x_2 = -2.330$ $y = +0.0007$

Proof of $x_3 = 0.2016$:

for $x_3 = 0.201$ $y = +0.0031$
for $x_3 = 2.202$ $y = -0.0018$

We are assured that in taking

$$x_1 = 2.128 \qquad x_2 = -2.330 \qquad x_3 = 0.201$$

these values are correct to 0.001.

EXAMPLE 2. Divide a hemisphere into two equivalent parts by a plane parallel to the base.

SOLUTION. Let R be the radius of the sphere, then the volume of the hemisphere is $\frac{2}{3}\pi R^3$, and that of the spherical segment with one base, which should be equal to one-half the volume of the hemisphere, is (931):

$$v = \frac{1}{3}\pi R^3.$$

If the altitude of the spherical segment is designated by x (931, REMARK):

$$v = \frac{1}{3}\pi x^2 (3R - x) = \frac{1}{3}\pi R^3,$$

$$x^3 - 3Rx^2 + R^3 = 0.$$

Taking $R = 1$,

$$x^3 - 3x^2 + 1 = 0. \tag{1}$$

To eliminate the term x^2 take (see note (*) page 446)

$$x = y + \frac{3}{3} = y + 1. \tag{2}$$

Equation (1) becomes:

$$y^3 - 3y - 1 = 0. \tag{3}$$

Comparing with the equation,

$$y^3 + py + q = 0,$$

it is seen that $$\frac{q^2}{4} + \frac{p^3}{27} < 0.$$

Thus we have the irreducible case of the third-degree equation. The equation (3) has three real roots.

Writing $\rho = \sqrt{\frac{27}{27}} = 1$, and $\cos\phi = \frac{-q}{2\rho} = \frac{+1}{2}$,

then $$\phi = 60^\circ \text{ and } \frac{\phi}{3} = 20^\circ.$$

The three roots are:

$$y_1 = 2\sqrt[3]{\rho}\cos\frac{\phi}{3},$$

$$y_2 = -2\sqrt[3]{\rho}\cos\left(60^\circ - \frac{\phi}{3}\right),$$

$$y_3 = \quad 2\sqrt[3]{\rho}\cos\left(120° - \frac{\phi}{3}\right).$$

Substituting the numerical values,

$$\begin{aligned} y_1 &= + \ 1.8793, \\ y_2 &= - \ 1.55208, \\ y_3 &= - \ 0.34729; \end{aligned}$$

then substituting in equation (2):

$$\begin{aligned} x_1 &= 1 + y_1 = 2.8793, \\ x_2 &= 1 + y_2 = - \ 0.55208, \\ x_3 &= 1 + y_3 = + \ 0.6527. \end{aligned}$$

The first value x_1 being greater than the radius $R = 1$, cannot be used as a solution.

The second x_2 being negative must also be rejected.

The third value x_3 being less than $R = 1$ and positive, is the solution which applies to the case in hand.

REMARK. If the radius of the sphere were R, the altitude of the required segment will be

$$x_3 = 0.6527 \ R.$$

SPHERICAL TRIGONOMETRY

Properties of spherical triangles.

1073. A spherical triangle is determined by three arcs of great circles drawn on the sphere. If the vertices are connected to the center of the sphere, a trihedral angle corresponding to the spherical triangle, the faces of which are measured by the sides of the spherical triangle, is formed.

Each side of the spherical triangles, which are treated in trigonometry, is less than a semi-circumference.

1074. *The measurement of the angles of a spherical triangle.* The angles A, B, C, of a spherical triangle are measured by tangents drawn to the sides a, b, c, of the triangle. These angles measure the dihedral angles of the trihedral angle corresponding to the spherical triangle.

A spherical triangle may be rectangular, bi-rectangular, or tri-rectangular.

1075. *Lengths of the sides of a spherical triangle.* R being the radius of the sphere, and n the number of degrees in the side of the triangle, we have:

$$a = \frac{\pi R n}{180^\circ}.$$

1076. *General geometrical properties of spherical triangles.* In a spherical triangle each side is smaller than the sum of the other two sides and greater than their difference.

The sum of the three sides is less than the circumference, 360°, of a great circle. The sum of the three angles, A, B, C, lies between two and six right angles.

1077. *Supplementary or polar spherical triangles.* Two triangles are supplementary when the sides of the first are supplements of the angles of the second, and conversely.

GENERAL FORMULAS

1078. *Formula containing the three sides and an angle.*

Theorem. The cosine of any side a is equal to the product of the cosines of the other two sides, increased by the product

of the sines of these two sides multiplied by the cosine of their included angle. Thus,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

1079. *Formula containing the three angles and one side.* This is the inverse of the preceding formula. Thus we have:

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

1080. *Theorem. The sines of the sides of a spherical triangle are to each other as the sines of the opposite angles.*

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

1081. *Formulas containing two sides, the angle included by them and an angle opposite one of them.* We have,

$$\begin{aligned}
\cot a \sin b &= \cos b \cos C + \sin C \cot A,\\
\cot a \sin c &= \cos c \cos B + \sin B \cot A,\\
\cot b \sin a &= \cos a \cos C + \sin C \cot B,\\
\cot b \sin c &= \cos c \cos A + \sin A \cot B,\\
\cot c \sin a &= \cos a \cos B + \sin B \cot C,\\
\cot c \sin b &= \cos b \cos A + \sin A \cot C.
\end{aligned}$$

RIGHT SPHERICAL TRIANGLES

In all cases that follow, A is the right angle, a the hypotenuse, and B and C are the oblique angles of the spherical triangle.

1082. *Theorem. The cosine of the hypotenuse is equal to the product of the cosines of the two sides.* We have,

$$\cos a = \cos b \cos c.$$

1083. *Theorem. The sine of each side is equal to the sine of the hypotenuse multiplied by the sine of the opposite angle.* We have,

$$\begin{aligned}
\sin b &= \sin a \sin B,\\
\sin c &= \sin a \sin C.
\end{aligned}$$

1084. *Theorem. The tangent of each side is equal to the tangent of the hypotenuse multiplied by the cosine of the adjacent angle.*

We have,

$$\begin{aligned}
\tan b &= \tan a \cos C,\\
\tan c &= \tan a \cos B.
\end{aligned}$$

1085. *Theorem. The tangent of each side is equal to the sine of the other side multiplied by the tangent of the angle opposite to the first side.* We have,

$$\tan b = \sin c \tan B,$$
$$\tan c = \sin b \tan C.$$

1086. *Theorem. The cosine of each oblique angle is equal to the cosine of the opposite side times the sine of the other oblique angle.* We have,

$$\cos B = \cos b \sin C,$$
$$\cos C = \cos c \sin B.$$

SOLUTION OF RIGHT SPHERICAL TRIANGLES

1087. These triangles have but one right angle. There are six cases to be considered.

CASE 1. *Solve a right spherical triangle when the hypotenuse a and the side b are given.*

GIVEN.	UNKNOWN.
$A = 90°;\ a,\ b$	$c,\ B,\ C.$

Substituting in the formulas,

$$\cos a = \cos b \cos c,$$
$$\sin b = \sin a \sin B,$$
$$\tan b = \tan a \cos C,$$

we obtain,

$$\cos c = \frac{\cos a}{\cos b},$$

$$\sin B = \frac{\sin b}{\sin a},$$

$$\cos C = \frac{\tan b}{\tan a}.$$

REMARK. The angle B and the side b are of the same species, that is, both are acute or obtuse.

In order that the problem be possible, the hypotenuse must be included between the given side and its supplement.

Another solution. The following formulas may also be used:

$$\tan \frac{1}{2} c = +\sqrt{\tan \frac{1}{2}(a + b) \tan \frac{1}{2}(a - b)},$$

$$\tan\left(45^\circ + \frac{1}{2}B\right) = \pm\sqrt{\frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}(a-b)}},$$

$$\tan\frac{1}{2}C = +\sqrt{\frac{\sin(a-b)}{\sin(a+b)}}.$$

1088. Case 2. *Solve a right spherical triangle having the hypotenuse a and one angle B given.*

Given.	Unknown.
a, $A = 90^\circ$, B.	b, c, C.

From the formulas

$$\sin b = \sin a \sin B, \tag{1}$$
$$\tan c = \tan a \cos B. \tag{2}$$

The angle C may be deduced from

$$\cos a = \cot B \cot C. \tag{3}$$

Transposing,

$$\cot C = \frac{\cos a}{\cot B}.$$

Remark. The side b and the angle B are of the same species, that is, both acute or obtuse.

The problem is always possible and has only one solution.

It may be commenced by determining c and C from (2) and (3), and then b is determined from the equation

$$\tan b = \sin c \tan B.$$

1089. Case 3. *Solve a right spherical triangle when two sides and the right angle are given.*

Given.	Unknown.
b, c, $A = 90^\circ$.	B, C, a.

The following formulas give:

$$\cos a = \cos b \cos c, \tag{1}$$
$$\tan B = \frac{\tan b}{\sin c}, \tag{2}$$
$$\tan C = \frac{\tan c}{\sin b}. \tag{3}$$

REMARK. The problem has only one solution and is always possible.

The angles B and C may be determined by the formulas (2) and (3), and are calculated from one of the following:

$$\tan c = \tan a \cos B,$$
$$\tan b = \tan a \cos C.$$

1090. CASE 4. *Solve a right spherical triangle when a side b and the angle B opposite are given.*

GIVEN.	UNKNOWN.
$b, B, A = 90°.$	$C, a, c.$

The following formulas give:

$$\sin a = \frac{\sin b}{\sin B}, \qquad \sin c = \frac{\tan b}{\tan B}, \qquad \sin C = \frac{\cos B}{\cos b}.$$

The following may also be used:

$$\tan\left(45° + \frac{1}{2}a\right) = \pm\sqrt{\frac{\tan\frac{1}{2}(B+b)}{\tan\frac{1}{2}(B-b)}}, \qquad (1)$$

$$\tan\left(45° + \frac{1}{2}c\right) = \pm\sqrt{\frac{\sin(B+b)}{\sin(B-b)}}, \qquad (2)$$

$$\tan\left(45° + \frac{1}{2}C\right) = \pm\sqrt{\cot\frac{1}{2}(B+b)\cot\frac{1}{2}(B-b)}. \qquad (3)$$

REMARK. B and b are of the same kind: both are acute or obtuse.

If $b > 90°$, then $B > 90°$, and in this case the radical (1) must be taken with a plus sign, $+$, and the two others (2) and (3) with minus signs, $-$.

If $b < 90°$, then $B < 90°$, and in this case the radical (1) must be taken with a minus sign, $-$, and the two others (2) and (3) with plus signs, $+$.

1091. CASE 5. *Solve a right spherical triangle when one side b and the adjacent angle C is given.*

GIVEN.	UNKNOWN.
$b, C, A = 90°.$	$a, c, B.$

The following formulas give:

$$\cos B = \cos b \sin C, \tag{1}$$

$$\tan a = \frac{\tan b}{\cos C}, \tag{2}$$

$$\tan c = \sin b \tan C. \tag{3}$$

a and c may be determined first, and then B calculated from the following:

$$\cos a = \cot B \cot C,$$
$$\tan b = \sin c \tan B.$$

The problem is always possible and has but one solution.

1092. CASE 6. *Solve a right spherical triangle when the two oblique angles are given.*

GIVEN.	UNKNOWN.
$A = 90°, B, C.$	$a, b, c.$

From the following formulas:

$$\cos a = \cot B \cot C,$$

$$\cos b = \frac{\cos B}{\sin C},$$

$$\cos c = \frac{\cos C}{\sin B}.$$

Another solution. The following formulas may also be used:

$$\tan \frac{1}{2} a = +\sqrt{\frac{-\cos(B+C)}{\cos(B-C)}},$$

$$\tan \frac{1}{2} b = +\sqrt{\tan\left(\frac{B-C}{2}+45°\right)\tan\left(\frac{B+C}{2}-45°\right)},$$

$$\tan \frac{1}{2} c = +\sqrt{\tan\left(\frac{C-B}{2}+45°\right)\tan\left(\frac{C+B}{2}-45°\right)}.$$

REMARK. In order that the problem be possible, $\frac{B+C}{2}$ must lie between 45° and 135°, and $\frac{B-C}{2}$ between $-45°$ and $+45°$. There is but one solution.

SOLUTION OF OBLIQUE SPHERICAL TRIANGLES

1093. There are six cases.

First and second case. Solve a spherical triangle when the three sides or three angles are given.

Case 1. *Let the sides a, b, and c be given.*
From the following formulas:

$$\tan\frac{1}{2}A=\sqrt{\frac{\sin(p-b)\sin(p-c)}{\sin p\sin(p-a)}}, \tag{1}$$

$$\tan\frac{1}{2}B=\sqrt{\frac{\sin(p-a)\sin(p-c)}{\sin p\sin(p-b)}}, \tag{2}$$

$$\tan\frac{1}{2}C=\sqrt{\frac{\sin(p-a)\sin(p-b)}{\sin p\sin(p-c)}}. \tag{3}$$

In these formulas we have,

$$p=\frac{a+b+c}{2},$$

and the radical should be taken with the sign $+$.

Remark. Each side should be less than the sum of the two others, and the whole sum less than 360°.

Case 2. *The three angles A, B, and C are given*, and it follows that the sides a', b', c', of the supplementary triangle are

$$\begin{aligned} a' &= 180° - A, \\ b' &= 180° - B, \\ c' &= 180° - C. \end{aligned}$$

The formulas (1), (2), and (3) with the sides a', b', and c', determine the angles A', B', and C' of the supplementary triangle; then the sides of the triangle in question are

$$\begin{aligned} a &= 180° - A', \\ b &= 180° - B', \\ c &= 180° - C'. \end{aligned}$$

The triangle is then solved. But the following formulas may be used, which give the three sides directly:

$$\tan\frac{1}{2}a=\sqrt{\frac{\sin\frac{1}{2}\Delta\sin\left(A-\frac{1}{2}\Delta\right)}{\sin\left(B-\frac{1}{2}\Delta\right)\sin\left(C-\frac{1}{2}\Delta\right)}},$$

$$\tan\frac{1}{2}b=\sqrt{\frac{\sin\frac{1}{2}\Delta\sin\left(B-\frac{1}{2}\Delta\right)}{\sin\left(A-\frac{1}{2}\Delta\right)\sin\left(C-\frac{1}{2}\Delta\right)}},$$

$$\tan\frac{1}{2}c = \sqrt{\frac{\sin\frac{1}{2}\Delta\sin\left(C-\frac{1}{2}\Delta\right)}{\sin\left(A-\frac{1}{2}\Delta\right)\sin\left(B-\frac{1}{2}\Delta\right)}}.$$

These radicals are taken with the sign $+$. In the preceding formulas Δ is the spherical excess; that is, the difference between the sum of the angles and 180°. Thus,

$$A + B + C - 180^\circ = \Delta.$$

Δ lies between 0 and 360°.

REMARK. The sum of the three angles should lie between two and six right angles.

1094. *Third and fourth case. Solve a spherical triangle when two sides and the included angle or one side and the adjacent angles are given.*

The solution of these two problems is given by the formulas of Napier.

CASE 3. *Two sides and the included angle given.*

GIVEN.	UNKNOWN.
$a, b, c.$	$c, A, B.$

The following formulas, known as *Napier's analogies*, will be used.

$$\tan\frac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{1}{2}C. \tag{1}$$

$$\tan\frac{1}{2}(A-B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{1}{2}C. \tag{2}$$

$$\tan\frac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{1}{2}c. \tag{3}$$

$$\tan\frac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{1}{2}c. \tag{4}$$

The formulas (1) and (2) give $A + B$ and $A - B$, from which A and B can be deduced. The values of $A + B$ and $A - B$ substituted in (3) or (4) give c.

Or c may be determined directly from

$$\cos c = \cos a \cos b + \sin a \sin b \cos C, \tag{5}$$

which is easily solved by logarithms when written in the form:

$$\cos c = \cos a\,(\cos b + \sin b \tan a \cos C).$$

Let $\tan \phi = \tan a \cos C$, then

$$\cos c = \cos a\,(\cos b + \sin b \tan \phi).$$

Substituting $\dfrac{\sin \phi}{\cos \phi}$ for $\tan \phi$, we have

$$\cos c = \frac{\cos a \cos (b - \phi)}{\cos \phi}.$$

Case 4. *One side and the two adjacent angles given.*

Given.	Unknown.
$c, A, B.$	$C, a, b.$

The formulas (3) and (4) give $a + b$ and $a - b$, and consequently the sides a and b. The quantities $a + b$ and $a - b$ substituted in (1) or (2) give C.

C may also be calculated directly. Thus,

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c,$$

or $$\cos C = -\cos A\,(\cos B - \sin B \tan B \cos c).$$

Let $$\tan B \cos c = \cot \phi,$$

then $$\cos C = -\cos A\,(\cos B - \sin B \cot \phi).$$

Substituting $\dfrac{\cos \phi}{\sin \phi}$ for $\cot \phi$, we have:

$$\cos C = \frac{-\cos A \cdot \sin (\phi - B)}{\sin \phi} = \frac{\cos A \cdot \sin (B - \phi)}{\sin \phi}.$$

1095. *Fifth and sixth case. Solve a spherical triangle when two sides and the angle opposite one of them or two angles and the side opposite one of them is given.*

Case 5. *Two sides and the angle opposite one of them given.*

Given.	Unknown.
$a, b, A.$	$c, B, C.$

Write

$$\frac{\sin B}{\sin A} = \frac{\sin b}{\sin a}, \tag{1}$$

from which the value of B is determined. The values c, C, are determined by the Napier formulas (see page 460 (1094)).

The formulas (2) and (4) of article (1094) give:

$$\cot \frac{1}{2} C = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} \tan \frac{1}{2}(A-B), \tag{2}$$

$$\tan \frac{1}{2} c = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tan \frac{1}{2}(a-b). \tag{3}$$

Case 6. *Two angles and the side opposite one of them given.*

Given.	Unknown.
$A, B, a.$	$C, b, c.$

The solution is the same as in case 5. Thus,

$$\frac{\sin b}{\sin a} = \frac{\sin B}{\sin A},$$

from which b is deduced. The values c and C are obtained from the relations (2) and (3) of case 5.

Remark. The values B and b are given by the sines, therefore, the $\sin B$ and $\sin b$ must be positive since the angles b and B are less than 180°.

Moreover, the values C and c are necessarily positive, since C and c are less than 180°; then $\frac{1}{2} C$ and $\frac{1}{2} c$ are less than 90°, and the corresponding tangents are positive. Because of this, in formulas (2) and (3) the differences $A - B$ and $a - b$ must have like signs (see case 5).

This condition may be used to determine whether the two supplementary values of the angle B, given by the equation (2), can be accepted.

All these conditions together may be used to determine if there is one or two solutions, or if it is impossible.

1096. *The measure of the surface of a spherical triangle.* It may be shown that the area of a spherical triangle is propor-

tional to its spherical excess, when the area of the surface of a tri-rectangular triangle, which is $\frac{1}{8}$ of the surface of the sphere, is taken as the unit of area. That is, Δ being the spherical excess, R the radius of the sphere, and T the area of any triangle, we have (A, B, C, being the angles of the triangle):

$$\Delta = A + B + C - 180°,$$

$$\frac{T}{\frac{\pi R^2}{2}} = \frac{\Delta}{1 \text{ rt.} \angle}.$$

In this relation $\frac{\pi R^2}{2}$ is $\frac{1}{8}$ of the surface of the sphere:

$$T = R^2 \Delta \frac{\pi}{2 \text{ rt.} \angle\text{s}}. \tag{1}$$

This formula proves itself in the tri-rectangular triangle, which gives,

$$\Delta = A + B + C - 180° = 3 \text{ rt.} \angle\text{s} - 2 \text{ rt.} \angle\text{s} = 1 \text{ rt.} \angle,$$

and formula (1) becomes:

$$T = \frac{\pi R^2}{2}.$$

which is $\frac{1}{8}$ of the area of the sphere.

Example. Let $A + B + C = 300°$;

then $\Delta = 300° - 180° = 120°.$

The area of the spherical triangle will be

$$T = R^2 \frac{120}{180} \pi = \frac{2}{3} \pi R.^2$$

The area of a spherical triangle in terms of its sides. Calculate the spherical excess by the formula:

$$\tan \frac{1}{4} \Delta = \sqrt{\tan \frac{1}{2} p \tan \frac{1}{2} (p - a) \tan \frac{1}{2} (p - b) \tan \frac{1}{2} (p - c)}.$$

Δ being determined, calculate the area as in the preceding example.

Note. $\frac{a + b + c}{2} = p.$

PROBLEMS IN SPHERICAL TRIGONOMETRY

PROBLEM 1. *Reduce an angle to the horizontal,* that is, find the projection of an angle formed by two straight lines in space, upon the horizontal.

Thus, if from a point O in space (Fig. 104, article 765) the axis of an instrument is directed toward the points A and B, and the angle $AOB = c$ is measured, it remains to determine the projection AGB on the horizontal. To this end the angles b and a which the radii OA and OB make with the vertical are measured. Now the three faces of a trihedron $OABG$ having O as vertex are known, or, which is the same thing, the three sides of a spherical triangle are given to determine the angle $AGB = C$, opposite one of the sides or the face $AOB = c$. From the formula (see case 1, *Oblique Spherical Triangles*):

$$\tan\frac{1}{2}C = \sqrt{\frac{\sin(p-a)\sin(p-b)}{\sin p\cdot\sin(p-c)}}.$$

Let

$$\begin{aligned}
a = 45^\circ 15';\ b &= 50^\circ 35';\ c = 91^\circ 32'\\
2p = a + b + c &= 187^\circ 22'\\
p &= 93^\circ 41'\\
p - a &= 48^\circ 26'\\
p - b &= 43^\circ\ \ 6'\\
p - c &= 2^\circ\ \ 9'\\
\log\sin 48^\circ 26' &= \bar{1}.8740085\\
\log\sin 43^\circ\ \ 6' &= \bar{1}.8345948\\
c^t\log\sin 93^\circ 41' &= 10.0008980\\
c^t\log\sin 2^\circ\ \ 9' &= 10.4257861\\
&\quad -20\\
\log\tan\frac{1}{2}C &= \frac{1}{2}(0.1352874)\\
\log\tan\frac{1}{2}C &= 0.0676437\\
\frac{1}{2}C &= 49^\circ 26' 38''\\
C &= 98^\circ 53' 16''
\end{aligned}$$

Thus the angle C is the projection on the horizontal of the angle c.

PROBLEM 2. *Distance from Paris to St. Petersburg,* that is, the shortest distance between two points on the surface of a sphere or the length of the arc of a great circle passing through the two

places. This distance is the side of a spherical triangle, two sides and the included angle of which are known. If two meridians are passed through these places, the portions of these meridians between these points and the pole are two sides of a spherical triangle, the third side of which is the required distance.

The dihedral angle between the two meridians is measured by the difference in longitude, and the two sides which include this angle are complements of the latitudes of the two places, provided they are in the same hemisphere as are Paris and St. Petersburg.

	Longitude, East	Latitude, North
St. Petersburg	27° 59′ 36″	59° 46′ 19″
Paris	0°	48° 50′ 49″

Let a and b be the distances from the above places to the pole, c the required distance between the cities, and C the included angle at the pole or the difference of the longitudes.

$$a = 90° - 48°\ 50'\ 49'' = 41°\ \ 9'\ 11''.$$
$$b = 90° - 59°\ 56'\ 19'' = 30°\ 13'\ 41''.$$
$$C = 27°\ 59'\ 36''.$$

Referring to case 3 and case 4 of spherical triangles (**1094**):

$$\cos c = \frac{\cos a \cdot \cos (b - \phi)}{\cos \phi}.$$
$$\tan \phi = \tan a \cdot \cos C.$$

CALCULATION OF THE AUXILIARY ANGLE ϕ

$$\begin{aligned}
\log \tan a &= \bar{1}.94150525 \\
\log \cos C &= \bar{1}.94596178 \\
\hline
\log \tan \phi &= \bar{1}.88746703 \\
\phi &= 37°\,39'\,30.7'' \\
b - \phi &= -\ (7°\,25'\,49.7'')
\end{aligned}$$

CALCULATION OF THE DISTANCE C

$$\begin{aligned}
\log \cos a &= \bar{1}.87676866 \\
\log \cos (b - \phi) &= \bar{1}.9963378 \\
c' \log \cos \phi &= 10.10146724 \\
&\quad -10 \\
\hline
\log \cos c &= \bar{1}.97457370 \\
c &= 19°24'53.4''
\end{aligned}$$

To obtain the distance in miles, reduce the side c to seconds; thus,

$$c = 69893.4'',$$

and
$$90^\circ = 324000''.$$

Taking a quadrant as 6250 miles we have:

$$\frac{90}{c} = \frac{6250}{x},$$

or
$$\frac{324000}{69893.4} = \frac{6250}{x},$$

$$x = 1348 \text{ miles.}$$

ANGLES FORMED BY THE FACES OF REGULAR POLYHEDRONS

Problem 3. There are only five regular polyhedrons (903): the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron.

Tetrahedron. The polyhedral angle of a tetrahedron is a trihedral angle, the three equal faces of which are measured by the angle of an equilateral triangle. Therefore a spherical triangle, the three sides of which are each equal to $\frac{2}{3}$ of a right angle 60°, is to be solved. This is the first case in the solution of spherical triangles (1093). Let C be the required dihedral angle, then using the formula:

$$\sin\frac{1}{2}C = \sqrt{\frac{\sin(p-a)\sin(p-b)}{\sin a \sin b}}, \tag{1}$$

we have
$$a = b = c = 60^\circ,$$

$$p = \frac{a+b+c}{2} = \frac{60 \times 3}{2} = 90^\circ,$$

$$p - a = p - b = 90^\circ - 60^\circ = 30^\circ.$$

Formula (1) gives

$$\sin\frac{1}{2}C = \sqrt{\frac{(\sin 30^\circ)^2}{(\sin 60^\circ)^2}} = \frac{\sin 30^\circ}{\cos 30^\circ} = \tan 30^\circ,$$

and
$$C = 70^\circ 31' 43.6''.$$

Cube. The dihedral angle of a cube is 90°.

Octahedron. This problem may be solved by spherical trigonometry, by dividing one of the polyhedral angles formed by four

equilateral triangles into two trihedrons. A much simpler method is as follows: a being the edge of the octahedron, and C one of the dihedral angles, considering one of the two pyramids with a square base, which compose the octagon, we have,

$$\tan\frac{1}{2}C = \frac{\frac{1}{2}a\sqrt{2}}{\frac{1}{2}a} = \sqrt{2},$$

which gives $$C = 108° \, 28' \, 1.6''.$$

Dodecahedron. The polyhedral angle of this polyhedron is a trihedral angle, the three faces of which are measured by the angles 108° of a regular pentagon. Thus the dihedral angle of a dodecahedron is obtained by solving a spherical triangle whose three equal sides are each measured by 108°.

The first case of spherical triangles (1093) gives:

$$\sin\frac{1}{2}C = \sqrt{\frac{\sin(p-a)\sin(p-b)}{\sin a \sin b}}.$$

We have $$a = b = c = 108°,$$

$$p = \frac{a+b+c}{2} = \frac{108 \times 3}{2} = 162°,$$

$$p - a = p - b = 162° - 108° = 54°,$$

$$\sin\frac{1}{2}C \sqrt{\frac{(\sin 54°)^2}{(\sin 108°)^2}} = \frac{\sin 54°}{\sin 108°},$$

$$\sin 108° = \sin(180° - 108°) = \sin 72^2.$$

Therefore, $$\sin\frac{1}{2}C = \frac{\sin 54°}{\sin 72°},$$

and $$C = 116° \, 33' \, 54''.$$

Icosahedron. It is readily seen that one of the dihedral angles of an icosahedron belong to a trihedral angle of which the three faces are known: two faces are formed by two equilateral triangles, and the third face is formed by a diagonal plane, which determines an isosceles triangle whose angle at the vertex is equal to the interior angle of a regular pentagon. The three faces of the trihedron are known.

$$a = b = \frac{2}{3}\text{ rt. } \measuredangle\text{s and } c = 108°.$$

The formula in article (21) may be used.

$$\sin\frac{1}{2}C = \sqrt{\frac{\sin(p-a)\sin(p-b)}{\sin p \sin b}}. \qquad (A)$$

$$p = \frac{a+b+c}{2} = \frac{60° + 60° + 108°}{2} = 114°.$$

$$p - a = p - b = 114 - 60 = 54°.$$

From formula (A) $\quad C = 138°\ 11'\ 22.8''.$

1097. *Formulas for transforming algebraic and trigonometric expressions into such a form that they may be solved by logarithms.*

Example 1.

Let $\quad x = A \pm B$ be given.

1st, considering

$$x = A + B,$$

we may write $\quad x = A\left(1 + \frac{B}{A}\right).$

Putting $\quad \frac{B}{A} = \tan^2 \alpha,$

we have $\quad \log \tan \alpha = \frac{1}{2}(\log B - \log A),$

and $\quad x = A(1 + \tan^2 \alpha) = A\left(1 + \frac{\sin^2 \alpha}{\cos^2 \alpha}\right).$

$$x = A\left(\frac{\cos^2 \alpha + \sin^2 \alpha}{\cos^2 \alpha}\right) = \frac{A}{\cos^2 \alpha}.$$

$$\log x = \log A - 2 \log \cos \alpha.$$

2d. If we consider $\quad x = A - B,$

and if B is less than A the ratio of B to A is less than unity, and we may write successively:

$$x = A\left(1 - \frac{B}{A}\right),$$

$$\frac{B}{A} = \sin^2 \alpha,$$

$$x = A(1 - \sin^2 \alpha) = A \cos^2 \alpha,$$

$$\log x = \log A + 2 \log \cos \alpha.$$

If B is greater than A we may write

$$\frac{B}{A} = \tan \alpha,$$

and therefore $$x = A\,(1 - \tan\alpha) = A\left(1 - \frac{\sin\alpha}{\cos\alpha}\right),$$

$$x = A\,\frac{(\cos\alpha - \sin\alpha)}{\cos\alpha} = \frac{A}{\cos\alpha}[\sin(90 - \alpha) - \sin\alpha].$$

Taking the formulas (1052)

$$\sin p - \sin q = 2\cos\frac{1}{2}(p + q)\sin\frac{1}{2}(p - q),$$

putting $$p = 90 - \alpha \text{ and } q = \alpha,$$

then $$\frac{1}{2}(p + q) = 45^\circ,$$

$$\frac{1}{2}(p - q) = 45^\circ - \alpha,$$

$$x = \frac{A}{\cos\alpha}\,2\cos 45^\circ \sin(45^\circ - \alpha).$$

This formula is logarithmic.

EXAMPLE 2. Having given:

$$x = \tan a \pm \tan b, \tag{1}$$

we may write

$$x = \frac{\sin a}{\cos a} + \frac{\sin b}{\cos b} = \frac{\sin a\cos b \pm \cos a\sin b}{\cos a\cdot\cos b},$$

or $$x = \frac{\sin(a \pm b)}{\cos a\cdot\cos b}. \tag{2}$$

EXAMPLE 3.

$$x = \cot B \pm \cot A, \tag{1}$$

or $$x = \frac{1}{\tan B} \pm \frac{1}{\tan A} = \frac{\tan A \pm \tan B}{\tan A\tan B},$$

$$x = \frac{1}{\tan A\tan B}(\tan A \pm \tan B). \tag{2}$$

Now $$\tan A \pm \tan B = \frac{\sin A}{\cos A} \pm \frac{\sin B}{\cos B},$$

or $$\tan A \pm \tan B = \frac{\sin A\cos B \pm \sin B\cos A}{\cos A\cos B} = \frac{\sin(A \pm B)}{\cos A\cdot\cos B}.$$

Therefore from (2) we may write

$$x = \frac{1}{\tan A\tan B}\,\frac{\sin(A \pm B)}{\cos A\cos B} = \frac{\cos A}{\sin A}\,\frac{\cos B}{\sin B\cdot}\,\frac{\sin(A \pm B)}{\cos A\cdot\cos B},$$

or $$x = \frac{\sin(A \pm B)}{\sin A\sin B}.$$

This formula is logarithmic

EXAMPLE 4.

$$x = \sqrt{2} + \sin a, \tag{1}$$

or

$$x = \sqrt{2}\left(1 + \frac{\sin a}{\sqrt{2}}\right). \tag{2}$$

Putting

$$\frac{\sin a}{\sqrt{2}} = \tan^2 \phi = \frac{\sin^2\phi}{\cos^2\phi}, \tag{3}$$

Therefore formula (2) becomes:

$$x = \sqrt{2}\,(1 + \tan^2\phi) = \sqrt{2}\left(\frac{\cos^2\phi + \sin^2\phi}{\cos^2\phi}\right) = \frac{\sqrt{2}}{\cos^2\phi}.$$

Formula (1) is therefore replaced by a logarithmic formula. The auxiliary angle ϕ is calculated from the following formula deduced from (3):

$$\log\tan\phi = \frac{1}{2}(\log\sin a - \log\sqrt{2}).$$

EXAMPLE 5.

$$x = \csc a + \sec b, \tag{1}$$

or

$$x = \frac{1}{\sin a} + \frac{1}{\cos b} = \frac{\cos b + \sin a}{\sin a \cos b},$$

or

$$x = \frac{\sin(90 - b) + \sin a}{\sin a \cos b}. \tag{2}$$

From (1052) we have

$$\sin p + \sin q = 2\sin\frac{1}{2}(p + q)\cos\frac{1}{2}(p - q).$$

Putting

$$90 - b = p,$$
$$a = q,$$

we have

$$\frac{1}{2}(p + q) = 45° - \frac{a - b}{2},$$

$$\frac{1}{2}(p - q) = 45° - \frac{a + b}{2},$$

and equation (2) becomes

$$x = \frac{2\sin\left(45° - \frac{a - b}{2}\right)\cos\left(45° - \frac{a + b}{2}\right)}{\sin a \cos b},$$

which may be calculated by logarithms.

PART V

ANALYTIC GEOMETRY

1098. *The purpose of analytic geometry is* the study of geometrical figures by means of *algebraic analysis*.

This branch of mathematics was invented by Descartes, who found that the properties of geometrical figures could be studied by algebraic methods; he also found graphic solutions for algebraic calculations. The latter are the more useful to the engineer.

Analytic geometry, like elementary geometry, is divided into two parts (610): *plane geometry* and *solid geometry*.

DETERMINATION OF A LINE

1099. We have seen that the position of a point in a plane or in space is fixed when its coördinates are known (1020, 1021). In order that a line be determined, it suffices to know the coördinates of its points.

When the same algebraic relation exists between the coördinates of each of the points of the line, as many points may be determined as one wishes, and therefore, by plotting the points which are thus obtained, the line may be drawn.

Thus, if the relation between the coördinates of a plane curve are known, by assuming any value for one coördinate the corresponding value of the other is found from the given relation which determines a point on the curve (504).

Suppose that the relation $y = 3x + 2$ exists between the coördinates, then if $x = 4$, $y = 3 \times 4 + 2 = 14$.

Giving x a new value, another corresponding value of y is found, and so on.

When the curve is not a plane curve, since only one coördinate may be chosen arbitrarily, the two others can only be determined when there are two equations (516).

1100. *Polar Coördinates.* A point M is also determined in a plane MOx, when the angle $MOx = \alpha$, which the line OM makes with the axis Ox, and the distance $OM = \rho$, called *radius vector*, from the *pole* O are given.

The two quantities α and ρ are called *polar coördinates.*

When the same algebraic relation exists between the polar coördinates of each of the points of a line, as in the preceding case, any number of points may be determined, and consequently, the line drawn.

1101. *Focal coördinates.* The position of a point M is also fixed in a plane, when the distances $MF = \rho$ and $MF' = \rho'$ from the point M to the two fixed points F and F' are known. The points F and F' are called *foci*, and the distances ρ and ρ' are called *radius vectors* or *focal coördinates.* These same coördi-

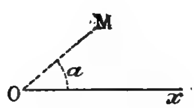

Fig. 271

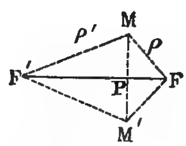

Fig. 272

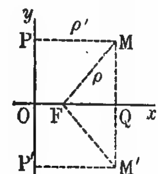

Fig. 273

nates, ρ and ρ', determine a point M' in the same plane and symmetrical to M with respect to the axis FF', and an equation between ρ and ρ', considering them as variables, determines a line made up of two parts symmetrical to each other with respect to the axis FF' (504).

A point M is also determined in a plane by the distances $MF = \rho$ and $MP = \rho'$, also called *radius vectors*, to the fixed point F and the fixed line Oy, which are respectively called the *focus* and the *directrix*. As in the preceding case, the two absolute lengths of the radius vectors determine two points, M and M', symmetrical to each other with respect to the axis Ox, drawn through the focus F perpendicular to the directrix Oy. Thus an equation between the two radius vectors, ρ and ρ', considered as variables, determines a line symmetrical with respect to the axis Ox.

1102. Curves are determined by the relations between their coördinates with respect to two axes (1099), or by those between their polar coördinates (1100) or by those between their focal coördinates (1101).

The study of the curves most often used in practice will make all this clear. The equation which expresses the relations between the coördinates of a curve is called the *equation of the curve.*

HOMOGENEITY

1103. A polynomial is said to be *homogeneous* when all its terms are of the same degree. The degree m of each term is the degree of homogeneity of the polynomial (**455**, **457**).

In general, we say that a function (**504**) is homogeneous and of the degree m, when in multiplying each of the letters which appear in the expression by a constant k raised to the power of that particular letter, the function is multiplied by k^m (**478**). Such are:

$$a^2 + 2\,ab, \frac{ab}{c} - \sqrt{dc}, \quad \frac{a + \sqrt{ab}}{a + c}, \quad \frac{a}{a^3 - b^3},$$

of which the degree is respectively 2, 1, 0, and -2.

A monomial is always an homogeneous function of a degree equal to that of the monomial.

If, in a function, letters appear which represent numerical coefficients, these letters are neglected in forming the degree of the homogeneity of the function. Thus, n being a numerical coefficient, the following function is homogeneous and of the first degree:

$$\frac{a^2 + (y + nx^2)}{\sqrt{ab - (ny + x)^2}}.$$

The transcendental functions, sin, cos, ..., log, of homogeneous functions of the degree 0, such as e^{α}, in which α is also an homogeneous function of the degree 0, are considered as numerical coefficients. Such are:

$$\sin \frac{ab}{a^2 + b^2}, \quad \log \frac{b + \sqrt{a^2 - b^2}}{a + b}, \quad e^{\frac{a^2 - e^2}{ab}}.$$

In multiplying each letter of a function of the degree o by k, the value of the function is not changed, and therefore k may be omitted; which, however, could not be done if the degree of the function were not 0.

Thus the following function is homogeneous and of the degree $\frac{1}{2}$:

$$\frac{a\sqrt{b} + b\sqrt{c}\sin\dfrac{c}{a}}{a + b}.$$

1104. From the above and the operations on polynomials, it follows:

1st. That the sum or difference of two homogeneous functions of the same degree is an homogeneous function of the same degree as the first (**460**, **461**).

2d. That the product of several homogeneous functions of any degree is an homogeneous function of a degree equal to the sum of the degrees of the given functions (**477**).

3d. That the quotient obtained in dividing one homogeneous function by another is an homogeneous function of a degree equal to the degree of the first less that of the second (**494**).

4th. That a power of an homogeneous function is an homogeneous function of a degree equal to the degree of the given function multiplied by the degree of the power (2d).

5th. That the root of an homogeneous function is an homogeneous function of a degree equal to the degree of the given function divided by the index of the root (4th).

1105. An *equation* is said to be *homogeneous* when its two members are homogeneous and of the same degree, or when one of its members is zero and the other is homogeneous (**1103**).

From this definition it follows:

1st. That an homogeneous equation remains homogeneous when all the letters which it contains are multiplied by the same factor k, with an exponent equal to that of each letter (**1103**).

2d. That an homogeneous equation between two concrete quantities of the same kind (**12**) — other quantities being considered as coefficients (**1103**) — is independent of the unit used to express these quantities. In changing the unit, all the concrete quantities are multiplied by the same factor whole or fractional.

Conversely, if a whole algebraic equation — the only case which need be considered (**447**) — between concrete quantities of the same kind exists, no matter what units are used, the equation is homogeneous, or comes from the addition of several homogeneous equations of different degrees (**1108**).

1106. Any algebraic equation may be transformed to one in which one of the members is zero, and the other a whole rational quantity (**447**).

If the equation is homogeneous and of the degree m, each of its terms contain m literal factors, not including the literal coefficients (**1103**).

Thus in general an equation may be written in the form of the function

$$f\,(a,\, b,\, x,\, y,\, \ldots\,.) = 0.$$

1107. In geometry, lengths are the only concrete quantities which have to be considered, because areas and volumes depend upon the linear dimensions.

To express algebraically a relation between several lengths, they must first be reduced to the same units, which are generally arbitrarily chosen (1109).

1108. All equations in geometry are homogeneous when the unit is indeterminate. This is of the greatest importance in analytic geometry: it serves as a means of proof during the course of the calculations; it aids one in memorizing the formulas; it establishes analogies between the expressions, and may suggest methods of calculation which are more simple and elegant.

Remark 1. When several homogeneous equations are combined by addition or subtraction, they should be of the same degree; because if they are not, the resulting equation, although exact, will not be homogeneous; and such a combination, in a well-conducted analysis, should be avoided.

Remark 2. The theorem of homogeneity is applicable to all the equations of geometry; but in remembering that areas are the products of two lengths, and volumes the products of three lengths, therefore, according as a letter A or V represents an area or a volume, it must be considered as being of the second or third degree. Thus, h, h', b, b', expressing lengths, A and A' areas, and V and V' volumes, the two following formulas are homogeneous:

$$A - A' = \frac{1}{2}(h - h')\,(b - b'), \quad V - V' = \frac{1}{3}(h - h')(A + A' + \sqrt{AA'}).$$

In general, according as the unknown of a problem is an area or a volume, the expression which is obtained is homogeneous and of the second or third degree. Thus we have,

$$A = ab \text{ or } V = abc.$$

1109. In all which has been said, the unit has been taken as arbitrary. This hypothesis should hold for the solution of all geometrical problems; because, otherwise, if, for example, a certain length was taken as unit, although homogeneous equations could be obtained they would not appear to be so.

Thus, taking an arbitrary unit, the area of a circle is:

$$A = \pi r^2.$$

If, on the contrary, we take the radius equal to one, we have

$$A' = \pi \times 1^2 = \pi,$$

equation in which the first member is of the second degree, and the second apparently of the degree 0, because π is an abstract number. In order to give the equation its usual homogeneous aspect, the radius is expressed in arbitrary units; r is substituted for 1, and we have r^2 in the second member. Thus,

$$A'r^2 \quad \text{or} \quad A = \pi r^2.$$

Taking the radius as unity, the volume of a sphere is:

$$V' = \frac{4}{3}\pi \times 1^3 = \frac{4}{3}\pi.$$

Substituting an arbitrary unit for the radius, which gives r instead of 1, the preceding equation becomes:

$$V'r^3 \quad \text{or} \quad V = \frac{4}{3}\pi r^3.$$

Half the major axis of an ellipse being taken as unity, the area of the ellipse is:

$$A' = \pi \times 1 \times b'. \qquad (1162)$$

Substituting an arbitrary unit, a, for 1, and comparing all the lengths to this same arbitrary unit, we have,

$$A'\,a^2 = \pi \times a \times ab',$$
$$A = \pi ab.$$

THE GEOMETRICAL CONSTRUCTION OF ALGEBRAIC FORMULAS

1110. From the law of homogeneity it follows that any homogeneous algebraic expression of the first degree, in which the different letters represent lengths, is an expression of a length x (1108, REMARK 2), and this length may always be determined geometrically, that is, with the aid of a rule and compass: *First*, when the expression is rational (447); *Second*, when, being irrational, it contains only radicals whose index is 2 or a power of 2.

1111. *Construction of rational expressions.* To construct,

$$x = a + b - c + d - e,$$

commencing at the point 0 on an indefinite straight line, take $OA = a$, $AB = b$, $BC = d$, $CE = -c$, and $EF = -c$. The distance $-OF$ is value of x (Fig. 274).

If we have

$$x = \frac{ab}{m},$$

construct the fourth proportional to the three lines, a, b, and m (969).

For $$x = \frac{abcd}{mnp}.$$

Construct the fourth proportional $x' = OX' = \frac{ab}{m}$ to the three lines, $m = OM$, $a = OA$, and $b = OB$; then the fourth proportional $x'' = OX'' = \frac{x'c}{n} = \frac{abc}{mn}$ to the three lines, $n = ON$, $x' = OX'$, and $c = OC$; finally, construct the fourth proportional,

$$x = Ox = \frac{x''d}{p} = \frac{abcd}{mnp},$$

to the three lines, $p = OP$, $x'' = OX''$, and $d = OD$.

The construction of the fourth proportionals in the preceding

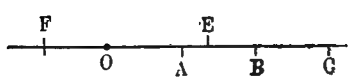

Fig. 274

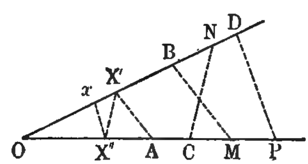

Fig. 275

example. After having drawn the indefinite lines OP and OD, lay off *alternately* on one and then the other, $AO = a$, $OB = b$, $OC = c$, $OD = d$, $OM = m$, $ON = n$, and $OP = p$; draw BM, CN, and DP, and AX', $X'X''$, and $X''x$ parallel respectively to the first; then Ox is the required length x.

The expressions

$$x = \frac{a^2}{m} = \frac{aa}{m}, \quad x = \frac{a^3}{m^2} = \frac{aaa}{mm}, \quad \text{etc.},$$

being the same as the above, except that the several factors are equal, x is found in the same way by constructing the fourth proportionals.

x being expressed by a fraction whose terms are polynomials, the construction is reduced to that given above by operating as follows:

Let
$$x = \frac{a^3b + 4\,a^2bc}{5\,ab^2 - b^2c}.$$

k being an arbitrary length, we may put the value of x in the form

$$x = \frac{k\left(\dfrac{a^3b}{k^3} + \dfrac{4\,a^2bc}{k^3}\right)}{\dfrac{5\,ab^2}{k^2} - \dfrac{b^2c}{k^2}}.$$

The exponent of k being one less than the degree of the terms which it divides, each of the resulting monomial fractions may be constructed from the preceding rule, and A, B, M, N being the lengths found, we have,

$$x = \frac{k\,(A + B)}{M - N}.$$

Determining $A + B = \alpha$, and $M - N = m$, we have,

$$x = \frac{k\alpha}{m};$$

and x, being the fourth proportional of the lengths k, α, and m, is constructed as shown above.

Remark. In the preceding problems, as in those of the next article, if the given quantities instead of being lines were numbers, taking a length as unity the given numbers could be represented by lengths which, being submitted to the constructions indicated by the formula, would give a length, which, expressed in the chosen units, would be the required result.

Thus, for example,

$$x = \frac{3 \times 7}{5}.$$

Taking the lengths a, b, m, equal respectively to 3, 7, and 5 times some chosen unit, and constructing the 4th proportional,

$$x = \frac{ab}{m},$$

the length x expressed on the given units would be,

$$x = \frac{3 \times 7}{5}.$$

1112. *Construction of irrational expressions.* Since the degree of homogeneity should be 1 (1110), if the radical is of the second degree, the quantity placed under the radical should be homogeneous and of the second degree; thus, when this quantity is fractional the degree of the numerator is two units greater than that of the denominator. $x = \sqrt{ab}$ is a mean proportional between the lines a and b (970).

For $x = \sqrt{5 \times 7}$, taking a length as unity (1111, REMARK), a and b being the lengths equal respectively to 5 and 7 times this unit, the mean proportional $x = \sqrt{ab}$ expressed in terms of the chosen unit is $\sqrt{5 \times 7}$. For $x = \sqrt{5}$, noting that $\sqrt{5} = \sqrt{5 \times 1}$, we have the same case as the preceding.

$x = \sqrt{a^2 + b^2}$ is the hypotenuse of a right triangle, the sides of which are a and b (703).

$x = \sqrt{a^2 - b^2}$ is one of the sides of a right triangle, having a for its hypotenuse and b for its second side (702); this is also a mean proportional $\sqrt{\alpha\beta}$ between the two lines,

$$\alpha = a + b \quad \text{and} \quad \beta = a - b. \tag{729}$$

$x = a\sqrt{2}$, or $x^2 = 2\,a^2$, is the hypotenuse of a right isosceles triangle, one leg of which is a (Fig. 276).

$x = \sqrt{ab + c^2}$. After having constructed a mean proportional $p = \sqrt{ab}$, we have,

$$x = \sqrt{p^2 + c^2}.$$

$x = \dfrac{a}{\sqrt{2}}$, from which $x^2 = \dfrac{a^2}{2}$, is the chord AB which subtends a quadrant whose diameter is a (706).

$x = \dfrac{2\,a}{\sqrt{3}}$. Squaring, we have $x^2 = \dfrac{4\,a^2}{3}$, and $\dfrac{x^2}{a^2} = \dfrac{4}{3}$; which shows that the problem reduces to finding the side x a square

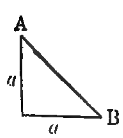

Fig. 276

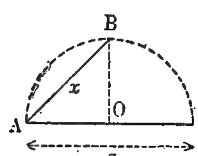

Fig. 277

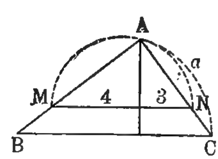

Fig. 278

which is to another square a^2 as 4 : 3. On a line MN, lay off lengths proportional to the numbers 4 and 3; on MN as a di-

ameter describe a semicircle; on AN take $AC = a$, and drawing CB parallel to MN, we have $AB = x$. From (1000),

$$AB : a = AM : AN \text{ or } AB^2 : a^2 = \overline{AM}^2 : \overline{AN}^2 = 4 : 3. \quad (732)$$

$x = \dfrac{a\sqrt{2}}{\sqrt{7}}$, and $\dfrac{x^2}{a^2} = \dfrac{2}{7}$, would also be solved by the preceding construction.

If the quantity under the radical is a fraction, as

$$x = \sqrt{\frac{a^5 + ab^4 - 5b^2c^3}{a^3 - b^2c}},$$

choosing an arbitrary length k, as in article (1111), we have,

$$x = \sqrt{\frac{k^2\left(\dfrac{a^5}{k^4} + \dfrac{ab^4}{k^4} - \dfrac{5\,b^2c^3}{k^4}\right)}{\dfrac{a^3}{k^2} - \dfrac{b^2c}{k^2}}}.$$

The quantity written within the parentheses is reduced to a line a, and the denominator to a line m; such that

$$x = \sqrt{\frac{k^2a}{m}} = \sqrt{k\frac{ka}{m}} = \sqrt{ku},$$

which shows that the construction of the 4th proportional $u = \dfrac{ka}{m}$ (1111), and the mean proportional $x = \sqrt{ku}$ (970), will give the required construction. If the index of the root were $2^2 = 4$, the quantity under the radical would be homogeneous and of the 4th degree.

Let
$$x = \sqrt[4]{\frac{a^6 + a^2b^4 - b^3c^3}{a^2 + bc}}.$$

To construct x, write

$$x = \sqrt{\frac{k^4\left(\dfrac{a^6}{k^5} + \dfrac{a^2b^4}{k^5} - \dfrac{b^3c^3}{k^5}\right)}{\dfrac{a^2}{k} + \dfrac{bc}{k}}}.$$

This formula may be reduced as was the one in the preceding case.

$$x = \sqrt[4]{\frac{k^4a}{m}} = \sqrt{k\sqrt{k\frac{ka}{m}}} = \sqrt{k\sqrt{ku}} = \sqrt{kv},$$

which shows that the 4th proportional of $u = \frac{ka}{m}$, the mean proportional $v = \sqrt{ku}$, and the mean proportional $x = \sqrt{kv}$ must be constructed.

Finally, x may be expressed by a quantity, one part of which is rational and the other part irrational; such as

$$x = \frac{-b \pm \sqrt{a^2 + b^2 - c^2}}{2}.$$

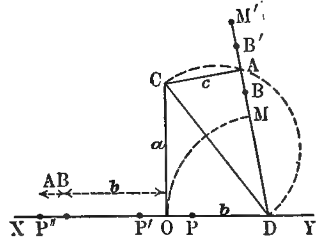

Fig. 279

First, the irrational part is constructed,

$$AD = \sqrt{\overline{CD}^2 - c^2} = \sqrt{a^2 + b^2 - c^2},$$

which is only possible when $a^2 + b^2 > c^2$.

Subtract b from AD, and the point B, in the middle of AM, gives:

$$AB = \frac{-b + \sqrt{a^2 + b^2 - c^2}}{2}.$$

This first value of x, considered as positive, is laid off from the origin O on OY and is equal to OP.

If $b > \sqrt{a^2 + b^2 - c^2}$, the point M would be at M', and we would have $x = \frac{AM'}{2} = AB'$. This value being negative, is laid off from O in the direction OX equal to OP'.

If the radical is preceded by the sign $-$, b is added to its value AD, and half of the line which results is the second value of x, which being negative is laid off in the direction OX from O.

Let it be required to construct

$$x = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} + a^2}. \qquad (a)$$

From the construction of the division of a in the extreme and mean ratio (971), we have the right triangle ABO (Fig. 280):

$$AO = \sqrt{\frac{a^2}{4} + a^2};$$

then

$$AI = AO - \frac{a}{2} = -\frac{a}{2} + \sqrt{\frac{a^2}{4} + a^2}.$$

This is the first value of x; it is positive, and is laid off in the positive direction AY from the origin A.

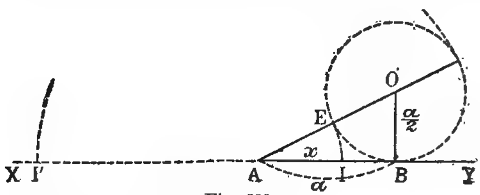

Fig. 280

The second value of x being

$$x = -\frac{a}{2} - AO,$$

it is negative, and is laid off from A in the negative direction AX.

The equation (a) becomes:

$$x = \frac{-a \pm \sqrt{5a^2}}{2} = \frac{-a \pm a\sqrt{5}}{2} = \frac{a}{2}(\pm \sqrt{5} - 1).$$

The two values of x represented by this expression are evidently the same as those represented by the expression (a), and are obtained by dividing a in the extreme and mean ratio.

THE GENERAL CONSTRUCTION OF CURVES REPRESENTED BY EQUATIONS.

1113. An equation between two variables, x and y, being given, if these variables are considered as coördinates, each pair of real values of x and y which satisfies the equation determines a point; varying x in a continuous manner between certain limits, the equation is ordinarily satisfied by real and continuous values of y, and then a continuous series of points, that is, a line, is obtained. Thus, in general, *an equation between two coördinates represents a line* (**1099**).

1114. To determine points of a curve, the values of x are ordinarily taken in arithmetical progression (**357**), and the corresponding values of y calculated from the equation. Above all, when the function is a whole algebraic function (**447**, **504**), it is wise to take this precaution, because, in order to shorten the computations, the differences between the successive values of y may be used in getting new values.

For example, let it be required to construct the equation $y = ax^2 + b$, a form which is met with in equations relative to the determination of the curve taken by the cables in suspension bridges. Suppose we have

$$a = 0.1 \quad \text{and} \quad b = 1, \quad \text{then} \quad y = 0.1\,x^2 + 1,$$

the following table shows that in giving successively to x the values 1, 2, 3, . . ., the values obtained for y are such that in taking their *first differences*, 0.1, 0.3, 0.5, . . ., the *second differences* between the first differences are equal. Thus, taking successively $x = 0$, $x = 1$, and $x = 2$, we have respectively $y = 1$, $y = 1.1$, and $y = 1.4$; the first differences are 0.1 and 0.3, and the constant second difference is 0.2. This second difference added to the last first difference gives the next following first difference, and each first difference added to the immediately preceding value of y gives the next following value of y; thus it is seen that by simple successive additions, the values of the first differences and then the values of the ordinates are obtained.

abscissas x . . .	0		1		2		3		4		5		6		7		8 . . .	
ordinates y . .	1		1.1		1.4		1.9		2.6		3.5		4.6		5.9		7.4 . . .	
1st differences . .		0.1		0.3		0.5		0.7		0.9		1.1		1.3		1.5 . . .		
2d differences . .		0.2		0.2		0.2		0.2		0.2		0.2		0.2		0.2 . . .		

The negative values of x would give the same values for y.

According as the function is of the 2d, 3d, 4th, . . ., degree, the constant differences are respectively the *second, third, fourth,* etc., differences, which are obtained by calculating from the equation, 3, 4, 5, . . ., ordinates, and taking their successive differences. Having the constant difference, the process is reversed as was done in the above example until the value of the next ordinate is obtained, and so on.

1115. Instead of calculating all the ordinates in constructing the curve $y = ax^2 + b$, the three first equidistant ordinates, AO, BP, CQ, may be calculated. Drawing the parallels AF and BG to the axis Ox, the two first differences, BH and CG, are determined, and prolonging AB, we have the second difference, $CC' = CG - C'G = CG - BH$, and is constant. To construct the fourth ordinate DR, prolong BC to D', and take $D'D = CC'$. In the same manner the next ordinate ES, and all ordinates following, may be constructed, and then joining the points A, B, C, D, etc., by a curve, we have the representation of the equation $y = ax^2 + b$.

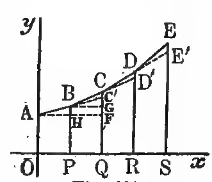

Fig. 281

1116. *Empiric functions.* In practice it happens daily that

observation or experiments furnish a series of corresponding values of two variables, without any algebraic equation to represent the law which governs these variables.

In this case, taking the values of one of the variables for abscissas and the corresponding values of the other variable for ordinates, and drawing a smooth curve through the points thus obtained, if the points are near enough together, this curve will represent with sufficient accuracy the law which governs these variables. Such a curve furnishes a picture of the observed phenomena; it may be used to find any intermediate points that were not directly observed; if it closely resembles some known curve, it may be expressed by an equation or formula known as *empiric;* any anomaly which breaks the continuity of the curve indicates an error in the observations or a peculiarity in the phenomena observed.

STRAIGHT LINE

1117. *The general equation of a straight line with reference to a rectangular coördinate system is*

$$y = ax + b.$$

Let any straight line AB be situated in the plane of the rectangular axis Ox and Oy.

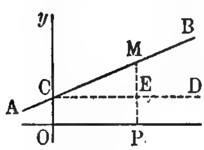

Fig. 282

From any point M on this line drop a perpendicular MP to the axis Ox; it determines the coördinates $MP = y$ and $OP = x$ of the point M. Through the point C where AB intersects the axis Oy, draw CD parallel to the axis Ox.

In the right triangle CME we have (1055),

$$ME = CE \times \tan BCD.$$

Adding EP to the two members of this equation, we have,

$$ME + EP = CE \times \tan BCD + EP.$$

Noting, first, that $ME + EP = y;$ second, that the angle BCD, which the line makes with CD or the axis Ox, is constant, and therefore its tangent, which may be represented by a, called an *angular coefficient*, or the *slope*, is also constant; third, that $CE = x$; fourth, that $EP = OC$ is also constant and may be

represented by b, called the *ordinate at the origin*, the preceding equation takes the form

$$y = ax + b,$$

which is the equation of a straight line, since it was established for any point in the line, and took into account the different signs which enter into the equation.

REMARK 1. When the straight line AB passes through the origin O, the ordinate at the origin $OC = b = 0$, and the equation becomes:

$$y = ax.$$

REMARK 2. When AB is parallel to Ox, the angle BCD is zero, then the tan $BCD = a = 0$ (1027), and the equation becomes:

$$y = b.$$

REMARK 3. In the case where $a = 0$ and $b = 0$, the equation becomes:

$$y = 0,$$

which indicates that the line coincides with the x-axis.

REMARK 4. If the line were parallel to the y-axis or coincided with it, its equation would be obtained by interchanging y and x in the last two equations given above. Thus, we would have

$$x = b \quad \text{and} \quad x = 0,$$

wherein b is no longer the ordinate at the origin, but the *abscissa at the origin.*

REMARK 5. It is seen that *the equation of a straight line is of the first degree* (510). Conversely, *any equation of the first degree between two variables is the equation of a straight line.* This is why straight lines are called *lines of the first degree.*

1118. *The equation of a straight line whose slope is given and passes through a point, the coördinates of which are x' and y'.*

For the point (x', y'), we have,

$$y' = ax' + b, \quad \text{and} \quad b = y' - ax'.$$

Substituting this value of b in the general equation, $y = ax + b$, we have,

$$y - y' = a\,(x - x').$$

1119. *The equation of a straight line passing through two given points* (x', y', *and* x'', y''). a being the unknown slope of the line, for the point (x', y'), we have (**1118**),

$$y - y' = a(x - x').$$

This equation should be satisfied by putting $y = y''$ and $x = x''$, which gives

$$y'' - y' = a(x'' - x').$$

Eliminating a by division, we have,

$$\frac{y - y'}{y'' - y'} = \frac{x - x'}{x'' - x'}.$$

If one of the points is on the x-axis, and the other on the y-axis, that is, if we have $x' = p$, $y' = 0$, and $y'' = q$, $x'' = 0$, the equation becomes:

$$\frac{y}{q} = \frac{x - p}{-p} \text{ or } \frac{x}{p} + \frac{y}{q} = 1.$$

If one of the points is at the origin, if for instance, $y'' = x'' = 0$, we have the equation of a straight line through the origin to a point (x', y'). Thus,

$$\frac{y - y'}{-y'} = \frac{x - x'}{-x'} \text{ or } \frac{y}{y'} = \frac{x}{x'}.$$

1120. *The intersection of two straight lines given by their equations.*

Any two lines, straight or curved, being given by their equations, by solving the system of two equations with x and y as the unknowns, which cease to be indeterminate variables, the values obtained are the coördinates of the points of intersection of the lines. Thus the point of intersection of two lines (**520**, **1117**) is

$$x = \frac{b' - b}{a - a'} \text{ and } y = \frac{ab' - a'b}{a - a'}.$$

Conversely, having a system of two equations involving two unknowns to solve, if the two lines represented by the equations are constructed, the coördinates of each point of intersection will be a solution of the system (**580**).

1121. *Two straight lines perpendicular to each other,* making

two angles with the x-axis whose difference is equal to 90°, the tangents of these angles give the relation in article (**1044**); from which it follows that

$$aa' = -1 \quad \text{or} \quad aa' + 1 = 0.$$

CIRCLE.

1122. The definition of a circle (**665**) may be expressed in polar coördinates. Thus, if we put (**1100**):

$$\rho = a\alpha + r,$$

and make $a = 0$ and r constant, we see that, no matter what the value of α, we always have

$$\rho = r,$$

an equation which is satisfied by any point in the circumference of a circle whose center is at the origin 0 and whose radius is r.

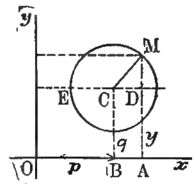

Fig. 283

1123. *General equation of a circle, with respect to a system of rectangular coördinates* (**1099**).

Let M be any point in the circumference of a circle whose center is C and whose radius is r.

Let $MA = y$ and $OA = x$, the coördinates of the point M, and $CB = q$ and $OB = p$, the coördinates of the center, which remain constant.

In the right triangle CDM (**730**):

$$\overline{MD}^2 + \overline{CD}^2 = r^2.$$

$$MD = y - q, \quad \text{or} \quad \overline{MD}^2 = y^2 + q^2 - 2\,qy; \qquad \textbf{(728)}$$

$$CD = x - p, \quad \text{or} \quad \overline{CD}^2 = x^2 + p^2 - 2\,px.$$

Adding the equations of $\overline{MD}^2$ and $\overline{CD}^2$ and replacing $\overline{MD}^2 + \overline{CD}^2$ by r^2,

$$y^2 + x^2 - 2\,qy - 2\,px + q^2 + p^2 = r^2,$$

or $$y^2 - 2\,qy = r^2 - x^2 + 2\,px - q^2 - p^2;$$

from which $$y = q \pm \sqrt{q^2 + r^2 - x^2 + 2\,px - q^2 - p^2}. \qquad \textbf{(572)}$$

Such is the general equation of the circle in rectangular coördinates.

When E is the origin and EC the x-axis, we have $q = 0$ and $p = r$, and the general equation becomes

$$y^2 + x^2 - 2\,rx + r^2 = r^2,$$

or $$y^2 = 2\,rx - x^2 \quad \text{and} \quad y = \pm\sqrt{2\,rx - x^2}.$$

If the center of the circle is at the origin, we have $q = 0$ and $p = 0$, and the equation becomes

$$y^2 + x^2 = r^2 \quad \text{and} \quad y = \pm\sqrt{r^2 - x^2}.$$

It is seen that in each of the three cases which we have just examined, two values of y correspond to each value of x; which is as it should be, since the equation of the circle is of the second degree. Furthermore, in the last two cases the values of y are equal and opposite in sign, which indicates that the curve is symmetrical with respect to the x-axis.

1124. *Draw a tangent to a circle at a point M taken on the circumference.*

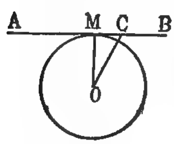

Fig. 284

Draw the radius OM, and the perpendicular AB at the extremity of this radius is the required tangent.

Proof. It suffices to prove that AB has only the point M in common with the circle, that is, that any point C on this line, other than M, is outside of the circle. Drawing OC, this line is oblique and greater than OM, which is a radius; therefore the point C is outside of the circle, and AB is the required tangent at the point M (**954**).

1125. Since AB is tangent to the circle, all its points except M are situated outside of the circle; therefore any straight line OC is greater than OM; therefore the radius OM, drawn to the point of contact, is perpendicular to the tangent (**620**), and consequently to the circumference (**678**). Thus, *to draw a normal at a certain point in the circumference, it suffices to connect this point to the center.*

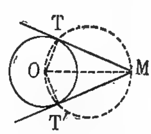

Fig. 285

1126. *Draw a tangent to a circle through a point M taken outside of the circle* (**954**).

Draw MO. On this line as a diameter describe a circumference which cuts the given circumference in the points T and T', then connecting these points with M, we have TM and $T'M$ as the required tangents.

Proof. Drawing the radii OT and OT', each of the angles OTM and $OT'M$ is a right angle, being inscribed in a semicircle (**684**), and the lines MT and MT', perpendicular to the tangent radii OT and OT' at their extremities, are tangent to the circle (**1124**).

ELLIPSE

1127. *The ellipse* is a curve such that the sum $MF + MF'$, of the distances of any point M to two fixed points, foci, F and F', is a constant quantity.

It is seen that an ellipse is defined by its equation in focal coördinates (1101). Designating the radius vectors of the points in the curve by the variables ρ and ρ', and the constant sum by $2a$, we have,

$$\rho + \rho' = 2a.$$

1128. As in the case of a circle (666), a portion of an ellipse is an *arc*, and the straight line which joins the extremities of the arc is a *chord*.

On an ellipse, and, in general, on any curve, an arc of one degree is one such that the normals erected at its extremities form an angle with each other of one degree. The chord AA', which passes through the foci, is the *major axis* of the ellipse.

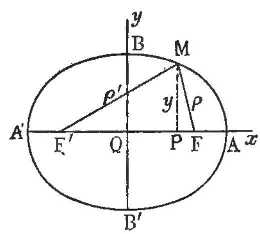

Fig. 286

The chord BB', which is the perpendicular bisector of the major axis, is the *minor axis* of the ellipse.

The point of intersection O of the two axes is the *center* of the ellipse.

Any chord which passes through the center is a *diameter* of the ellipse.

The extremities A, A', B, and B' of the axes are the *vertices* of the ellipse.

1129. *The foci are equally distant:*

1st. *From the vertices,* $AF = A'F'$ and $AF' = A'F$;

2d. *From the center,* $OF = OF'$.

1st. The vertices A and A' are part of the ellipse, the sums of their radius vectors are each equal to the constant $2a$ (1127), and consequently equal to each other; therefore

$$AF + AF' \text{ or } 2\,AF + FF' = A'F' + A'F \text{ or } 2\,A'F' + F'F.$$

Subtracting FF' from both members, we have $2\,AF = 2\,A'F'$, and $AF = A'F'$, and for the same reason $AF' = A'F$.

2d. Having $OA = OA'$ and $AF = A'F'$,

we also have, $OA - AF = OA' - A'F'$ or $OF = OF'$.

1130. *The constant sum $2a$ of the radius vectors is equal to the major axis.*

Since the point A is part of the ellipse, we have,

$$AF + AF' = 2a.$$

Replacing AF' by its equal $A'F$, we have,

$$AF + A'F = 2a = AA'.$$

1131. *The equation of an ellipse when the major and minor axes are taken as the coördinates* (**1099, 1128**).

Let $2a = AA'$, the major axis, and $2c = FF'$, the distance between the foci. We always have

$$2a > 2c \quad \text{or} \quad a > c.$$

In the right triangles MPF' and MPF, we have respectively,

$$\overline{MF'}^2 \text{ or } \rho'^2 = \overline{MP}^2 + \overline{PF'}^2, \text{ and } \overline{MF}^2 \text{ or } \rho^2 = \overline{MP}^2 + \overline{PF}^2.$$

Since $$MP = y,$$

$$PF' = OF' + OP = c + x, \text{ or } \overline{PF'}^2 = c^2 + x^2 + 2cx, \quad (727)$$

and $$PF = OF - OP = c - x, \text{ or } \overline{PF}^2 = c^2 + x^2 - 2cx. \quad (728)$$

Substituting these values in the formulas for ρ^2 and ρ'^2,

$$\rho'^2 = y^2 + x^2 + c^2 + 2cx \text{ and } \rho^2 = y^2 + x^2 + c^2 - 2cx. \quad (a)$$

Subtracting these two equations, we have

$$\rho'^2 - \rho^2 \text{ or } (\rho' + \rho)(\rho' - \rho) = 4cx;$$

from which $$\rho' - \rho = \frac{4cx}{\rho' + \rho} = \frac{4cx}{2a} = \frac{2cx}{a}.$$

Adding this equation to

$$\rho' + \rho = 2a,$$

we obtain $$2\rho' = \frac{2cx}{a} + 2a, \text{ from which } \rho' = \frac{cx}{a} + a,$$

and therefore $$\rho'^2 = \frac{c^2x^2}{a^2} + a^2 + 2cx. \quad (727)$$

Putting this value of ρ'^2 and the value in (a) equal to each other, and eliminating the denominator a^2,

$$a^2y^2 + a^2x^2 + a^2c^2 + 2a^2cx = c^2x^2 + a^4 + 2a^2cx.$$

Canceling the term $2a^2cx$ and grouping the terms,

$$a^2y^2 + (a^2 - c^2)x^2 = a^2(a^2 - c^2),$$

representing the constant $(a^2 - c^2)$ by b^2 (1133), we have for *the equation of the curve :*

$$a^2y^2 + b^2x^2 = a^2b^2 \quad \text{or} \quad \frac{y^2}{b^2} + \frac{x^2}{a^2} = 1;$$

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}; \tag{571}$$

which shows that for every value of x there are two equal values of y opposite in sign, and consequently the curve is symmetrical with respect to the x-axis. In expressing the value of x in terms of y, it will be seen that for every value of y there are two equal values of x opposite in sign, and consequently the curve is also symmetrical about the y-axis (1138).

REMARK. In the case where $a = b = r$ the equation of the ellipse becomes

$$y^2 + x^2 = r^2,$$

which is nothing other than the equation of a circle (1123).

Thus, the circle is a special case of the ellipse, in which the semi-axes are equal to the radius r. Therefore the properties of the ellipse are also those of the circle.

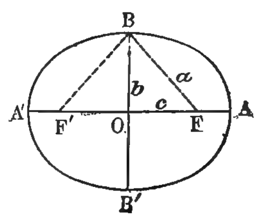

Fig. 287

1132. *The straight lines BF and BF', which join the extremities of the minor axis to the foci, are each equal to the semi-major axis a.*

These lines are equal since they cut off equal distances from the foot of the perpendicular BO (620) Furthermore, we have,

$$BF + BF' \quad \text{or} \quad 2\,BF = 2\,a \quad \text{and} \quad BF = a.$$

1133. Having $BF = a$, $OF = c$, if the semi-minor axis OB is represented by b, the right triangle BOF gives (730):

$$b^2 = a^2 - c^2.$$

Thus, *in the equation of the ellipse* (1131), *the constant quantity b is the semi-minor axis.*

1134. The distance $FF' = 2c$ between the foci is called the

focal distance, and the ratio $\frac{2c}{2a} = \frac{c}{a}$ of the focal distance to the major axis is called *the eccentricity of the ellipse.*

Designating this eccentricity by e, we have,

$$e = \frac{c}{a} = \sqrt{\frac{a^2 - b^2}{a^2}}.$$

The eccentricity of the ellipse lies always between 0 and 1; at the limit 0 the ellipse is a circle, and at the limit 1 the curve is flattened to a straight line joining the vertices and the foci.

1135. *The foci and one of the axes of an ellipse being given to find the other axis* (Fig. 287).

1st. AA' being the major axis, and F and F' the foci (1128), the perpendicular bisector BB' of AA' coincides with the minor axis; and if from one of the foci F as center and $AO = a$ as radius, an arc is described, it will cut BB' in the points B and B', which are the extremities of the minor axis (1132).

2d. If the minor axis BB' and the foci F and F' are given, to find the major axis, lay off to the right and left of the point 0 on FF', the distance $BF = a$.

1136. *The axes AA' and BB' of an ellipse being given, to find the foci* (Fig. 287). From one of the extremities B of the minor axis, with the semi-major axis for radius, describe an arc which cuts AA' in the points F and F', which are the foci of the ellipse (1132).

1137. *The ellipse is the geometrical locus of all the points the sum of whose radius vectors is equal to the major axis* $2a$ (609, 1130).

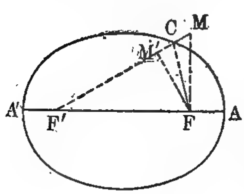

Fig. 288

1st. M being a point situated outside of the ellipse, we have $MF + MF' > 2a$. Drawing CF, the point C being on the ellipse, we have $CF + CF' = 2a$. Replacing CF by the greater quantity $MC + MF$, we have,

$$MF + MC + CF' \quad \text{or} \quad MF + MF' > 2a.$$

2d. The point M' being situated within the ellipse, we have,

$$M'F + M'F' < 2a.$$

Because drawing CF, the point C being on the ellipse, we have,

$$CF + CM' + M'F' = 2a.$$

Replacing $CF + CM'$ by a smaller quantity $M'F$, we have,

$$M'F + M'F' < 2a.$$

CoROLLARY. The converse statements of the above are also true.

1138. *The major and minor axis both divide the ellipse into two equal and symmetrical parts.*

1st. M being a point on the ellipse, its corresponding symmetrical point M' with respect to the major axis AA' (836) is also on the ellipse.

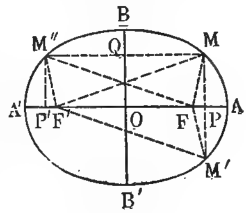

Fig. 289

This follows from the equation of the curve (1131); furthermore, the two equal right triangles, MPF and $M'PF$, giving $MF' = M'F'$, we have,

$$M'F + M'F' = MF + MF' = 2a,$$

and the point M' is on the ellipse (1137).

From this it follows, that if the part of the ellipse AMA' be turned about the axis AA', it would come into coincidence with the part $AM'A'$; therefore they are equal and symmetrical.

2d. The point M'', symmetrical to M with respect to the minor axis BB', is also on the ellipse. This follows directly from the equation, and may also be proved as follows: Having $OP = OP'$ as quantities each equal to $QM = QM''$, and $OF = OF'$, it follows that $FP' = F'P$ and $FP = F'P'$; and since $MP = M''P'$, the two equal right triangles MPF', $M''P'F$, give $MF' = M''F$, and the two other equal triangles MPF, $M''P'F'$, give $MF = M''F'$; it follows that

$$M''F + M''F' = MF' + MF = 2a;$$

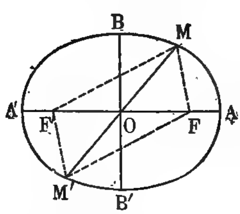

Fig. 290

therefore M'' is on the ellipse, and the ellipse is also divided into two equal and symmetrical parts by the minor axis.

1139. *The center of the ellipse divides all the diameters into two equal parts.*

The point M being on the ellipse, prolonging MO to M', making $M'O = MO$, and drawing MF, MF', $M'F$, and $M'F'$, in the quadrilateral $MFM'F'$, the diagonals cutting each other in two equal parts, the figure is a parallelogram (660), and we have,

$$M'F + M'F' = MF + MF' = 2a.$$

Therefore the point M' is on the ellipse (**1137**), and MM' is a diameter divided into two equal parts at the point 0.

1140. *Any diameter MM', other than the major and minor axes, divides the ellipse into two equal parts but not symmetrical with respect to that diameter* (837).

Bringing the part MBM' upon the part $M'B'M$ by turning it about O as a center until M coincides with M' and M' with M, and considering any diameter BB', after the change, the part OB will coincide with the part OB', since the angle $BOM = B'OM'$, and since $OB = OB'$, the point B will coincide with the point B'. The point B being any point, it is seen that all the points on the part MBM' fall upon the curve $M'B'M$; therefore any diameter divides the ellipse into two equal parts.

1141. From the equation of the ellipse (**1131**), we may deduce, that for any point M,

$$\frac{y^2}{a^2 - x^2} \text{ or } \frac{y^2}{(a+x)(a-x)} = \frac{b^2}{a^2}, \qquad (a)\ (729)$$

or noting that $a + x = A'P$ and $a - x = AP$,

$$\frac{y^2}{AP \times A'P} = \frac{b^2}{a^2},$$

which shows that *the ratio of the square of an ordinate to the product of the corresponding segments of the major axis is equal to the*

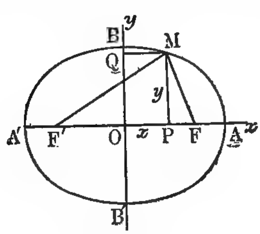

Fig. 291

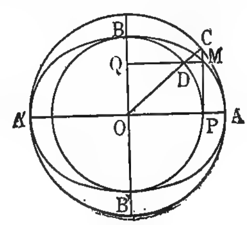

Fig. 292

ratio of the squares of the minor and major axes, and therefore this ratio is constant.

For another point, we would have,

$$\frac{y'^2}{AP' \times A'P'} = \frac{b^2}{a^2};$$

and then

$$\frac{y^2}{y'^2} = \frac{AP \times A'P}{AP' \times A'P'}.$$

Thus the squares of the ordinates are to each other as the products of the corresponding segments of the major axis.

From the equation of the ellipse, and by the same process of reasoning, the same properties are found for the abscissas and the corresponding segments of the minor axis:

$$\frac{x^2}{BQ \times B'Q} = \frac{a^2}{b^2}, \qquad \frac{x^2}{x'^2} = \frac{BQ \times B'Q}{BQ' \times B'Q'}.$$

1142. Describing a circle on the major axis as diameter, and drawing any corresponding ordinates $MP = y$ and $CP = Y$ of the ellipse and of this circle (Fig. 292), we have,

$$\frac{y}{Y} = \frac{b}{a}.$$

Proof. The right triangle OPC gives

$$\overline{OC}^2 - \overline{OP}^2 = \overline{CP}^2 \quad \text{or} \quad a^2 - x^2 = Y^2.$$

Substituting in equation (a) of the preceding article, we have,

$$\frac{y^2}{Y^2} = \frac{b^2}{a^2} \text{ or } \frac{y}{Y} = \frac{b}{a}. \tag{a}$$

Describing a circle upon the minor axis, the same relation is found to hold, thus,

$$\frac{MQ}{DQ} \text{ or } \frac{x}{X} = \frac{a}{b}. \tag{b}$$

1143. From the equation (a) of the preceding article, we may consider *any ellipse having* $2\,a$ *and* $2\,b$ *for its axes, as being a projection of a circle of the diameter* $2\,a$ *upon the plane of the ellipse,* and from the equation (b) *that any circle of the diameter* $2\,b$ *may be considered as being the projection on its plane of different ellipses having a common minor axis* $2\,b$.

From these relations, diverse interesting consequences relative to the supplementary chords, to the conjugate diameters, to the circumscribed parallelograms, and to the area of the ellipse, may be deduced (11 2).

Thus the ellipse $ABA'B'$, which has $AA' = 2\,a$ and $BB' = 2\,b$ for its axes, is the projection of a circle $aba'b'$, having $2\,a$ for its diameter, and its plane making an angle θ, whose cosine is $\frac{b}{a}$, with the plane of the ellipse.

The diameter aa' being parallel to the plane of the ellipse, its true length is projected upon AA', and for any point m the projection of the perpendicular mp to aa' is

$$MP = mp, \cos\theta = mp \times \frac{b}{a};$$

from which it follows that M is part of an ellipse whose major axis is AA' and minor axis is

$$BB' = \overline{bb'} \cos\theta = \overline{aa'} \times \frac{b}{a}.$$

Each of the elements mp of a circle having its surface multiplied by $\cos\theta$ for its projection, the projection of the entire circle is equal to the surface of the circle multiplied by $\cos\theta$. This is not only true of the projection of a circle, but also of the *projection of any plane surface.*

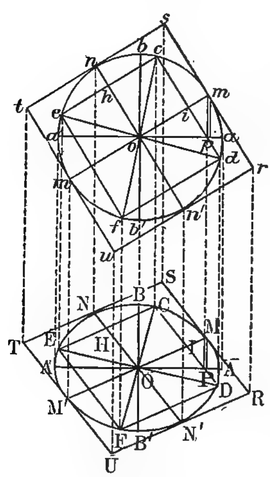

Fig. 293

Drawing any diameter de of the circle, and the two chords cd and ce, which are perpendicular to each other (684), the diameters mm', nn', which pass through the middle points i and h of these chords, are also perpendicular to each other, and each one divides all the chords, which are parallel to the other, into two equal parts. Moreover, the tangents m, n, m', n' form a circumscribed square $rstu$.

Projecting these lines which have just been discussed upon the plane of the ellipse, and representing the different points by the same letters, written as capitals, the chords CD and CE, which start from the same point C in the curve and end at the extremities of the same diameter DE, are called *supplementary chords;* the diameters MM', NN', are parallel to these chords, and each divides into two equal parts all chords parallel to the other, which property gives them the name of *conjugate diameters;* moreover, the projection of the square $rstu$ is a circumscribed parallelogram $RSTU$, the sides of which are parallel to the conjugate diameters MM' and NN' passing through the points of contact; finally, the

square *rstu* being constant, the area of the circumscribed parallelogram $RSTU$ is also constant and equal to $4a^2 \cos\theta = 4a^2 \frac{b}{a} = 4ab$.

1144. Thus, *in an ellipse, any diameter MM' which divides a chord AA' into two equal parts, divides all the chords NN', BB' . . ., parallel to the first, in the same manner,* and the two diameters MM' and NN' of the ellipse are said to be *conjugate diameters* when each divides into two equal parts the chords parallel to the other.

1145. *A diameter MM'* (Fig. 294) *being given, find its conjugate.* Draw a chord CC' parallel to MM', and, drawing the diameter NN' through the middle point D of the chord CC', we have the conjugate diameter of MM'.

When the major axis of the ellipse (Fig. 295) is known, drawing a chord AB parallel to MM' through its extremity A and

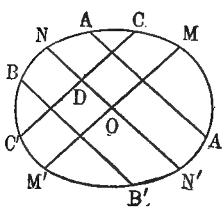

Fig. 294

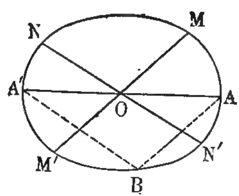

Fig. 295

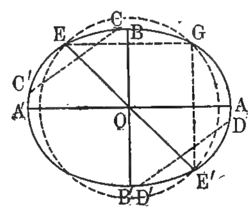

Fig. 296

joining B to A', the diameter NN' parallel to BA' is the conjugate of MM' (1143). This construction is more simple than the preceding one.

1146. *An ellipse being given, to determine: first, its center; second, its axes; third, its foci.*

1st. Drawing two parallel chords CC' and DD', the straight line EE' which joins the middle points of these chords is a diameter, the middle point O of which is the center of the ellipse.

2d. From the center O, describe a circle with a radius sufficiently long to cut the ellipse in four points; then the line which joins E and G and the line which joins G and E' are respectively parallel to the major and minor axes, and these axes may be drawn.

3d. Having the axes, the foci are determined as in article (1136).

To determine the center of an arc of an ellipse, inscribe two parallel chords in the arc; draw a line through the middle points of

these chords, then this line having the direction of a diameter, will pass through the center. Now by drawing in two new chords parallel to each other, and repeating the first construction, the intersection of the two bisectors will give the center of the ellipse.

In case the arc is long enough, so that the circle GEE' can cut it in two points G and E or G and E', the major or minor axis may be drawn, and, erecting a perpendicular to this axis at the center, the second axis is obtained.

1147. From article (1142) *an easy method of constructing an ellipse by points* may be deduced.

Describing circles on the axes as diameters (Fig. 292), drawing any radius OC and through the points C and D drawing parallels to the axes, these parallels meet at a point M on the ellipse. Thus, MP being parallel to OP, we have,

$$\frac{MP}{CP} = \frac{OD}{OC} \quad \text{or} \quad \frac{y}{Y} = \frac{b}{a}.$$

It is evident that as many such points may be constructed as desired, and when enough have been determined and connected by a smooth curve we have an ellipse (1099).

1148. *Another method by points* (1147).

AA' being the major axis of an ellipse, and F and F' the foci,

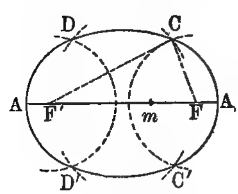

Fig. 297

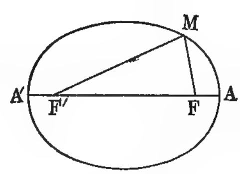

Fig. 298

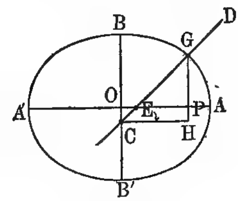

Fig. 299

with F and F' as centers and a radius equal to Am, which may vary from AF to AF', describe two arcs; then from the same centers with a radius equal to $A'm$ describe arcs which cut the first arcs in the points D, D', C, and C', which are on the ellipse, because $Am + A'm = AA' = 2a$ (1137). In this manner as many points may be determined as is desired, and a smooth curve connecting them is the required ellipse.

1149. *A method used by gardeners for constructing an ellipse* (Fig. 298).

AA' being the major axis, and F and F' the foci, fasten the ends

of a cord at F and F', making the length of the cord $FM + MF'$ equal to the major axis AA'; hold the cord taut with a pointed stick at M, and walk around making a mark in the soil with the stick. If the cord is held taut, the sum of the radius vectors FM and $F'M$ is always constant and equal to the major axis AA', and we have an ellipse. The same method may be used on paper by substituting a pencil for the sharp stick (1137).

1150. *Construction of an ellipse with a rule* (Fig. 299).

Marking three points C, E, and G, on the edge of a thin rule, such that $CG = OA = a$ the semi-major axis, and $EG = OB = b$ the semi-minor axis, from which $CE = a - b$; moving the rule in such a way that the point E remains constantly upon AA', and C upon BB', G will follow the curve of the ellipse whose major and minor axes are respectively AA' and BB'.

This method is used for constructing the intrados and extrados of arches which have the form of an ellipse.

The point G follows the curve of an ellipse because, drawing CH parallel to OA, we have,

$$\frac{GP}{GH} = \frac{GE}{GC} \text{ or } \frac{y}{\sqrt{a^2 - x^2}} = \frac{b}{a};$$

and

$$y = \frac{b}{a}\sqrt{a^2 - x^2}. \qquad \textbf{(1131)}$$

If from the point C as center, $CE = a - b$ for radius, an arc of a circle had been described, the point E would have been determined; then drawing CE and prolonging it to G, making $EG = b$, the point G upon the ellipse would have been found.

1151. *The elliptic-compasses* are constructed according to the principle demonstrated in the preceding article, and permit the construction of an ellipse by a continued motion. It consists of two slots or guides assembled in the form of a cross (Fig. 300) so that they may be made to coincide with the axes AA' and BB' of the ellipse; a rod CD carrying two slides

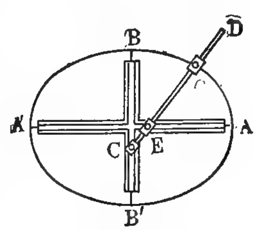

Fig. 300

E and G, which may be fastened at any two points. The slide E carries a pivoted foot, which fits in the slot AA' and G a point or pencil which traces the curve when the rod is moved. At the extremity of the rod is another pivoted foot, which is fixed and

slides in the slot BB'. Having fixed the slides in such a manner that $CG = OA$ and $EG = OB$, and placed the instrument so that the guides coincide with the axes of the ellipse, an ellipse is traced by turning the rod CD.

The middle point of CE describes a circle about the center O. Therefore if this point is joined to the center by a link, one of the feet C or E may be left off.

1152. Instead of spacing the points on the rule in such a manner as was done in article (1150), we may take $GC = b$ and $GE = a$ (Fig. 301). Moving the rule so that C follows the major axis and E the minor axis, the point G will describe an ellipse.

$$\frac{GP}{EQ} = \frac{GC}{GE} \text{ or } \frac{y}{\sqrt{a^2 - x^2}} = \frac{b}{a}. \tag{1131}$$

1153. *Draw a tangent to an ellipse through a point M taken on the curve.*

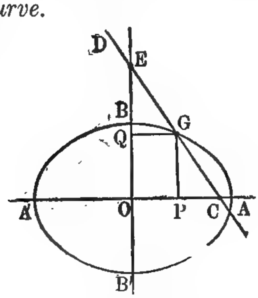

Fig. 301

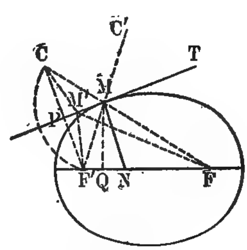

Fig. 302

Draw the radius vectors MF' and MF, prolonging the latter so that $MC = MF'$; draw CF', and the perpendicular TP dropped from the point M on CF' is tangent to the ellipse, that is, that any other point M' taken on TP lies outside the ellipse.

Proof. Joining $M'F$, $M'F'$, and $M'C$, the triangle MCF' being isosceles, the straight line MP is perpendicular to CF' at its middle point, and we have $M'F' = M'C$. In the triangle FCM', we have $M'F + M'C$ or $M'F + M'F' > CF$ or $MF + MF'$ or $2a$; therefore the point M′ is situated outside the ellipse (1137), and TP is the tangent at the point M.

REMARK 1. The tangent TP bisects the angles, formed by each radiu vector with the prolongation of the other.

Proof. The triangle $F'MC$ being isosceles, the perpendicular

MP bisects the angle $F'MC$ at the vertex and also its vertical angle FMC'.

REMARK 2. The preceding method for drawing a tangent to an ellipse, and those which follow, except that in (1159), do not require that the ellipse be constructed. This is a great advantage where the ellipse is constructed by points; because, as soon as a point is found, its tangent may be drawn, and in this manner the curve is blocked out, making it possible to draw it in with a lesser number of points.

1154. *Draw a normal to an ellipse at a point M* (Fig. 302).

Join M to the foci, then the bisector MN of the angle formed by the two radius vectors is normal to the ellipse, that is, perpendicular to the tangent TM (678, 946).

Proof. The angles CMF' and $C'MF$ being equal, their halves are equal, and we have $PMF' = TMF$; since $F'MN = NMF$, adding these two equations, we obtain $PMN = NMT$; therefore MN is perpendicular to TM (614).

Prolonging MN to FF', and projecting the point M on FF', the projection NQ of MN on FF' is called a *subnormal.*

Since, when radius vectors FM and $F'M$ are drawn from any point M, the angle of incidence formed with the tangent is equal to the angle of reflection (950), it follows that on an elliptic billiard table, a ball shot from one focus to any point on the cushion will pass through the other focus, then, after touching the cushion the second time, will pass through the first focus, and so on. The same is true of rays of heat or light which radiate from one focus of an elliptical mirror.

Because of this reciprocal action of each focus they are called *conjugate foci.*

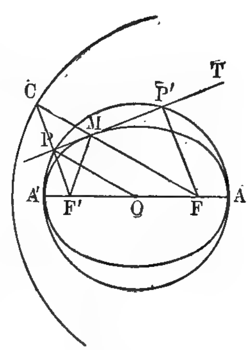

Fig. 303

1155. MT being the tangent to the ellipse at the point M, drawn according to the construction in article (1153), and O the center of the ellipse, in the triangle $FF'C$, the straight line OP bisects FF' and CF', and we have $OP = \dfrac{FC}{2} = a$ (699) ; which shows that the circle described on the axis AA', as diameter, passes through the point P, and is *the geometrical locus of the projections P, P', of the foci on the tangents* (609, 715).

Describe a circle from the focus F as a center, with $AA' = 2a$ for a radius, then drawing any radius FC we have $MC = MF'$.

Therefore *an ellipse may be defined as a curve such that all its points are equally distant from the circumference of a circle* (670) *and a fixed point within the circle.*

From this definition a method of constructing the ellipse by points may be deduced (1147 to 1152).

The circle described on the major axis AA' as diameter is often called the *principal circle of the ellipse*, and the one described from one of the foci as centers, with the major axis $FC = AA'$ for radius, is called the *directrix circle.*

From that which was said above, in order to *draw a tangent to an ellipse at the point M* (Fig. 303), describe a circle on AA' as

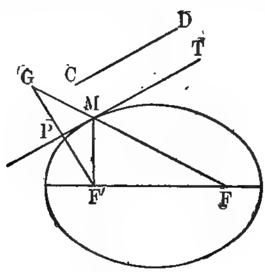

Fig. 304

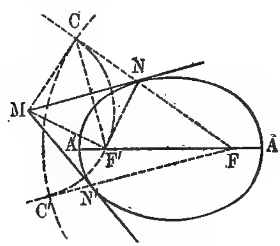

Fig. 305

diameter, and another having F as center and AA' for its radius; draw the radius FC passing through M, then CF' which will cut the circumference of the principal circle in P, and joining M to P, we have the required tangent.

1156. *To draw a tangent to an ellipse parallel to a given straight line CD.*

From the focus F' draw $F'G$ perpendicular to CD; from the other focus F with a radius $FG = 2a$ (1131), describe an arc which determines the point G; drawing FG, we have the point of contact M; the required tangent is now obtained by drawing a parallel to CD through M, or a perpendicular to $F'G$ through M.

To draw a tangent to an ellipse making any given angle with a given line, draw a tangent parallel to a line which makes the required angle with the given line (955).

1157. *Draw a tangent to an ellipse through a point M taken outside of the ellipse.*

From the point M as center, and with the distance from the point M to the nearest focus F' as radius, describe an arc; from the other focus F, with the major axis of the ellipse as radius, describe a second arc, which cuts the first in the points C and C'; draw CF and $C'F$, which determine the points N and N', and drawing MN, MN', these lines are tangent to the ellipse at N and N'.

Proof. From $NF' + NF = CF$, major axis, we have $NF' = NC$; since $MF' = MC$, the line MN is perpendicular to CF' at its middle point (621), and bisects the angle $F'NC$; therefore it is tangent to the ellipse at the point N (1153).

1158. Noting (Fig. 293) that the point of meeting of the tangent to the ellipse at M with the axis AA' is the projection of the point of meeting of the tangent to the circle at m with the diameter aa', it follows (Fig. 306) that all ellipses having the same major axis AA', and the circle which has this major axis as its diameter, have the following property: namely, that the tangents drawn through the points M, N, ... where a plane perpendicular to the major axis cuts the ellipses and the circle, meet in the same point T on a prolongation of the major axis.

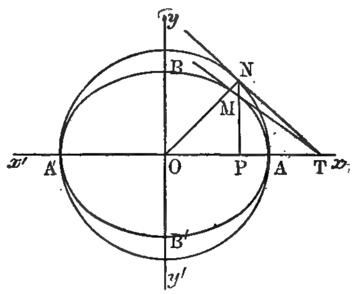

Fig. 306

This property reduces the difficulty of drawing a tangent to an ellipse to that of drawing one to a circle (1124, 1126). When the point through which the tangent is to be drawn is outside of the ellipse, it should be on the axis xx'.

In the right triangle ONT, we have (705),

$$OP : ON = ON : OT, \text{ and } OT = \frac{\overline{ON}^2}{OP},$$

which determines the point T where the tangent to the ellipse at the point M meets the prolongation of the major axis AA'.

Describing a circle on the minor axis BB', the tangent drawn to this circle at the point where it cuts ON and the tangent TM meet in the same point on the prolongation of the minor axis BB'. In this case the two points of contact lie in the same

plane perpendicular to the minor axis BB'. From this follows a method, analogous to the preceding one, for drawing a tangent to an ellipse at the point M. These two methods taken together give the points where the tangents to the ellipse meet the two axes; which may be used to verify the correctness of the construction of the ellipse.

1159. *Another method of drawing a tangent to an ellipse at a point taken on the curve.*

Through the point M draw two chords MC and MD; through the points C and D draw two others CE and DG parallel to MD and MC respectively, then the parallel MT to EG drawn through M is the required tangent.

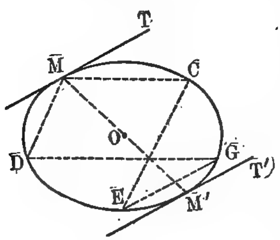

Fig. 307

Drawing the diameter MM', the chord EG and all parallel to it are bisected; MT is therefore parallel to the conjugate diameter of MM' (1141), which is still another method of determining the direction of MT; but it is easier to construct EG than the conjugate diameter of MM'.

Remark. The parallel $M'T'$ to EG or MT is also tangent to the ellipse. Thus, as is the case with the circle, tangents drawn at the extremities of the diameter of an ellipse are parallel (1143).

1160. *Two ellipses are said to be similar* when their axes are proportional, that is, when $a : a' = b : b'$ (1131).

As is the case for two similar polygons (695) or circles (749, 750), if two ellipses are similar, the ratio of their axes is equal to the ratio of any homologous linear dimensions straight or curved.

The surfaces of two similar ellipses are to each other as the squares of their axes ($s : s' = a^2 : a'^2$), and in general as the squares of any homologous linear dimensions.

Two portions of similar ellipses whose perimeters are formed of homologous lines are also similar, and their surfaces are to each other as the surfaces of the ellipses.

Similar ellipses have the same eccentricity e, since we have $c : a = c' : a'$ (1134).

1161. *The length of an ellipse or an arc of an ellipse* is not given exactly by any elementary geometrical construction (951); but, considering the ellipse or arc to be made up of a series of

very short straight lines, the length is equal to the sum of these lines (1111).

l being the length of a semi-ellipse, whose major and minor axes are respectively a and b, we have,

$$l = \pi a \left[1 - \left(\frac{1}{2} e\right)^2 - \frac{1}{3}\left(\frac{1 \cdot 3}{2 \cdot 4} e^2\right)^2 - \frac{1}{5}\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} e^3\right)^2 - \cdots\right],$$

in which e is the eccentricity of the ellipse (1134):

$$e = \frac{c}{a} = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{(a+b)(a-b)}{a^2}}. \qquad (729)$$

When $a = b = r$, we have $e = 0$, and therefore $l = \pi r$; which is as it should be, since the semi-ellipse becomes a semi-circle (752).

e being put in the form $\sqrt{\frac{(a+b)(a-b)}{a^2}}$, with the aid of logarithms, the value of e is easily computed; and letting σ represent the sum of the quantities within the parentheses, we have,

$$l = \pi a (1 - \sigma);$$

taking $a = 1$,

$$l = \pi \ (1 - \sigma).$$

This gives the value of l with sufficient approximation, and is used in calculating the values in the fourth column of the following table. Multiplying these tabular values by a expressed in feet or inches, we obtain l in feet or inches.

Taking the axes of the ellipse as coördinate axes (Fig. 292), x being the abscissa $MQ = OP$, and y the ordinate MP, of any point M on the curve, and calling the angle corresponding to COA, θ, we have,

$$\cos\theta = \frac{x}{a} \text{ and } \sin\theta = \frac{y}{b} \text{ or } y = b\sin\theta.$$

For the point M, the value of the *subnormal* (1154) is:

$$s = \frac{b^2}{a}\cos\theta.$$

The slope of the normal with reference to the x-axis, designating the angle which the normal makes with the axis as α (1030), is:

$$\tan\alpha = \frac{y}{s} = \frac{b\sin\theta}{\frac{b^2}{a}\cos\theta} = \frac{a}{b}\tan\theta.$$

It is useful to know this slope in constructing elliptical arches (1150).

EXAMPLE. Having $a = 15$ ft., and $b = 10$ ft., for a point in the curve whose abscissa is $x = 11.49060$ ft., we have,

$$\cos \theta = \frac{11.49060}{15} = 0.76604.$$

From the table, the value of $\sin \theta$ which corresponds to this $\cos \theta$ is

$$\sin \theta = 0.64279.$$

Therefore, we have $y = 10 \times 0.64279 = 6.4279$ ft. In this manner any number of points may be determined and the curve drawn.

The subnormal is

$$s = \frac{100}{15} \times 0.76604 = 5.10693,$$

and *the slope of the normal* is

$$\tan \alpha = \frac{15}{10} \times 0.83910 = 1.25865.$$

Having $\tan \alpha$, the table (**1071**) gives $\alpha = 51° 32'$.

If the ratio $\frac{x}{a} = \cos \theta$ were not contained in the following table the table (**1071**) could be used. Having $a = 15$ ft., and $b = 10$ ft., l may be obtained as follows:

Putting $$\frac{b}{a} = \cos \theta = \frac{10}{15} = 0.66667.$$

The angle θ is constant for a given ellipse, and is equal to the angle COA (Fig. 292), wherein $OP = OB = b$ (**1143**).

When $a = 1$ we have $b = \cos \theta$. This is indicated in the second column of the table.

Taking $a = 1$, the table gives the length l of the perimeter of the semi-ellipse by interpolation (**755**):

$$2.64768 - 0.01823 \frac{0.66913 - 0.66667}{0.66913 - 0.65606} = 2.64768 - 00343 = 2.64425.$$

Therefore, in feet, we have,

$$l = 15 \times 2.64425 = 39.66375 \text{ ft.}$$

Remark. If, instead of b, the semi-focal distance c (**1134**) had been given, we would have,

$$\frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \cos^2 \theta} = \sin \theta.$$

Then the table would give the value of l corresponding to $\sin \theta$ when $a = 1$.

Designating the *radius of curvature* at the point whose abscissa is x, by ρ, we have,

$$\rho = \frac{a^2}{b}\left(1 - \frac{c^2x^2}{a^4}\right)^{\frac{3}{2}};$$

or, putting $\frac{b}{a} = \sin \alpha$ or $\frac{c}{a} = \cos \alpha$, and $\frac{c}{a} \times \frac{x}{a} = \cos \beta$,

we have, $$\rho = \frac{a \sin^3 \beta}{\sin \alpha}.$$

Designating the *abscissa of the center of curvature* by x', we have,

$$x' = \frac{c^2x^3}{a^4}.$$

Table for the construction of the ellipse by points, for the determination of the normal at any of these points and the calculation of the semi-perimeter.

Angle θ	$b = \cos \theta$	$\sin \theta$	$\frac{1}{2}$ perim. l for $a = 1$	Differences.	Angle θ	$b = \cos \theta$	$\sin \theta$	$\frac{1}{2}$ perim. l for $a = 1$	Differences.
0°	1.00000	0.00000	3.14159	0.00024	45°	0.70711	0.70711	2.70128	0.01767
1	0.99985	0.01745	3.14135	0.00072	46	0.69466	0.71934	2.68361	0.01787
2	0.99939	0.03490	3.14063	0.00120	47	0.68200	0.73135	2.66573	0.01806
3	0.99863	0.05234	3.13944	0.00167	48	0.66913	0.74314	2.64768	0.01823
4	0.99756	0.06976	3.13776	0.00215	49	0.65606	0.75471	2.62945	0.01838
5	0.99619	0.08716	3.13561	0.00262	50	0.64279	0.76604	2.61107	0.01852
6	0.99452	0.10453	3.13299	0.00310	51	0.62932	0.77715	2.59255	0.01865
7	0.99255	0.12187	3.12989	0.00357	52	0.61566	0.78801	2.57390	0.01876
8	0.99027	0.13917	3.12632	0.00404	53	0.60182	0.79864	2.55514	0.01885
9	0.98769	0.15643	3.12228	0.00451	54	0.58779	0.80902	2.53629	0.01894
10	0.98481	0.17365	3.11777	0.00498	55	0.57338	0.81915	2.51735	0.01899
11	0.98163	0.19081	3.11279	0.00544	56	0.55919	0.82904	2.49836	0.01904
12	0.97815	0.20791	3.10736	0.00590	57	0.54464	0.83867	2.47932	0.01907
13	0.97437	0.22495	3.10146	0.00635	58	0.52992	0.84805	2.46025	0.01908
14	0.97030	0.24192	3.09510	0.00680	59	0.51504	0.85717	2.44117	0.01906
15	0.96593	0.25882	3.08830	0.00726	60	0.50000	0.86603	2.42211	0.01904
16	0.96126	0.27564	3.08104	0.00771	61	0.48481	0.87462	2.40307	0.01898
17	0.95630	0.29237	3.07333	0.00814	62	0.46947	0.88295	2.38409	0.01892
18	0.95106	0.30902	3.06519	0.00858	63	0.45399	0.89101	2.36517	0.01882
19	0.94552	0.32557	3.05661	0.00902	64	0.43837	0.89879	2.34625	0.01870
20	0.93969	0.34202	3.04759	0.00944	65	0.42262	0.90631	2.32765	0.01856
21	0.93358	0.35837	3.03815	0.00986	66	0.40674	0.91355	2.30909	0.01840
22	0.92718	0.37461	3.02829	0.01028	67	0.39073	0.92050	2.29069	0.01821
23	0.92050	0.39073	3.01801	0.01069	68	0.37461	0.92718	2.27248	0.01799
24	0.91355	0.40674	3.00732	0.01110	69	0.35837	0.93358	2.25449	0.01774
25	0.90631	0.42262	2.99622	0.01149	70	0.34202	0.93969	2.23675	0.01747
26	0.89879	0.43837	2.98473	0.01188	71	0.32557	0.94552	2.21928	0.01716
27	0.89101	0.45399	2.97285	0.01227	72	0.30902	0.95106	2.20212	0.01682
28	0.88295	0.46947	2.96058	0.01265	73	0.29237	0.95630	2.18530	0.01645
29	0.87462	0.48481	2.94793	0.01302	74	0.27564	0.96126	2.16885	0.01604
30	0.86603	0.50000	2.93492	0.01337	75	0.25882	0.96593	2.15281	0.01560
31	0.85717	0.51504	2.92154	0.01373	76	0.24192	0.97030	2.13721	0.01510
32	0.84805	0.52992	2.90781	0.01408	77	0.22495	0.97437	2.12211	0.01456
33	0.83867	0.54464	2.89373	0.01441	78	0.20791	0.97815	2.10755	0.01398
34	0.82904	0.55919	2.87932	0.01474	79	0.19081	0.98163	2.09357	0.01335
35	0.81915	0.57338	2.86458	0.01506	80	0.17365	0.98481	2.08022	0.01265
36	0.80902	0.58779	2.84952	0.01538	81	0.15643	0.98769	2.06757	0.01189
37	0.79864	0.60182	2.83414	0.01567	82	0.13917	0.99027	2.05568	0.01106
38	0.78801	0.61566	2.81847	0.01596	83	0.12187	0.99255	2.04462	0.01015
39	0.77715	0.62932	2.80251	0.01623	84	0.10453	0.99452	2.03447	0.00915
40	0.76604	0.64279	2.78628	0.01651	85	0.08716	0.99619	2.02532	0.00803
41	0.75471	0.65606	2.76977	0.01677	86	0.06976	0.99756	2.01729	0.00678
42	0.74314	0.66913	2.75300	0.01701	87	0.05234	0.99863	2.01051	0.00535
43	0.73135	0.68200	2.73599	0.01724	88	0.03490	0.99939	2.00516	0.00366
44	0.71934	0.69466	2.71875	0.01747	89	0.01745	0.99985	2.00150	0.00150
45	0.70711	0.70711	2.70128		90	0.00000	1.00000	2.00000	

Table of the perimeters, of ellipses, whose minor axes $2b$ *are all equal to* 100.

This second table is less rigorous in the decimal part, but gives the required results more directly.

Major Axis. $2a$	Perimeter. $2l$	Major Axis. $2a$	Perimeter. $2l$	Major Axis. $2a$	Perimeter. $2l$
101	315.7478	350	762.0212	680	1400.0412
102	317.3364	360	780.9768	690	1419.6200
103	318.9249	370	799.9512	700	1439.2084
104	320.5135	380	819.0084	710	1458.8072
105	322.1021	390	838.0740	720	1478.4116
106	323.6907	400	857.1708	730	1498.0284
107	325.2792	410	876.2972	740	1517.6476
108	326.8678	420	895.4524	750	1537.2756
109	328.4564	430	914.6324	760	1556.9120
110	330.0450	440	933.8376	770	1576.5548
120	346.2680	450	953.0668	780	1596.2048
130	362.7856	460	972.3192	790	1615.8624
140	379.5624	470	991.5944	800	1635.5248
150	396.5712	480	1010.8896	810	1655.1948
160	413.7792	490	1030.2064	820	1674.8704
170	431.1732	500	1049.5404	830	1694.5504
180	448.7276	510	1068.8901	840	1714.2392
190	466.4488	520	1088.2616	850	1733.9332
200	484.2652	530	1107.6492	860	1753.6321
210	502.2223	540	1127.0492	870	1773.3359
220	520.2924	550	1146.4672	880	1793.0446
230	538.4560	560	1165.8968	890	1812.7580
240	556.7612	570	1185.3452	900	1832.4772
250	575.0624	580	1204.8044	910	1852.2020
260	593.4832	590	1224.2776	920	1871.9300
270	611.9944	600	1243.7604	930	1891.6640
280	630.5401	610	1263.2568	940	1911.4004
290	649.1640	620	1282.7656	950	1931.1452
300	667.8392	630	1302.2852	960	1950.8916
310	686.5904	640	1321.8172	970	1970.6404
320	705.3808	650	1341.3571	980	1990.3943
330	724.2152	660	1360.9096	990	2010.1525
340	743.0984	670	1380.4708	1000	2029.9192

EXAMPLE. For $2a = 30$ ft., and $2b = 20$ ft., making $2b = 100$, we have,

$$2a = 100\frac{30}{20} = 150.$$

For this value of $2a$ the table gives,

$$2l = 396.5712.$$

Therefore the value in feet is

$$2l = 396.5712 \times \frac{20}{100} = 79.31424 \text{ feet, or } l = 39.65712 \text{ feet,}$$

which is not greatly different from that obtained from the first table.

1162. *Surface of the ellipse.* Since we may consider an ellipse whose major axis is $2a$ and minor axis $2b$, as being a projection

of a circle whose diameter is $2a$, upon the plane of the ellipse; the angle between the plane of the circle and that of the ellipse being θ and $\cos\theta = \frac{b}{a}$ (1143), the area S of the surface of the ellipse is,

$$S = S' \cos\theta = \pi a^2 \frac{b}{a} = (\pi ab),$$

wherein S' is the area of the circle.

For $a = 3$ ft., and $b = 2$ ft., we have,

$$S = 3.1416 \times 3 \times 2 = 18.85 \text{ sq. ft.}$$

Thus we have $S : S' = b : a$.

Therefore the surface of an ellipse is equivalent to that πr^2 of a circle the radius of which is a mean proportional between the semi-major axis a and the semi-minor axis b, that is, $r^2 = ab$ (753, 970).

When the two foci of the ellipse approach each other until they coincide, the radius vectors of all points become equal to the semi-major axis which is equal to the semi-minor axis. The ellipse is then a circle having $a = b = r$ for its radius, and therefore πr^2 for its area. (See Part VI.)

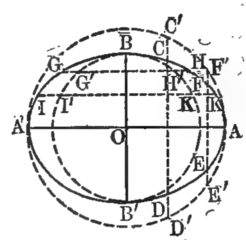

Fig. 308

1163. That portion of an ellipse included between two parallel chords is a *segment*.

The area of a segment included between two chords parallel to either the major or minor axis.

1st. Describe a circle on the major axis AA' as diameter; then, after having determined the area S' of the circular segment $C'D'E'F'$ (760), the area of the segment of the ellipse $CDEF$ is found from the proportion

$$S : S' = b : a,$$

from which

$$S = S' \times \frac{b}{a}.$$

Proof. Since the entire ellipse may be considered as being the projection of a circle (1162), we may also consider the segment of an ellipse as being the projection of the segment of a circle, and we have,

$$S = S' \cos\theta = S' \frac{b}{a}.$$

2d. The chords GH and IK, which bound the segment, being parallel to the major axis, describing a circle on the minor axis BB' as diameter, the area of the segment of the ellipse is given by the proportion

$$GHIK \text{ or } S:G'H'I'K' \text{ or } \frac{S}{S'} = a:b, \text{ and } S = S'\frac{a}{b}.$$

When the parallel chords are perpendicular to the minor axis at its extremities, the segment becomes the ellipse, and that of the circle a circle of radius b, and we still have the ratio

$$S : S' = a : b.$$

1164. *The ellipsoid of revolution* is a solid generated by the revolution of an ellipse about one of its axes.

1165. *The surface of an ellipsoid* is not given by any elementary algebraic expression. It may be computed by considering the generating ellipse as being made up of short straight lines, which generate cylinders, frustums, and cones of revolution; measuring all these lateral surfaces (**906**, **912**, **908**), and summing them, we have the approximate area of the ellipsoid. (See (**1355**) integral calculus.)

1166. *The volume of an ellipsoid. When the ellipsoid has three unequal axes*, that is, when a plane drawn through the center perpendicular to the major axis $2a$, does not determine a circle of diameter $2b$, as in the ellipsoid of revolution, but an ellipse having $2b$ and $2c$ for its axes, its volume is,

$$V = \frac{4}{3}\pi abc.$$

For an ellipsoid of revolution, according as the ellipse turns upon its major or minor axis, it suffices to make $c = b$ or $c = a$ in the preceding formula, and we have respectively,

$$V = \frac{4}{3}\pi ab^2 \text{ or } V = \frac{4}{3}\pi a^2 b. \qquad \text{(See Part VI.)}$$

When $a = b = r$, that is, when the generating ellipse is a circle, we have,

$$V = \frac{4}{3}\pi r^3,$$

which is as it should be, since the ellipsoid is a sphere of radius r (**924**).

HYPERBOLA

1167. *The hyperbola* is an open curve of two branches (Fig. 309), such that the difference $MF' - MF$ between the distances of each of its points from two fixed points, called the *foci* F and F', is constant.

It is seen that, like the ellipse (1127), the hyperbola is defined by its equation in focal coördinates (1101); designating the radius vectors of each point by the variables ρ and ρ' and the constant difference by $2\,a$, we have,

$$\rho' - \rho = 2\,a.$$

1168. The straight line which passes through the foci F,F', of the hyperbola is the *principal axis* (Fig. 309).

The segment AA' of the principal axis, intercepted by the curve, is called the *transverse axis.*

The points A and A' are the *vertices* of the hyperbola.

The perpendicular bisector of AA' is called the *conjugate axis.*

1169. *The distances of the foci to the nearer vertices are equal,* and therefore *so are the distances from the foci to the center:*

$$AF = A'F' \quad \text{and} \quad FO = F'O.$$

Proof. The vertices A and A' being on the hyperbola, we have,

$$AF' - AF \text{ or } AA' + A'F' - AF = A'F - A'F' \text{ or } A'A + AF - A'F'.$$

Canceling the quantity AA' common to both members of the equation, and transposing the like quantities to the same side of the equation, we have,

$$2\,A'F' = 2\,AF \text{ or } AF = A'F';$$

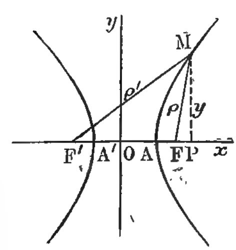

Fig. 309

adding the quantity AA' to both members of this equation, we have $A'F = AF'$, which shows that the distances from the foci to the farther vertices are equal.

Since $AO = A'O$, we have also $FO = F'O$.

1170. *The constant difference $2\,a$ of the radius vectors is equal to the transverse axis AA'.*

The point A being on the hyperbola, we have,

$$AF' - AF \text{ or } AA' + A'F' - AF = 2\,a;$$

from which, noting that $A'F' = AF$ (1169),

$$AA' = 2a.$$

1171. *The equation of the hyperbola, taking the axes of the curve as coördinate axes* (1168).

Let $AA' = 2a$ and $FF' = 2c$. We always have $2a < 2c$ or $a < c$.

Since $F'P = x + c$ and $FP = x - c$, the right triangles MPF' and MPF give respectively (730):

$$\rho'^2 = y^2 + (x + c)^2 \text{ and } \rho^2 = y^2 + (x - c)^2; \qquad (a)$$

developing (727, 728) and simplifying,

$$\rho'^2 - \rho^2 = y^2 + x^2 + c^2 + 2cx - y^2 - x^2 - c^2 + 2cx = 4cx;$$

that is (729),

$$(\rho' + \rho)(\rho' - \rho) = 4cx,$$

and

$$\rho' + \rho = \frac{4cx}{\rho' - \rho} = \frac{4cx}{2a} = \frac{2cx}{a};$$

and, since

$$\rho' - \rho = 2a,$$

adding these two equations, we have,

$$2\rho' = \frac{2cx}{a} + 2a \text{ or } \rho' = \frac{cx}{a} + a,$$

and therefore,

$$\rho'^2 = \frac{c^2x^2}{a^2} + a^2 + 2cx.$$

Putting this value of ρ'^2 equal to that in equation (a), and eliminating the denominator a^2,

$$a^2y^2 + a^2x^2 + a^2c^2 + 2a^2cx = c^2x^2 + a^4 + 2a^2cx.$$

Canceling $2a^2cx$, and transposing,

$$a^2y^2 + x^2(a^2 - c^2) = a^2(a^2 - c^2).$$

Representing the constant quantity $(a^2 - c^2)$, which is necessarily negative, by $-b^2$ (1186), we have for the equation of the hyperbola,

$$a^2y^2 - b^2x^2 = -a^2b^2 \text{ or } \frac{y^2}{b^2} - \frac{x^2}{a^2} = -1,$$

and

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}. \qquad \textbf{(571)}$$

From this equation it follows that, like the ellipse (1131, 1138), the hyperbola is divided into two equal and symmetrical parts by each of its axes (839). This equation shows furthermore that x cannot be less than a, and, according as x varies from $\pm a$ to $\pm \infty$, y varies from 0 to $\pm \infty$. Thus the curve is composed of two infinite branches.

1172. The distance $2c = FF'$ between the foci is called the *focal distance*, and the ratio e of the focal distance to the transverse axis $2a$ is called the *eccentricity* (1134). Thus we have,

$$e = \frac{c}{a} = \sqrt{\frac{a^2 + b^2}{a^2}}.$$

1173. From the equation of the hyperbola (1171), we find for any point M (Fig. 309):

$$\frac{y^2}{x^2 - a^2} \text{ or } \frac{y^2}{(x+a)(x-a)} = \frac{b^2}{a^2}. \qquad (1141)$$

Noting that $x + a = \pm A'P$ and $x - a = \pm AP$,

$$\frac{y^2}{A'P \times AP} = \frac{b^2}{a^2}.$$

This shows that *the ratio of the square of an ordinate to the product of the corresponding segments of the principal axis is equal to the ratio of the square of the conjugate axis to the square of the transverse axis, and is therefore constant.*

For another point we would have,

$$\frac{y'^2}{A'P' \times AP'} = \frac{b^2}{a^2},$$

therefore, $\dfrac{y^2}{y'^2} = \dfrac{A'P \times AP}{A'P' \times AP'}$.

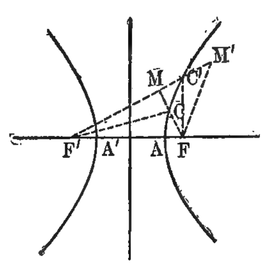

Fig. 310

Thus, *the squares of the ordinates of two points are to each other as the products of the corresponding segments of the principal axis.*

1174. *The hyperbola is the geometrical locus of the points the difference of whose radius vectors is equal to the transverse axis $2a$ of the curve* (1137).

1st. The point M being situated between the two branches of the hyperbola, we have $MF' - MF < 2a$.

Proof. Drawing CF', the point C is on the hyperbola, and we have,

$$CF' - CF = 2a.$$

Having $CF' > MF' - MC$ (637), replacing CF' by this smaller quantity,

$$MF' - MC - CF \text{ or } MF' - MF < 2a.$$

2d. The point M' not being between the two branches of the hyperbola, we have,

$$M'F' - M'F > 2a.$$

Proof. Drawing $C'F$, the point C' is on the hyperbola,

$$C'F' - C'F = 2a;$$

replacing the quantity $C'F$ by the smaller quantity $M'F - M'C'$, we have,

$$C'F' - M'F + M'C' \text{ or } M'F' - M'F > 2a.$$

COROLLARY. The converse statements of the above are also true.

1175. *The parts OM, OM', of the same straight line MM', included between the center O and the branches of the hyperbola, are equal.*

Drawing MP perpendicular to Ox, and taking $PN = PM$, the point N is on the hyperbola (1171). Drawing NQ perpendicular to Oy, and prolonging it until it meets MO at the point M'; since NM' is parallel to PO and $PN = PM$, we have $MO = OM'$. From this equation, and since OQ is parallel to MN, we have $QM' = QN$, and N is on the hyperbola, as is also its symmetrical point M'; therefore the point M', which gives $OM' = OM$, is situated on the hyperbola.

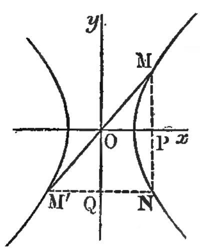

Fig. 311

From this it is seen that the point O may be considered as the *center* of the hyperbola, and straight lines, such as MM', as *diameters*.

Straight lines which pass through the center and do not cut the hyperbola are called *infinite diameters*.

Since any diameter cannot cut the hyperbola in more than two points, it cannot cut one of the branches in more than one point, and a chord in one of the branches does not meet the other.

1176. When the center O is joined to the middle i of a chord,

the diameter BB', which coincides with this line, bisects all chords EG, GH, etc., parallel to CD.

The infinite diameter IK which connects the center O to the middle e of the chord GC, bisects all chords HD parallel to GC (1144).

As was the case with the ellipse, the two diameters BB' and IK, each of which bisects the chords parallel to the other, are called *conjugate diameters.*

Having a diameter of an hyperbola given, its conjugate is found in the same way as is that of the ellipse (1145, 1189).

1177. An hyperbola or an arc of an hyperbola being given, to find its center and its axes, operate as with an ellipse (1146).

1178. *To trace an hyperbola by points.*

F and F' being the foci of an hyperbola, and A and A' the vertices, with F and F' as centers and $A'M$ as radius, which

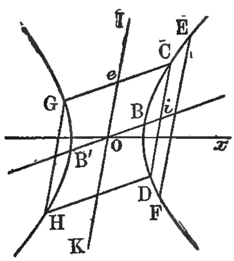

Fig. 312

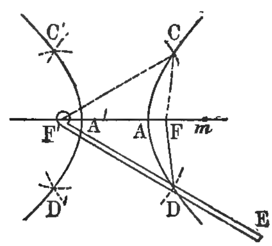

Fig. 313

may vary from AF to ∞, describe arcs; then with the same centers F and F', with a radius equal to Am, describe arcs cutting each of the first in the points CD, which belong to one branch of the hyperbola, and $C'D'$, which belong to the other branch.

Proof. Any of these points gives $CF' - CF = A'm - Am = AA' = 2a$ (1167).

Varying the position of m on the prolongation of AF, as many points may be determined as are desired, and the smooth curve drawn through these points form the two branches of the hyperbola.

1179. *To trace an hyperbola by a continuous motion.*

Let (Fig. 313) $F'E$ be a rule with a small hole at one end placed in line with one edge, and EDF be a string fastened at the other end of this same edge. Taking the length of this string EDF such that $EF' - (ED + DF) = AA' = 2a$, fastening the ex-

tremity F' with a pivot at one focus and the end of the string F at the other focus, and turning the rule while holding the string taut with a pencil D pressed tightly against the edge of the rule, a branch of an hyperbola is traced.

Proof. For any position D of the pencil,

$$DF' - DF = EF' - (ED + DF) = AA' = 2a.$$

The other branch of the hyperbola is traced in the same manner.

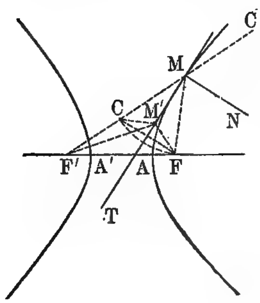

Fig. 314

1180. *To draw a tangent to an hyperbola through a point M taken on the curve* (1153).

Draw the radius vectors MF, MF'; take $MC = MF$, draw CF, and the perpendicular MT, dropped from the point M on CF, is the required tangent; that is, that any point M', other than M, taken on this line, gives

$$M'F' - M'F < AA' \text{ or } 2a. \qquad (1174)$$

Proof. MT being perpendicular to CF at its middle point, the triangle MCF is an isosceles triangle, and we have,

$$F'C + CM' - M'F = F'C = MF' - MF = 2a.$$

But

$$F'C + CM' > M'F';$$

therefore,

$$M'F' - M'F < 2a.$$

Remark. The triangle MCF being isosceles, it is seen that the tangent bisects the angle included by the radius vectors.

1181. As in the ellipse (1159), the tangent to the hyperbola is parallel to the conjugate of the diameter drawn through the point of contact (1176); which gives a *second method for drawing a tangent to an hyperbola.*

1182. *To draw a normal to an hyperbola through a point M* (Fig. 314).

The bisector MN of the angle FMC' formed by the radius vector MF and the prolongation MC' of the other radius vector, is the normal to the curve at the point M. Reasoning as in (1154), it may be proved that MN is perpendicular to MT at M.

1183. Two hyperbolas, and in general two curves, are said to be *homofocal* when they have the same foci.

An ellipse and an hyperbola, which are homofocal, cut each other at right angles.

As bisector of the angle FMC', MT is both tangent to the ellipse and normal to the hyperbola, and, as bisector of the angle FMF', MN is both normal to the ellipse and tangent to the hyperbola, and MN and MT are perpendicular to each other whether we consider the ellipse (**1154**) or the hyperbola (**1182**).

The method of determining the point T has been given (**1158**).

1184. *Hyperbolic mirrors* (**1154**). A ray of light or heat emanating from the focus F of a hyperbolic mirror (Fig. 315) strikes any point M and is reflected in the direction MC' and appears to come from the focus F'. As is seen, all the reflected rays, instead of meeting at the same point, as in the elliptical mirror, appear to come from the same point F', which is a *virtual focus* and not a conjugate focus.

The space in front of the mirror in the angle $DF'D'$ receives both the direct rays, from the source at F and those reflected by

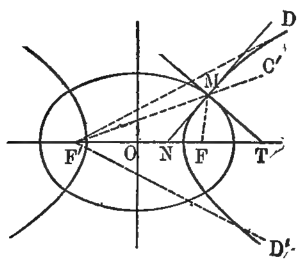

Fig. 315

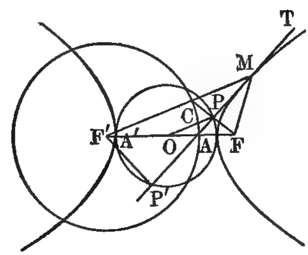

Fig. 316

the mirror. Thus it is seen that when a large area is to be lighted, a hyperbolic mirror should be used.

1185. What was said in article (**1155**) concerning the ellipse holds good for the hyperbola.

MT being the tangent drawn to the hyperbola at M, according to the construction of article (**1180**), and O the center of the hyperbola, in the triangle $FF'C$ the straight line OP bisecting FC and FF', we have $OP = \frac{F'C}{2} = a$; which shows that the circle described upon AA' as diameter passes through the point P, and that it is the geometrical locus of the projections P, P', of the foci upon the tangents (**1155**) (Fig. 316).

The circle described from one of the foci F' as center, with

$AA' = 2a$ as radius, has the property that when any radius $F'C$ is prolonged to the hyperbola $MC = MF$. Therefore, *an hyperbola may be defined as a curve such that all of its points are equally distant from the circumference of a circle and a fixed point outside of that circle.*

From this definition a method may be deduced for the construction of the hyperbola by points, but it is quite complicated.

The circle described on AA' as a diameter is called the *principal circle* of the hyperbola, and that described from one of the foci as center with the transverse axis AA' as radius is called the *directrix circle.*

From that which has been said, *in order to draw a tangent to an hyperbola at the point M,* describe a circle on AA' as diameter, and another with F' as a center with AA' for a radius; draw $F'M$, then CF, which will intersect the circumference of the principal circle at P, and connecting M to P we have the required tangent.

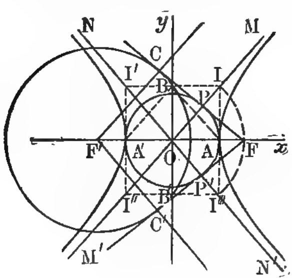

Fig. 317

1186. *Asymptotes.* The branches of the hyperbola extend to infinity, and the diameters increase to a maximum angle with the principal axis, at which angle they extend from $+\infty$ to $-\infty$ (1175). The two infinite diameters which meet the hyperbola at infinity are called the *asymptotes.* They are tangent to the branches at infinity. When the point of contact M (Fig. 316) moves along the curve, the point P describes the principal circle and the point C the directrix circle, whose center is at the focus F' (1185).

Since the straight lines OP and $F'C$ are always parallel (1185), the angles OPF and $F'CF$ are always equal; and if one of the angles OPF becomes a right angle, the other $F'CF$ also becomes a right angle, and FC is tangent to the principal circle and also to the directrix circle. Then (Fig. 317) the tangent MP perpendicular to FC at its middle point and the radius OP are in the same straight line; and since the point of contact is at the intersection of the two parallels OP and $F'C$, which is at infinity, the line OM is an asymptote.

Therefore, *to trace an asymptote,* connect the center to the

point of contact P of the tangent to the principal circle drawn through F. The other tangent FP' drawn to the principal circle gives the other asymptote ON', and the tangents drawn from F' to the same circle determine the asymptotes ON, OM', of the second branch of the hyperbola; but, since the figure is symmetrical, the asymptotes of the second branch are prolongations of those of the first. Therefore the hyperbola has two asymptotes.

Erecting perpendiculars to AA' at A and A', and completing a rectangle whose vertices are on the asymptotes, the two right triangles OPF OAI having an acute angle O common and the side $OP = OA$, being radii of the same circle, are equal, and $OI = OF = c$. Therefore, *to trace the asymptotes*, from one of the vertices A' as center, with OF as radius, describe an arc which cuts the transverse axis in B and B'; draw the rectangle $I\,I'I''I'''$ on AA' and BB', and the diagonals of this rectangle are the asymptotes; they may be traced without constructing the rectangle $I\,I'I''I'''$, by simply drawing parallels to $A'B$ and to AB through the center O.

In the right triangle $A'OB$ we have $\overline{OB}^2 = \overline{A'B}^2 - \overline{A'O}^2 = c^2 - a^2 = b^2$ (1171). This is why $BB' = 2\,b$ is taken as the *length of the conjugate axis.*

1187. An hyperbola is *equilateral* when the asymptotes are perpendicular to each other. Then the rectangle $I\,I'I''I'''$ (Fig. 317) is a square, and the two axes $2\,a$ and $2\,b$ are equal.

1188. *Two hyperbolas are said to be conjugate* when, having the same asymptotes and equal focal distances, $FF' = ff'$, the transverse axis of one is the conjugate axis of the other. From that which has been said, the points F, I, f, are on an arc of the same circle, whose center is O and radius is $OF = c$. The transverse axis $AA' = 2\,a$ and the conjugate axis $BB' = 2\,b$ of the hyperbola FF' are respectively the conjugate axis $2\,b'$ and the transverse axis $2\,a'$ of the conjugate hyperbola ff'. We have,

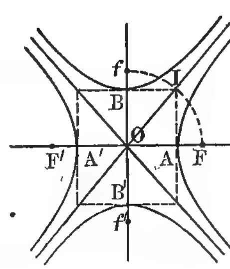

Fig. 318

$$a'^2 = b^2 = c^2 - a^2 \text{ and } b'^2 = a^2 = c^2 - b^2.$$

When one of the hyperbolas is equilateral (1187), its conjugate

is also. We have,

$$a^2 = b^2 = a'^2 = b'^2, \qquad c'^2 = c^2 = 2\,a^2 = 2\,b^2;$$

thus the two hyperbolas are identical.

1189. When the asymptotes are traced (1186), *to draw the conjugate to a given diameter LL'* (1176), through L, draw a parallel LD to the farther asymptote; it cuts the other asymptote in E; take $EG = EL$, and GO is the required conjugate diameter.

This construction is based upon the fact that *each asymptote bisects the parallels to the other which are included between two conjugate diameters.* Thus, the asymptote MM' bisects GL and all lines parallel to it and included between the conjugate diameters LL' and GG'; it also bisects all parallels AB', $A'B'$, ..., included between the other two conjugate diameters AA', BB'.

1190. *To draw a tangent to an hyperbola through a point M exterior to the hyperbola* (1157).

From the point M as center, with a radius equal to the distance MF to the nearer focus, describe an arc; from the other

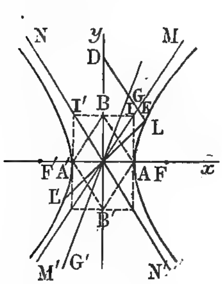

Fig. 319

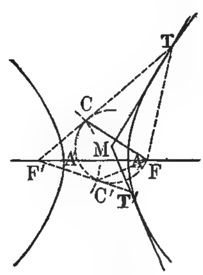

Fig. 320

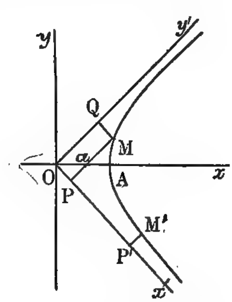

Fig. 321

focus F', with a radius $2\,a = AA'$, describe another arc which cuts the first in the two points C and C'; draw FC and FC', and the perpendiculars MT, MT', dropped from the point M to the middle points of these chords, are tangents to the hyperbola at the points T and T'.

The points of contact T and T' may be obtained directly, by drawing $F'C$ and $F'C'$ and prolonging these lines until they cut the hyperbola; because, if it was desired to draw a tangent at the point T where $F'C$ meets the hyperbola, we would lay off TF on TF', thus determining the point C; then T would be on the hyperbola, and we would have $TF' - TF = 2\,a = CF'$; we

would then draw FC, and the perpendicular dropped from the point T to the middle of FC would be the tangent (1180). This perpendicular coinciding with that which was drawn through M, the latter is also tangent to the hyperbola at the point T. In the same way it may be shown that MT' is tangent at T'.

1191. Taking the asymptotes Ox' and Oy' of the hyperbola as coördinate axes, the equation of the curve becomes (1171, 1186),

$$x'y' = \frac{a^2 + b^2}{4},$$

which shows that the product of the coördinates, perpendicular or oblique, $MQ = x'$ and $MP = y'$, is constant, and that the parallelogram $OPMQ$ formed by the coördinates of any point and the asymptotes is also constant, since, designating the angle included by the asymptotes by θ, the base of this parallelogram is x', its altitude is $y' \sin \theta$ and the area of its surface is

$$S = x'y' \sin \theta = \frac{a^2 + b^2}{4} \sin \theta.$$

When the hyperbola is equilateral, $\theta = 90°$ and $\sin \theta = 1$; that is, $OPMQ$ becomes a rectangle (Fig. 321),

$$S = x'y' = \frac{a^2 + b^2}{4}.$$

1192. *The area of an hyperbola.*

Making the constant quantity

$$x'y' = \frac{a^2 + b^2}{4} = m^2, \tag{1191}$$

the area A of the figure $MM'P'P$ included by the arc MM' the asymptote and the two ordinates y' and y'' is

$$A = m^2 \sin \theta \text{ L. } \frac{x''}{x'},$$

wherein $x' = OP$, $x'' = OP'$, and L. = Napierian logarithm (407, 408, and 1796).

When the hyperbola is equilateral (1187), we have $\sin \theta = 1$, and therefore,

$$A = m^2 \text{ L. } \frac{x''}{x'}.$$

If we take m as unity,

$$A = \text{L.}\ \frac{x''}{x'},$$

and in the case where the point M is at the vertex A of the hyperbola, since $x' = 1$ and $x' = y'$, $x'y' = m^2 = 1$, and

$$A = \text{L}_\circ\ x''.$$

This property of the Napierian logarithms gives them the name, *hyperbolic logarithms.*

1193. According as an hyperbola revolves about its conjugate axis or its transverse axis (1168), it generates an *un-parted hyperboloid or a bi-parted hyperboloid.*

PARABOLA

1194. A parabola is an open-branched curve (Fig. 322), all points of which are equally distant from a fixed point or *focus* F, and a fixed straight line or *directrix* OD.

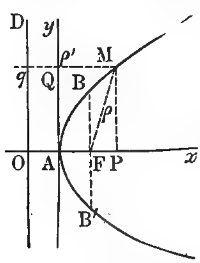

Fig. 322

The parabola, like the ellipse and hyperbola, is defined in focal coördinates (1127, 1167). Designating the radius vectors of different points on the curve by the variables ρ and ρ', we have,

$$\rho = \rho'.$$

Two parabolas having the same focus are said to be confocal (1183).

1195. The perpendicular Fx to the directrix drawn through the focus is the axis of the parabola.

The point A, where the axis cuts the curve, is the *vertex* of the parabola.

Twice the constant distance FO between the focus and the directrix is called the *parameter* of the parabola; it is represented by $2\,p$, and determines the parabola.

The vertex, being part of the curve, bisects the distance FO, and we have,

$$OA = AF = \frac{1}{2}p.$$

1196. *The chord BB' drawn through the focus perpendicular to the axis is called the latus rectum and is equal to the parameter* $2\,p$. From the definition of a parabola and the fact that parallels

comprehended between parallels are equal, we have $FB = FB' = OF = p$ and $BB' = 2\,p$.

1197. *The equation of the parabola referred to coördinate axes, when one coincides with the axis Ax and the other passes through the vertex A parallel to the directrix of the curve OD.*

In the right triangle MFP (730),

$$\rho^2 = \overline{MP}^2 + \overline{FP}^2 = y^2 + \left(x - \frac{1}{2}p\right)^2;$$

also,

$$\rho^2 = \overline{OP}^2 = \left(x + \frac{1}{2}p\right)^2;$$

Putting these two values of ρ^2 equal to each other,

$$y^2 + x^2 + \frac{1}{4}p^2 - px = x^2 + \frac{1}{4}p^2 + px.$$

Simplifying, we have the equation of the curve,

$$y^2 = 2\,px,$$

and (571)

$$y = \pm \sqrt{2\,px}.$$

For every value of x there are two equal values of y opposite in sign, therefore the curve is symmetrical about its x-axis.

Solving the equation for x,

$$x = \frac{y^2}{2\,p}.$$

y^2 being necessarily positive (537), x is always positive, and the curve is situated entirely on one side of the y-axis.

When x varies from 0 to ∞, y varies from 0 to $\pm\ \infty$; consequently the curve has one branch extending to infinity on both the $+\,y$ and the $-\,y$ side of the x-axis. If p is negative, the curve is open on the left side.

1198. *The squares of the ordinates of the parabola are to each other as the corresponding abscissas* (1141, 1173).

From the equation of the parabola (1197),

$$y^2 = 2\,px \quad \text{and} \quad y'^2 = 2\,px'$$

and

$$\frac{y^2}{y'^2} = \frac{x}{x'}.$$

1199. From the equation $y^2 = 2\,px$, we have,

$$\frac{y^2}{x} = 2\,p,$$

which shows that the ratio of the square of an ordinate to the corresponding abscissa is constant and equal to the parameter $2p$.

For $x = \frac{p}{2}$, we have $y^2 = p^2$ or $y = p$. *Thus the ordinate which corresponds to the focus is equal to the distance from the focus to the directrix* (1196).

1200. *The parabola is the geometrical locus of the points equally distant from the focus and the directrix* (1137, 1174).

1st. The point M being outside the parabola, we have $MQ < MF$.

Proof. Prolonging QM, and drawing CF, we have,

$$CF - CM < MF;$$

replacing CF by its equal CQ,

$$CQ - CM \quad \text{or} \quad MQ < MF.$$

2d. The point M' being inside the curve, we have $M'Q > M'F$; because, having

$$M'C + CF > M'F,$$

replacing CF by CQ,

$$M'C + CQ \quad \text{or} \quad M'Q > M'F.$$

Corollary. The converse statements of 1st and 2d are both true.

1201. *The axis of the parabola divides the curve into two equal and symmetrical parts.*

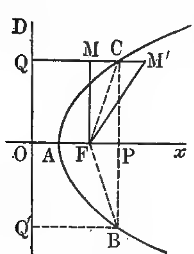

Fig. 323

C being any point in the curve (Fig. 323), drawing the perpendicular CP to Ox, and taking $PB = PC$, the point B symmetrical to C is on the parabola.

Proof. Drawing BF, we have $CF = BF$ (621); furthermore, since $CF = CQ$ and $CQ = BQ'$, we have $BF = BQ'$; which cannot be unless the point B is on the curve (1200); therefore the two parts of the curve are symmetrical with respect to the axis and equal each to each (839). This was proved in article (1197).

1202. The ellipse being the geometrical locus of the points, such as M, which are equally distant from the focus F and the

directrix circle whose center is at the other focus F' (1155), the vertex A and the focus F remaining fixed, according as the vertex A', the focus F', and the center ω move farther away the ellipse becomes flatter and the directrix circle becomes larger. When the vertex, the focus, and the center reach infinity, the directrix circle becomes a straight line OD and the ellipse becomes a parabola EAE', the points of which are equally distant from the focus F and the directrix OD.

Thus *the parabola may be considered as being the limit of an ellipse when one focus and vertex remain fixed and the other focus and vertex approach infinity.*

It is seen that a parabola may also be considered as the limit of an hyperbola when one focus and vertex remain fixed while the other vertex approaches infinity.

1203. The parabola being considered as a special case of the ellipse, all diameters meet in the center; but since the center

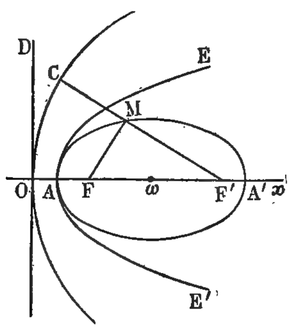

Fig. 324

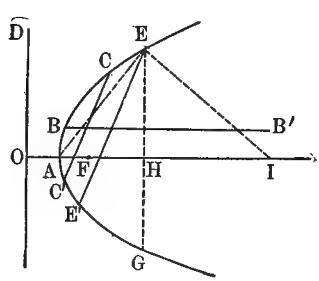

Fig. 325

is at infinity on the axis, all the diameters are parallel to the axis.

1204. As in the ellipse and the hyperbola (1144, 1176), any diameter BB', which bisects a chord CC', also bisects all chords EE' parallel to CC' (1207, 1214).

The axis, which is a diameter, bisects the chords EG which are perpendicular to it (1201).

1205. From the equation $y^2 = 2\,px$ (1197), it follows that any semi-chord EH perpendicular to the axis is a mean proportional between its distance from the vertex AH and the parameter $2\,p = 2\,OF =$ the *chord drawn through the focus perpendicular to the axis* (1196). Thus we have,

$$AH : EH = EH : 2\,p.$$

From this it follows that *in order to obtain the parameter* $2p$, draw a semi-chord EH perpendicular to the axis, draw AE, erect the perpendicular EI to AE at E, and we have $HI = 2p$. The right triangle AEI (Fig. 325) gives (705),

$$AH : EH = EH : HI.$$

1206. *A parabola being given, trace its axis, its focus, and its directrix.*

Drawing two parallel chords CC' and EE' (Fig. 325), the line BB' which joins their middle points is a diameter of the parabola and is parallel to the axis (1203). The middle point H of the chord EG lies on the axis, which is obtained by drawing a parallel to BB' through H. The parameter $2p = HI$ is obtained by the construction given in article (1205); and laying off a quarter of the parameter on the axis at the right and left of the vertex, the focus F is found, and the point O determines the directrix OD (1214).

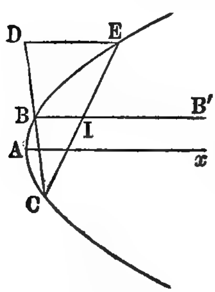

Fig. 326

1207. All diameters of the parabola being parallel to each other (1203), any one of them BB' has no conjugate (1176); but the direction of the parallel chords which are bisected by BB' may be considered as being the *conjugate direction of this diameter.*

A diameter BB' being given, to find its conjugate direction, connect B to any point C of the curve, prolong CB so that $BD = BC$, draw DE parallel to BB', and CE has the required direction.

Proof. Having $BD = BC$, we have $IE = IC$ (699).

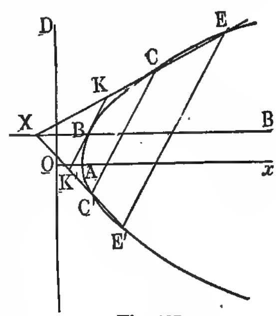

Fig. 327

1208. CC' and EE' being two parallel chords bisected by the diameter BB', the chords EC and $E'C'$ meet at the same point X in BB' (694). This being true no matter what the distance between CC' and EE' may be, it must be true for tangents drawn at the extremities of the same chord EE', and, in general, at the extremities of any chord parallel to EE'.

The chords parallel to EE' become shorter as they approach B, and at this point the chord is an element of the curve and coincides with the tangent KK' at this point, which is also parallel to EE'. Since $BK = BK'$, it is seen that *the tangent KK' parallel to the chord drawn between the points of contact E and E' of the two tangents to the parabola is bisected at its point of contact.*

This property of the tangent is only a special case of the more general property given below.

1209. *Any one of three tangents EX, $E'X$, and KK', to a parabola divides the other two into inversely proportional segments.*

Thus we have,

$$\frac{EK}{KX} = \frac{XK'}{K'E'}.$$

Drawing parallels to the axis through the points K, X, K', the chords of contact EE', EJ, and JE' are bisected at the points G, I, and L, and therefore we have,

$$GE = GE', \quad FE = FC,$$
$$HC = HE'.$$

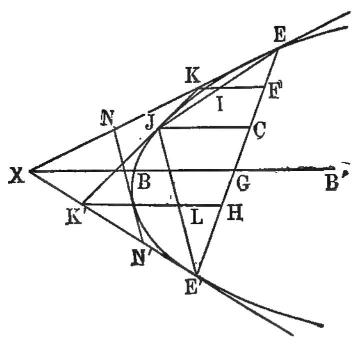

Fig. 328

In the triangles EGX and $E'GX$, we have respectively (699),

$$\frac{EK}{KX} = \frac{EF}{FG} \text{ and } \frac{XK'}{K'E'} = \frac{GH}{HE'}.$$

But the second members of these proportions are equal, since,

$$GH = GE' - HE' = GE - HC = CE - GH,$$

from which $$GH = \frac{CE}{2} = EF,$$

and $$HE' = GE' - GH = GE - EF = FG\,;$$

therefore $$\frac{EK}{KX} = \frac{XK'}{K'E'} \text{ or } \frac{EK}{XK'} = \frac{KX}{K'E'}. \qquad (345)$$

From this proportion we have (324),

$$\frac{EK + KX}{XK' + K'E'} \text{ or } \frac{EX}{XE'} = \frac{EK}{XK'} = \frac{KX}{K'E'}.$$

REMARK. Any tangent KK' giving

$$\frac{EK}{KX}=\frac{XK'}{K'E'},$$

and if the tangent is drawn through B, it is parallel to EE', and we have,

$$\frac{EK}{KX}=\frac{K'E'}{XK'}.$$

Those two proportions having a common ratio, we have,

$$\frac{XK'}{K'E'}=\frac{K'E'}{XK'},\quad \text{then}\quad XK=K'E',$$

and $$XB=BG.$$

Thus, the middle point B of the line joining the intersection X of any two tangents to the middle point G of the chord of contacts of these tangents, is part of the parabola.

1210. *No matter how many tangents are drawn to a parabola, upon any two of them EX, $E'X$* (Fig. 328), *the others determine proportional segments.* Thus we have,

$$\frac{EK}{XK'}=\frac{KN}{K'N'}=\frac{NX}{N'E'}.$$

Proof. Considering successively the tangents KK', NN', as cutting those EX, $E'X$, we have (1209),

$$\frac{EK}{XK'}=\frac{EX}{XE'}\text{ and }\frac{EN}{XN'}=\frac{EX}{XE'};$$

and $$\frac{EK}{XK'}=\frac{EN}{XN'}=\frac{EX}{XE'}.$$

Subtracting from the terms of each ratio the terms of the preceding ratio does not change the value of the ratios (349), and we have,

$$\frac{EK}{XK'}=\frac{KN}{K'N'}=\frac{NX}{N'E'}.$$

REMARK. If one of the tangents is bisected, all of them are.

1211. *To trace a parabola by points.* F being the focus, Ox the axis, OD the directrix, and A, which gives $AF = OF$, the vertex, erecting a perpendicular CC' to the axis at the point B taken at the right of the vertex A, and with the focus as center,

and the distance OB as radius, describe an arc; it cuts CC' in two points C and C', both of which are on the parabola.

Proof. From the construction, each of the points is equally distant from the directrix and focus, and is therefore part of the parabola (1200).

In this manner as many points may be obtained as is desired, and when connected by a smooth curve we have a parabola.

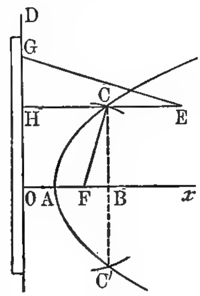

Fig. 329

1212. *To trace a parabola by a continuous motion* (Fig. 329). EGH being a triangle, and ECF a string of a length equal to EH, one end of which is fastened at the point E and the other end at the focus F, if the triangle is slid along a straight edge which coincides with the directrix OD, and the string held taut by pressing a pencil-point C against the edge of the triangle, the point C will trace the upper part of a parabola. Reversing the triangle, the lower part is drawn in the same manner.

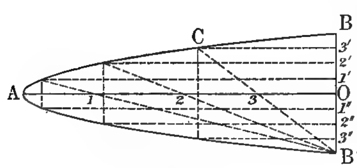

Fig. 330

Any position of C is on the parabola; because, having $EC + CF = EH$, we have $CF = CH$ (1200).

1213. *Another method of construction by points.*

This method is used in calculating the form of beams of uniform resistance, such as walking-beam of an engine, etc.

Let A be the vertex, AO the axis, and BO half the height of the beam, then the parameter is

$$2\,p = \frac{\overline{OB}^2}{OA}. \tag{1199}$$

Having the parameter, the focus and directrix are determined (1206), and the parabola may be traced as in (1211); or, choosing different values of x, the corresponding values of y may be calculated from the equation $y^2 = 2\,px$.

In practice the geometrical construction shown in (Fig. 330) is often used.

From the point B drop a perpendicular to the axis and prolong it beyond O so that $OB' = OB$; divide BO and AO into the same

number of equal parts, four for example; through the points of division on BO draw parallels to the axis; then joining B' to the points of division 1, 2, 3, on AO, and prolonging these lines until they cut the parallel to the axis which has the corresponding number 1′, 2′, or 3′, the point of intersection is on the parabola. Repeating this operation for the part OB', the lower part of the curve may be drawn.

Proof. From the construction $O\,3 = \dfrac{OA}{m}$ and $B\,3' = \dfrac{OB}{m}$, and $O\,3$ being parallel to $3'C$, we have (699),

$$O\,3 : 3'C = B'O : B'3'.$$

Representing OA by a and OB by b,

$$O\,3 = \frac{a}{m}, \quad 3'C = a - x, \quad B'O = b, \quad B'3' = b + y,$$

the above proportion may be written,

$$\frac{a}{m} : (a - x) = b : (b + y);$$

or, noting that $3'B$ or $\qquad \dfrac{b}{m} = b - y$

gives $\qquad m = \dfrac{b}{b - y},$

$$\frac{a\,(b - y)}{b} : (a - x) = b : (b + y).$$

Putting the product of the means equal to the product of the extremes (729),

$$\frac{a\,(b^2 - y^2)}{b} = b\,(a - x),$$

or $\qquad ab^2 - ay^2 = ab^2 - b^2x,$

and $\qquad y^2 = \dfrac{b^2}{a}\,x,$

which is the equation of the parabola, whose parameter is $\dfrac{b^2}{a}$.

A method of constructing a parabola on a large scale, often used in surveying, is shown in Fig. 331.

AT and *BT* being two lines to be connected by a parabola tangent to these lines at the points C and D, divide CT and DT

into the same number of equal parts; connect the points whose numbers correspond, and draw a curve tangent to AC at C, to BD at D, and to the lines 11, 22, and 33. This curve is the required parabola.

This same method may be used to construct an arc of a parabola which is normal to two lines CE and DF at two given points C and D.

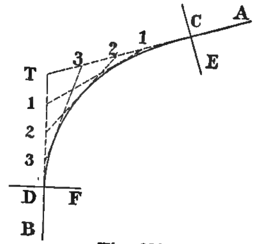

Fig. 331

1214. *To draw a tangent to a parabola through a point M taken on the curve* (1153, 1180) (Fig. 332).

Draw FQ, and the perpendicular MT, dropped from the point M to FQ, is the tangent. Thus, any point M', taken on MT, is outside the curve, that is, $M'Q' < M'F$ (1200).

Proof. The triangle MFQ being isosceles, MT is the perpendicular bisector of FQ, from which it follows that $M'Q = M'F$; but $M'Q > M'Q'$ (620), and therefore $M'Q' < M'F$.

REMARK 1. Since the triangle MFQ is isosceles, it follows *that the tangent MT bisects the angle FMQ and the radius vectors.*

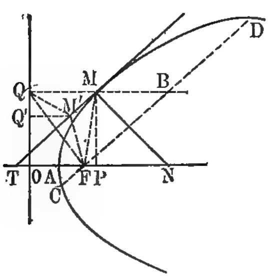

Fig. 332

REMARK 2. The angle $QMT = MTF$, being alternate interior angles; and $QMT = TFM$ being base angles of an isosceles triangle.

The triangle MTF being isosceles, it follows that in order *to draw a tangent at the point M,* lay off from the focus $FT = FM$ and draw MT.

Having $FT = FM = MQ = OP$, and $AO = AF$, it follows that we also have $AT = AP$.

REMARK 3. Taking $MB = MQ = MF = FT$, the chord CD, which passes through F and B, is parallel to the tangent MT, and is bisected at the point B.

From this we have a method for drawing a chord through F which is bisected by a given diameter MB (1204, 1207).

Drawing through the extremity of this diameter a parallel to the chord, it will be tangent to the curve; which gives a *third method for drawing a tangent to a parabola* (1208).

REMARK 4. Having drawn the diameter MB, and the axis of

the parabola, as per (1206), drawing the tangent MT, the triangle MTF is isosceles, and the perpendicular bisector of its base determines the focus F at the intersection of this line with the axis, and the point Q at the intersection of this same line with the diameter MB determines the directrix. This is a second method for determining the focus and directrix of a parabola (1206).

REMARK 5. Having $AO = AF$, the perpendicular erected at A to the axis AN of the parabola passes through the middle point of FQ (699), that is, at the point where the tangent cuts the line FQ, which is perpendicular to it; therefore the geometrical locus of the projection of the focus on the tangents is the perpendicular erected at the vertex A (1155, 1185).

1215. *To draw a normal to the parabola.* The bisector MN of the angle FMB, which is included by one radius vector and the prolongation of the other, is normal to the curve at the point M. It may be proved that MN is perpendicular to the tangent MT, as was done in article (1154).

Having $FT = OP$ (1214, REMARK 2), we have $FP = OT$, and since $AF = AO$, we have $AP = AT = x$, and $TP = 2x$.

This being true, the point M is on the curve, and we have (1197),

$$y^2 = 2px.$$

Representing the subnormal PN by s, the right triangle TMN gives (705),

$$y^2 = s \times TP = s \times 2x.$$

Putting these two values of y^2 equal to each other,

$$2sx = 2px, \quad \text{then} \quad s = p.$$

Thus, for the parabola, *the subnormal is constant and equal to the semi-parameter* $p = OF$. This furnishes an easy method of drawing a normal or a tangent to the parabola at any given point M.

1216. *Parabolic mirror, ear-trumpet, megaphone, etc.* In a parabolic mirror, all rays FM (Fig. 332) emanated from the focus are reflected along lines MB, parallel to the axis. All rays parallel to the axis which strike the mirror from outside are reflected to the focus.

This property is utilized in ear-trumpets. The sound which enters the trumpet is reflected to the focus, and, the end being removed, the focus is brought inside the ear (Fig. 333).

The megaphone is sometimes made by combining an ellipsoid and a paraboloid (Fig. 334) so that they have a focus F in common, the mouth being placed at the other focus F' of the ellipse.

1217. The path of a projectile would be a parabola were it not for the resistance of the air which modifies the curve. The cables on suspension bridges have a curvature which is very nearly parabolic, and in practice may be taken as such.

1218. *To draw a tangent to a parabola parallel to a given straight line CD* (Fig. 332), follow the same course as for the ellipse (**1156**). Thus, draw FQ from the focus perpendicular to CD, and the perpendicular bisector of FQ is the required tangent.

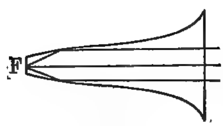

Fig. 333

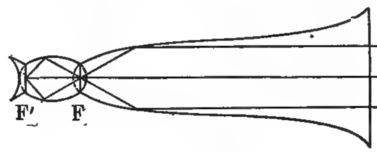

Fig. 334

To obtain the point of contact, draw QM parallel to the axis.

It is seen that the tangent and its point of contact may be determined without constructing the parabola, when the axis, focus, and directrix are given.

The problem is impossible when CD is parallel to the axis, because then the perpendicular FQ meets the directrix at infinity.

1219. *To draw a tangent to a parabola through a point M outside the curve.*

From the point M as center, and with MF as radius, describe an arc which cuts the directrix in the points D and D'; draw FD and FD'; then the perpendiculars to these lines, dropped from the point M, are tangents to the parabola at the points T and T', which are given directly by drawing parallels to the axis through D and D'.

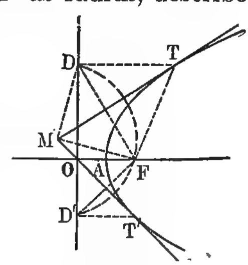

Fig. 335

If a tangent to the curve was to be drawn at the point T, a perpendicular would be dropped from this point to the middle point of FD (**1214**); but this perpendicular would coincide with that which was drawn from the point M; because, the triangle MDF being isosceles, this perpendicular also passes through the middle point of FD.

1220. As was the case with the ellipse and hyperbola, there is no method in elementary geometry by which the length of an arc of a parabola can be accurately determined.

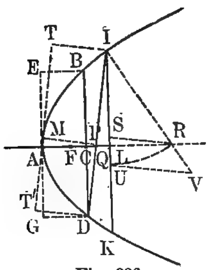

Fig. 336

1221. *The surface of a parabolic segment ABCD, included between the vertex and the chord BD perpendicular to the axis, is equal to* $\frac{2}{3}$ *of the rectangle EDBG, whose altitude is BD and whose base is AC*; thus we have (1329) (Fig. 336),

$$\text{surface } ABCD = \frac{2}{3} AC \times BD,$$

or

$$\text{surface } \quad ABC = \frac{2}{3} AC \times BC.$$

From this,

$$\text{surface } ABE = \frac{1}{3} \text{ surface } ACBE = \frac{1}{3} AC \times BC.$$

Noting that the segment $BIKD$, included between the two chords BD, IK, perpendicular to the axis, is the difference between two segments $AILK$ and $ABCD$, we have,

$$\text{surface } BIKD = \frac{2}{3} AL \times IK - \frac{2}{3} AC \times BD$$

$$= \frac{2}{3}(AL \times IK - AC \times BD).$$

M being the point of contact of the tangent MT parallel to ID (1214, Remark 3), the surface of the segments $AIQD$ is $\frac{2}{3}$ of the surface of the rectangle $IDT'T$, which has the same base ID and the same altitude MP as the segment; thus we have,

$$\text{surface } AIQD = \frac{2}{3} MP \times ID.$$

The segment whose base is perpendicular to the axis is simply a special case of the general theorem.

1222. The solid generated by the revolution of a parabola about its axis is called a *paraboloid.*

1223. *The surface of the paraboloid generated by the rotation of an arc AI upon the axis* (Fig. 336).

Take $LR = 2\,AF$, and $IS = 3\,AF$; draw SR; then take $IU = IR$, and draw UV parallel to SR; from which we have,

$$IS : IR = IU \text{ or } IR : IV.$$

The surface s of the paraboloid is equal to the lateral surface of a right cylinder having IR for its diameter and IV for its altitude, less $\frac{8}{3}$ of the surface of a circle having AF for its radius; thus we have (753, 906, and 1340),

$$s = \pi \cdot IR \cdot IV - \frac{8}{3}\pi\overline{AF}^2. \tag{a}$$

Representing the ordinate IL by y, since we have $LR = 2\,AF = p$, (1205), and $IS = 3\,AF = \frac{3}{2}p$, the right triangle ILR gives,

$$IR = \sqrt{y^2 + p^2}.$$

From the above proportion,

$$IV = \frac{\overline{IR}^2}{IS} = \frac{y^2 + p^2}{\frac{3}{2}p} = \frac{2\,(y^2 + p^2)}{3\,p}.$$

Since

$$\overline{AF}^2 = \frac{p^2}{4},$$

substituting these values in the formula (a),

$$s = \pi\sqrt{y^2 + p^2} \times \frac{2\,(y^2 + p^2)}{3\,p} - \frac{2}{3}\pi p^2.$$

This expression permits the calculation of s without any geometrical construction when the values of p and y are known.

Since, representing AL by x, $y^2 = 2\,px$ (1197), s may also be expressed in terms of x, thus:

$$s = \pi\sqrt{2\,px + p^2} \times \frac{4\,x + 2\,p}{3} - \frac{2}{3}\pi p^2.$$

1224. *The volume of a paraboloid generated by the rotation of the parabolic segment AIL about the axis, the base IL being per-*

pendicular to the axis (Fig. 336), is equal to that of a right cylinder having AL for its radius and $2\,AF$ for its altitude. Representing the volume by v (907 and 1340),

$$v = \pi \cdot \overline{AL}^2 \cdot 2\,AF.$$

Making $AL = x$ and $2\,AF = p$ (1195),

$$v = \pi x^2 p.$$

Replacing x^2 by $\frac{y^4}{4\,p^2}$ (1197),

$$= \frac{\pi y^4}{4\,p}.$$

CURVES OF THE SECOND DEGREE, OR CONIC SECTIONS

1225. A parabola may be considered as the limit of an ellipse when its major axis approaches infinity, the distance between one vertex and focus remaining constant (1202).

The parabola may also be considered as the limit of the hyperbola.

Placing the origin at the vertex of the ellipse, of the hyperbola and of the parabola, these three curves are represented by the general equation,

$$y^2 = 2\,px + qx^2,$$

wherein $$p = \frac{b^2}{a} \text{ and } q = \frac{p}{a} = \frac{b^2}{a^2}.$$

According as $q < 0$, $q > 0$, or $q = 0$, the equation becomes,

1st. $$y^2 = 2\frac{b^2}{a}x - \frac{b^2}{a^2}x^2, \text{ ellipse;}$$

2d. $$y^2 = 2\frac{b^2}{a}x + \frac{b^2}{a^2}x^2, \text{ hyperbola;}$$

3d. $$y^2 = 2\frac{b^2}{a}x, \text{ parabola.}$$

Changing the origin to the vertex at the *left*, and thus changing x to $x - a$ in the general equation (1131), equation 1st is obtained.

In a like manner, changing the origin to the vertex at the *right*, and thus changing x to $x + a$, the general equation of the hyperbola (1171) becomes equation 2d.

1226. The ellipse, hyperbola, and parabola are called *second-degree curves*; because the equations of these curves are of the second degree (**1131**, **1171**, **1197**), and all equations of the second degree involving two variables represent these curves.

1227. The curve of intersection of any secant plane with a right cone of revolution (**841**) is of the second degree, unless the plane passes through the vertex.

The section is an ellipse if the plane cuts all the elements of the cone; and if the plane is perpendicular to the axis, the section is a circle (**843**).

The section is an hyperbola when the plane is parallel to two elements of the cone; one of the branches is on one nappe and the other branch on the other nappe of the cone.

When the plane is parallel to only one element, it cuts only one nappe, and the section is a parabola.

All planes which cut the elements of a cylinder of revolution determine an ellipse, which is as it should be, since a cylinder may be considered as a cone whose vertex is at infinity. Since the plane which determines the parabola or hyperbola is parallel to one or two elements, and therefore to the axis, it cannot cut the lateral surface except along an element (**842**), and therefore determines no curve.

Any ellipse or parabola may be laid out upon the lateral surface of a given cone of revolution. The same is true of the hyperbola when the angle between the asymptotes is less than the angle between the opposite elements of the cone. Because of these properties, the name *conic sections* is often given to curves of the second degree.

1228. *The ellipse, the hyperbola, and the parabola are convex curves;* that is, that a straight line cannot cut them in more than two points (**648**). This follows from the determination of the points common to a given straight line and an ellipse, hyperbola, or parabola.

1st. F and F' being the foci of the ellipse, if a point M of the given line MM' is on the curve, prolonging $F'M$ to C, making $MC = FM$, the point C is on the directrix circle described from the focus F' as center (**1155**), and determining the point f sym-

metrical to F with respect to MM' (836), it is seen that M is the center of a circle tangent to the directrix circle and passing through the two points F and f. Then (964) describing a circle passing through F and f and cutting the directrix circle in any two points I and I', if from the point of intersection E of Ff and II' a tangent to the directrix circle is drawn and the point of contact C connected to the focus F', the line CF' cuts MM' in the required point M.

Thus, to find the point M, describe the directrix circle, drop a perpendicular from F upon MM', draw an arbitrary circle through

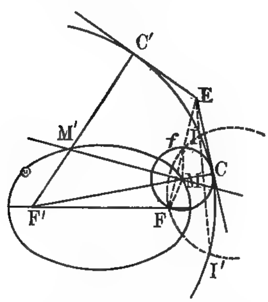

Fig. 337

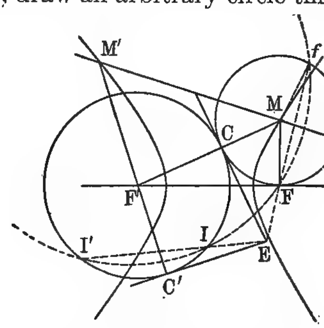

Fig. 338

F and its symmetrical f, and from the intersection E of Ff and II' draw a tangent EC to the directrix circle, then draw CF', which cuts MM' in M.

The second tangent EC' drawn through E to the directrix circle determines in the same way a second point M' common to the straight line MM' and the ellipse.

Since evidently there are as many common points as there are tangents to the directrix circle which pass through the point E, there are two, one, or none, according as the point E is outside of, upon, or inside of, the directrix circle. The line MM' is a secant in the first case, a tangent in the second, and does not meet the ellipse in the third.

2d. For the hyperbola the same course is followed. Thus, for the construction, from the focus F' as center describe the directrix circle (1185), determine the point f symmetrical to F with respect to MM', describe a circle passing through F and f and cutting the directrix circle in two points I and I', draw the chords fF and II', and through their point of intersection E draw the tangents EC and EC' to the directrix circle; then con-

necting the points of contact C and C' to the focus F', these lines cut the given line in the required points M and M'.

As in the preceding case, MM' has two, one, or no points common with the hyperbola, according as the point E is outside, upon, or within the directrix circle.

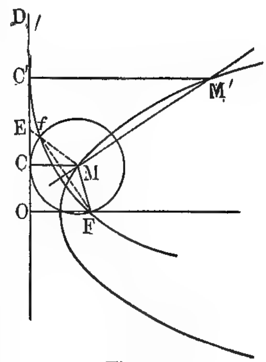

Fig. 339

3d. For the parabola, taking f symmetrical to the focus F with respect to MM', if the point M is on the curve, OD being the directrix, M is the center of a circle tangent to OD and passing through F and f. M may be determined without drawing the circle (960). Thus, draw FE perpendicular to MM', take EC a mean proportional between EF and Ef, or $\overline{EC}^2 = EF \times Ef$, and the perpendicular drawn through C to OD determines the point M. M' is also the center of a circle tangent to OD at C' and passing through F and f.

The tangent EC' to this circle gives $\overline{EC'}^2 = EF \times Ef = \overline{EC}^2$, then $EC' = EC$. Thus the same mean proportional laid off above and below E determines the two points M and M'.

When MM' passes through the focus, f coincides with the focus F, the points C and C' are obtained by erecting the perpendicular FE to MM' and taking $EC = EC' = EF$.

When MM' is parallel to the axis and consequently perpendicular to the directrix, C being the intersection of MM' with OD, draw FC and erect its perpendicular bisector which will cut MM' in the point M equally distant from F and C or OD, and is therefore on the parabola. M is the only point common to MM' and the curve; because any other point is unequally distant from F and OD, since it is not on the perpendicular bisector of FC.

If the point f is on OD, there is but one point in common, and MM' is tangent to the curve; and if f is on the other side of OD, there is no point in common, and MM' does not meet the curve.

LEMNISCATE. CISSOID. STROPHOID. LIMAÇON

1229. Although these four curves are of no great practical import, they nevertheless deserve to be mentioned.

1st. The *lemniscate* is the locus of the points M, such that the product $MF \times MF'$ of the radius vectors is equal to the square of half the focal distance FF'.

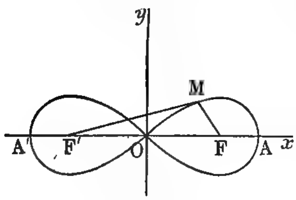

Fig. 340

Designating the constant FF' by $2a$, and MF and MF' by ρ and ρ', the equation of the curve in focal coördinates and in rectangular coördinates is respectively (1102),

$$\rho\rho' = a^2 \text{ and } y^2 = a\sqrt{4x^2 + a^2} - (x^2 + a^2). \tag{1111}$$

2d. A circle of diameter OA and a tangent to the circle at the extremity of this diameter being given, laying off on any secant OC, which passes through O, $OM = CD$, the locus of the positions of the point M is the *cissoid of Diocles*.

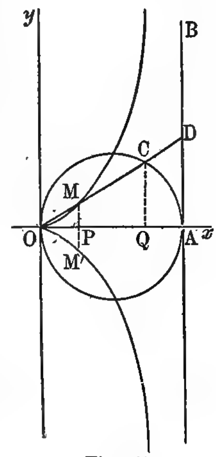

Fig. 341

Designating the diameter OA by a, the variable angle COx by α, and the variable distance OM by ρ, the equation of the curve in polar coördinates and rectangular coördinates is respectively,

$$\rho = \frac{a \sin^2 \alpha}{\cos \alpha} \text{ and } y = \pm x\sqrt{\frac{x}{a - x}}.$$

The curve has two symmetrical branches with respect to OA, and is included between Oy and AB, having AB for its asymptote.

3d. A right angle yOx and a fixed point A on one of its sides being given, draw any line AD through A, and from the point D at its intersection with Oy lay off $DM = DN = DO$; the locus of the points M and N is the *strophoid*.

Designating the constant OA by a, the variable angle DAx by α, and the variable distance AM or AN by ρ, the equation of the curve in polar coördinates and rectangular coördinates is respectively,

$$\rho = \frac{a(1 \pm \sin \alpha)}{\cos \alpha} \text{ and } y = \pm x\sqrt{\frac{a + x}{a - x}}.$$

The curve is symmetrical with respect to Ax. When the moving line occupies the position Ax, the two points M and N

coincide in O. When the ordinate OD becomes $\pm\infty$, the point N is at A and the point M at infinity; and since $DM = DN$, it is seen that in taking $OB = OA$, the perpendicular BE to Ax is asymptote to the two branches of the curve.

The tangents to the curve at O form angles of 45° with Ox, and are therefore perpendicular to each other. The perpendicular to Ax erected at A is also tangent to the curve.

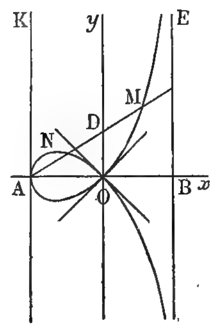

Fig. 342

4th. Through a point A on the circumference of a circle, draw any secant AD; starting from D, lay off on this secant a constant distance $DN = DM$. The locus of the points M and N is the *limaçon of Pascal.*

Designating the constant $DM = DN$ by a, the diameter AB by b, the variable angle DAx by α, and the variable distances AM and AN by ρ, A being the origin, the equation of the curve in polar coördinates and rectangular coördinates is respectively,

$$\rho = b \cos \alpha \pm a \text{ and } (y^2 + x^2 - bx)^2 = a^2 (y^2 + x^2).$$

The curve is symmetrical with respect to Ax. Fig. 343 shows a special case where $a < b$. When AD coincides with

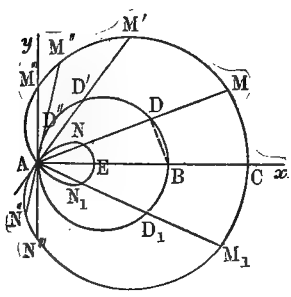

Fig. 343

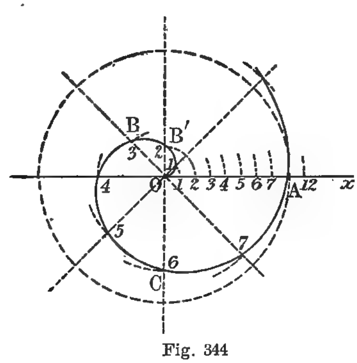

Fig. 344

Ax we have $BC = BE = a$, and one of the two branches starts from C and the other from E. The line AD turning comes into the position AD', which gives $AD' = a$; then the point N is at A, and AD' is tangent to the curve. Since beyond the position AD' we have $AD'' < a$, the point N'' is below Ax. For the posi-

tion $M'''N'''$ perpendicular to Ax, we have $AM''' = AN''' = a$. The angle α varying from 0° to 90° in the direction BD, the point M generates the arc CMM''' and the point N the arc $ENAN'''$ and these two arcs form half of the curve. α varying also from 0° to 90° in the direction BD_1, the point M describes the arc CM_1N''' and the point N the arc EN_1AM'''; these arcs are symmetrical to the first two with respect to Ax, and therefore meet and form a smooth, continuous curve.

NOTE. If $a = O$, the equation becomes $\rho = b \cos \alpha$, which is that of the circle AB.

THE SPIRAL ARCHIMEDES

1230. The spiral of Archimedes is a plane curve, traced by a point which moves about a fixed point O in such a manner that any two radius vectors are in the same ratio as the angles they make with the initial line Ox. Thus the spiral is defined by its equation in polar coördinates (**1100**).

Designating the coördinates by ρ and α,

$$\rho = a\alpha + b,$$

wherein:

ρ is the variable distance of the generating point from the pole or the radius vector;

α is the variable angle which the radius vector makes with the axis Ox;

a is the constant coefficient expressing the augmentation of ρ corresponding to the augmentation of α of one unit, of a degree for example;

b is a constant which expresses the value of ρ when $\alpha = O$; thus b is the distance from the pole O to the point in the axis Ox where the generating point starts.

In Fig. 344 the point starts from the pole, therefore $b = O$, and the equation of the curve is,

$$\rho = a\alpha.$$

1231. Each arc of the curve described by the point during one revolution about the pole, is called a *spire*.

The distance between any two consecutive spires, measured on the radius vector ρ, is constant, and is called the *pitch*. It represents the distance which the generating point travels away

from or toward the pole for each spire. Thus, for $\alpha = 360°$, and corresponding to $1°$, if we represent the pitch by p, we have,

$$p = a \times 360 \quad \text{or} \quad a = \frac{p}{360}.$$

1232. *To construct the spiral of Archimedes* (Fig. 344). Ox being the axis, O the pole, assuming $b = O$, that is, that the generating point starts from the pole O, lay off the pitch OA from O on Ox; divide OA into a certain number of equal parts, 8 for example; from the point O as center, with OA as radius, describe a circle, and divide it into the same number, 8, of equal parts. Drawing the radii to these points of division, and laying off on radius 1 the distance $O\,1$; on radius 2, the distance $O\,2$; on radius 3, the distance $O\,3$, etc., all the points thus determined lie upon the spiral.

Proof. Any of these points B gives,

$$OB : OA = BOA : 360, \text{ and } OB = \frac{OA}{360} \times BOA,$$

or

$$\rho = \frac{p}{360} \times \alpha = a\alpha. \qquad \textbf{(1230)}$$

To trace the second spire, prolong the radius vectors, and lay off the pitch OA upon each one, starting from the first spire. Thus, on $O\,1$ lay off OA from 1; on $O\,2$ lay off OA from 2, etc.

1233. *To draw a tangent to the spiral at a point M taken on the curve* (Fig. 345).

The following construction is based upon the general principle: *That the tangent to any curve generated by a point, whose motion has two components, is the diagonal of a parallelogram whose sides have the same directions as the two components of the motion and are equal to the distances passed through along the lines of these motions in the process of generation.*

For the spiral of Archimedes, the motion of the generating point M is composed of two components: one along a straight line OM, the other along a circle whose radius is OM, that is, along the perpendicular MC to the radius OM. Starting from M, lay off on MO the length MD equal to the pitch p of the spiral, and on the perpendicular MC lay off a length MC equal to the circumference $2\,\pi \times OM$ of the circle whose radius is OM; then completing the parallelogram $MDTC$, which in this case is

a rectangle, and has p and $2\pi \times OM$ for its sides, the diagonal MT is tangent to the curve at the point M.

Laying the length MC' equal to the arc MB described with the radius MO, off on MC, and completing the parallelogram $MOT'C'$, the diagonal MT' is also tangent to the curve, that is, MT' coincides with MT; and we have the following proportion:

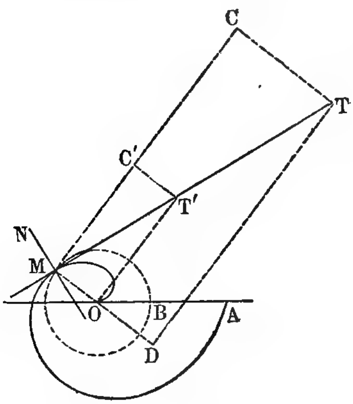

Fig. 345

$$OM : MC' = MD : MC,$$

or OM : arc $MB = p : 2\pi \times OM$.

1234. *To draw a normal to the spiral at the point M* (Fig. 345), draw the tangent MT, and the perpendicular MN erected to MT at the point M is the required normal.

1235. *The surface of a segment of a spiral OBB′O included by the radius vector OB and the arc BB′O subtended by it* (Fig. 344) is equal to one-third of the product of the surface of the circle whose radius is the radius vector $OB = \rho$ and the ratio of this radius to the pitch $OA = p$. Thus, s being the surface, we have (753)

$$s = \frac{1}{3}\pi \times \overline{OB}^2 \times \frac{OB}{OA} = \frac{1}{3}\pi\rho^2 \times \frac{\rho}{p} = \frac{\pi\rho^3}{3p}. \qquad (a)$$

From this it follows that the area of the surface included by the first spire is equal to one-third the surface of the circle whose radius is the pitch $OA = p$. Making $\rho = p$ in equation (a),

$$s = \frac{1}{3}\pi p^2. \qquad (a)$$

The surface of the first two spires is $\frac{8}{3}$ of the area of the circle whose radius is the pitch p. Putting $\rho = 2p$ in the general equation (a),

$$s = \frac{8}{3}\frac{\pi p^3}{p} = \frac{8}{3}\pi p^2.$$

Subtracting the area of the first spire from that of the two spires, we have the area of the second spire,

$$\frac{8}{3}\pi p^2 - \frac{1}{3}\pi p^2 = \frac{7}{3}\pi p^2.$$

Finally, to obtain the surface of the spiral S included between two radius vectors $OB = \rho$ and $OB' = \rho'$, take the difference between the segments which terminate at these radius vectors, thus:

$$S = \frac{\pi\rho^3}{3p} - \frac{\pi\rho'^3}{3p} = \frac{\pi}{3p}(\rho^3 - \rho'^3).$$

1236. *Volutes.* Having traced the first spiral with $b = 0$ (1232), if a second one is traced with $b = O\,1$, for example (Fig.

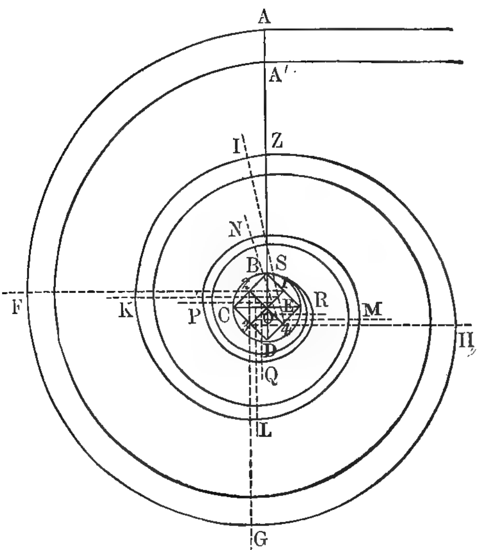

Fig. 346

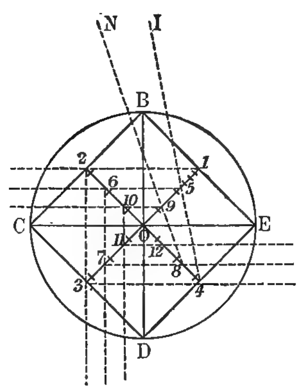

Fig. 347

344), the distance between the two spirals measured along any radius vector is constant and equal to $b = O\,1$.

Volutes are spiral ornaments which form the principal distinction of the Ionic capital. But they terminate in a central eye, and are made up of arcs of circles instead of spirals of Archimedes.

Let it be required to construct an Ionic volute according to the method of Vignole. Let the center O and the upper part AA' be given. From the point O as center, with a radius equal to $\frac{1}{9}$ the vertical distance OA', describe a circle, which is the *eye of the volute* (Fig. 347 represents this eye drawn to a larger scale); inscribe a square $BCDE$ in this circle so that the diagonal BD is

vertical and in line with AA'; divide each of the lines, which join the middle points of the opposite sides of the square, into 6 equal parts, and number the points of division as indicated in Figs. 346 and 347; draw the lines 1, 2; 2, 3; 3, 4; 4, 5; . . ., 11, 12.

That done, describe a series of arcs: AF from the point 1 as center, FG from the point 2 as center, GH from the point 3, HI from the point 4, IK from the point 5, etc., until RS from the point 12 has been described which terminates at the circumference of the eye. The successive arcs meet each other on a common tangent, since their centers are on the same line passing through the point of contact. The last arc RS is not tangent to the eye, but the angle is so small that the effect is not bad.

The second spiral is traced in the same manner; but starting from A', making AA' equal to one-fourth of AZ. Divide into four equal parts each of the three equal parts of $O\,1$, $O\,2$, $O\,3$, and $O\,4$ (Fig. 347), and take as centers for the successive arcs the points which lie nearest the first centers 1, 2, 3, . . ., 12.

INVOLUTE. EVOLUTE. RADIUS OF CURVATURE

1237. The *involute* of any curve is the curve $CC'C''C'''$ . . ., generated by the point C of a tangent CA, whose point of contact changes continually in such a manner that the distance from the point C to the point of contact is constantly equal to the distance traveled through by the point of contact along the curve; thus, $B'C'$, $B''C''$ . . ., being different successive positions of the tangent, we have $B'C' = B'C$, $B''C'' = B''C$ . . . The curve $CB'B''$ . . ., upon which the tangent rolls, is called the *evolute* of $CC'C''$ . . . The point C, where the evolute meets the involute, is the *origin*.

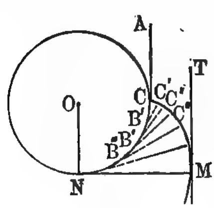

Fig. 348

1238. *The construction of the involute of a circle by points* (Fig. 348). C being the origin, if at different points B', B'' . . ., on the circumference of the circle, tangents are drawn, and lengths $B'C' = \text{arc } B'C$, $B''C'' = \text{arc } B''C$ . . ., laid off, the points C, C', C'' . . ., belong to the involute. (See its equation, **1270**.)

1239. *The construction of an involute by means of the radius of curvature.* When the points C, B', B'' . . ., are very close together (Fig. 348), the arcs CB', $B'B''$ . . ., may be considered

as straight lines, and we have $B'C' = B'C$. From this it follows that CC' may be considered as the arc of a circle having B' for its center and $B'C$ for its radius; for the same reason, $B''C''$ = arc $B''C = B''B' + B'C$, and $C'C''$ is the arc of a circle having B'' for its center and $B''C''$ for its radius, etc. Thus the involute may be considered as being made up of a series of arcs of circles, the centers and radii of which are determined.

This method is not very acceptable, since the radius of curvature is different for every point. However, although there is no instrument in common use by which the radius of curvature can be uniformly varied, this method is often used in practice.

Taking $B'B'' = B''B''' = \ldots$, the radii of curvature make equal angles with each other when the evolute is a circle.

1240. *To trace an involute by continuous motion.* Suppose a thread to be wound upon the curve CB''' (Fig. 348), and a pencil point fastened at the end C; if the thread is unwound and kept taut by pulling the pencil at C, the point C will describe an involute of the curve passing through the axis of the thread, which is very near to the evolute curve when the thread is very fine.

It may be noted that any other point on the thread describes a second involute everywhere equally distant from the first, and equal to it if the evolute is a circle.

1241. *To draw a normal and a tangent to an involute.* Drawing a tangent to the evolute at any point N, this tangent is normal to the involute. Then drawing a perpendicular MT to this normal MN at the point M, we have the required tangent. All tangents to the evolute are normals to the involute, and vice versa. Furthermore, the tangent MT to the involute and the normal NO to the evolute, both drawn at the extremities of the same radius of curvature, are parallel.

1242. *A curve CM being given to find its evolute* (Fig. 348). Take a series of points C, C', C'' . . ., on CM, and the normals to CM drawn through these points being tangent to the evolute (1241), inscribing a curve in the polygon $CB'B''$ . . ., whose vertices are the intersections of the consecutive normals, this curve may be taken as the evolute of CM.

CYCLOID

1243. If instead of the line rolling upon the circle as in the generation of the involute of a circle (1237), the circle rolls upon

the line AA', each point of the circumference of the circle describes a curve known as a cycloid between each consecutive contact with the line.

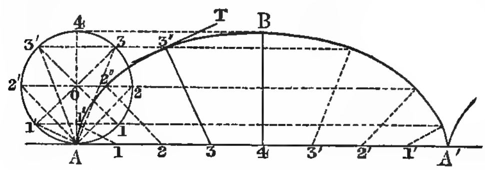

Fig. 349

Fig. 348 represents a cycloid ABA' described by the point A during one turn of the circle on AA'.

1244. The line AA', included between two consecutive contacts A and A' of a certain point A, is the *base* of the cycloid ABA' described by the point A. This base is equal to the circumference of the generating circle; d being the diameter of the circle, we have,

$$AA' = \pi d.$$

The perpendicular $B\,4$ at the middle of the base is the *axis* of the cycloid, and is equal to the diameter d; consequently we have (751, 752),

$$\frac{AA'}{B\,4} = \frac{\pi d}{d} = \pi = 3.1416 \cong \frac{22}{7};$$

$$AA' = 3.1416 \times d = \frac{22}{7}d \quad \text{and} \quad d = \frac{AA'}{3.1416} = \frac{7}{22}AA'.$$

1245. *The construction by points of the cycloid generated by the point A on the circumference of a circle of diameter d* (Fig. 349).

Draw a line AA' equal to the base πd of the cycloid; describe a circle O of diameter d, tangent to AA' at A; divide the base and the generating circle into the same number of equal parts, 8 for example, which are numbered as indicated in the figure. Through these points of division of the circle O draw parallels to the base AA', and through the points of division 1, 2, 3 . . . of the base draw parallels to the chords $A\,1'$, $A\,2'$ . . ., drawn from A to the different points of division 1, 2, 3 . . .; these parallels meet the parallels to the base in the points 1″, 2″, 3″ . . ., which are on the cycloid.

Proof. Considering any one of these points, 1″, when the point of contact is at 1, the diameter 1 3′ is perpendicular to AA' and the generatrix A occupies the same position with reference to the diameter as the point 1′ with reference to $A\,4$ in the figure; since this condition is fulfilled by 1″, this point is on the cycloid.

If the base AA' of the cycloid had been given instead of the diameter of the generating circle d, d would have been determined thus:

$$d = \frac{AA'}{\pi} \cong \frac{7}{22} AA'.$$

In the movement of the circle, the point A upon the circumference describes a cycloid ABA', the center O a parallel to AA', all points between the center O and the circumference a *prolate or inflected cycloid*, and all points outside of the circle a *curtate cycloid.*

1246. *To trace a cycloid by a continuous motion.* Take a circular plate with a pencil point fastened on the circumference, and roll the plate without sliding along the edge of a rule which coincides with the base AA'. Then the point A (Fig. 349) will describe the required cycloid ABA'.

1247. *To draw a normal and a tangent to a cycloid.* When the generating point A of the cycloid occupies any position $3''$ (Fig. 349), the point of contact being 3, the element $3''$ of the curve may be considered as coinciding with an element of an arc of a circle with its center at 3 and its radius $33''$; consequently the $33''$, being normal to the arc of the circle, is also normal to the curve. The perpendicular $3''T$ to $33''$ erected at $3''$ is tangent to the cycloid.

From that which has been said, it follows that in order to draw a normal and therefore a tangent to a cycloid, or an arc of a cycloid, at a point M, it suffices to determine the point of contact of the generating circle and the base corresponding to the point M. At a distance equal to the radius of the generating circle draw a parallel EE' to the base AA'; it is the locus of the center of the generating circle. When the generating point A is at M, the center of the circle is at a distance from M equal to the radius of the generating circle $\frac{1}{2}d$; consequently, from the point M as center, with $\frac{d}{2}$ as radius, describing an arc of a circle, it cuts the parallel EE' in a point O, which is the required center; and dropping a perpendicular OP upon AA', the point P is the

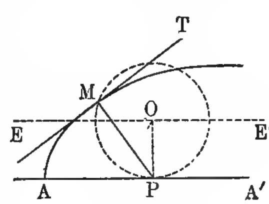

Fig. 350

point of contact. Then the line MP is the normal to the cycloid at the point M, and the perpendicular MT is the tangent.

1248. Drawing normals to the different points of the cycloid, its evolute is obtained (1242).

The radius of curvature at any point M of the cycloid (Fig. 350) *is double that portion of the normal MP included between the curve and its base;* from which it follows that the evolute may be traced by points. *The evolute of a semicycloid AB* (Fig. 349) *is a semicycloid equal to AB;* from which it follows that the semicycloid AB is also equal to its involute (1237).

1249. *The length of a cycloid* is equal to four times its axis or diameter d of the generating circle (1244). Thus, l being the length, we have,

$$l = 4\,d.$$

1250. *The surface S included by the cycloid and its base is three times that of the generating circle.* Thus, d being the diameter of the circle, we have (753),

$$S = \frac{3}{4}\pi d^2.$$

1251. The cycloid being reversed, that is, traced on the under side of the base, is a *tautochrone*; that is, a curve such that a body rolling down it under the influence of gravity, assuming that there is no friction, will always reach the lowest point in the same time, no matter from which point it may start.

In its normal position the cycloid is also a *brachystochrone*; that is, a curve such that a body starting from any point, impelled solely by the force of gravity, will reach another point of it in a shorter time than it could by any other path. It is sometimes called the *curve of quickest descent.*

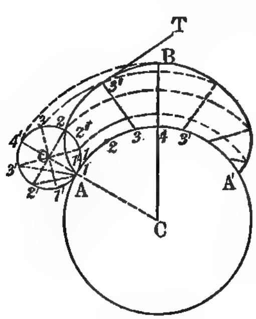

Fig. 351

EPICYCLOID

1252. If the generating circle O, instead of rolling on a straight line as in (1243), rolls on a circle C, any point A on the circumference of O describes between the two consecutive points of contact A and A', a curve ABA' called an *epicycloid.*

When the circle O rolls on the inside of the circumference C, each of the points of O describes a curve called an *hypocycloid.*

1253. The arc AA' of the circle C included between the two points of contact A and A' is the *base* of the epicycloid. This base is equal to the circumference πd of the generating circle O.

The straight line CB, drawn through the center C and the middle of the base, is the axis of the epicycloid, and we have $B4 = d$. Thus (1244),

$$\frac{AA'}{B4} = \frac{\pi d}{d} = \pi \simeq \frac{22}{7}.$$

$$AA' = \pi d \simeq \frac{22}{7} d \text{ and } d = \frac{AA'}{\pi} = \frac{7}{22} AA'.$$

The point B where the axis cuts the curve is the *vertex* of the curve.

1254. *To trace an epicycloid by points.* This method is analogous to that for the cycloid in (1245). Thus, taking the base $AA' = \pi d$, and describing the circle O of diameter d, tangent to the circle C at A, divide the base AA' and the circumference of O into the same number of equal parts, 8 for example, numbered as shown in Fig. 351. From the point C as center, with the distances from the center C to the points of division on the circle O as radii, describe the arcs concentric with AA', and from the points of division 1, 2, 3 . . ., of AA' as centers with radii equal respectively to the distances A to the points of division $1'$, $2'$, $3'$ . . ., of the circle O, describing arcs, these arcs cut the concentric arcs in points $1''$, $2''$, $3''$ . . ., on the epicycloid. If the base AA' had been given instead of the diameter d, we would have,

$$d = \frac{AA'}{\pi} \simeq \frac{7}{22} AA'.$$

Any point situated between O and A describes a *prolate* epicycloid, and any point outside the circle O describes a *curtate* epicycloid (1245).

1255. *To trace an epicycloid by a continuous motion.* C and O being circular plates, and A a point of a pencil fixed in the circumference of O, rolling the plate O upon the plate C, the point C describes an epicycloid.

1256. *To draw a normal and a tangent to an epicycloid* (Fig. 351).

For the same reason as in (1247) the line $33''$, which joins any point $3''$ of the curve to the corresponding point of contact 3, is normal to the curve at $3''$. The perpendicular $3''T$ to this normal is tangent to the epicycloid.

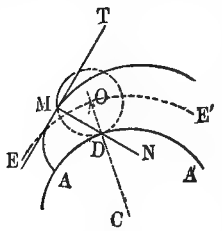

Fig. 352

From this it follows that in order to draw a normal and therefore a tangent to an epicycloid or an arc of an epicycloid at any given point M, it suffices to find the point of contact corresponding to this point M.

Describing an arc EE' concentric to the base AA' of the epicycloid and at a distance from AA' equal to the radius $\frac{d}{2}$ of the generating circle O. EE' is the locus of the center of the generating circle. When the generating point A is at M, the center of the circle is at a distance $\frac{d}{2}$ from M; consequently, describing an arc from M as a center, with $\frac{d}{2}$ as radius, this arc cuts EE' at the center O corresponding to M, and joining O and C the point of contact D is obtained. Then MD is the normal to the curve, and the perpendicular MT to MD at M is the tangent.

1257. *Length of the epicycloid.* The length $\frac{l}{2}$ of the semi-epicycloid AB (Fig. 351) is a fourth proportional to the three lengths CA, $CA + AO$, and $2\,A\,4'$; thus, making $CA = r$ and $A\,4'$ equal to d, we have,

$$r : \left(r + \frac{d}{2}\right) = 2\,d : \frac{l}{2} \text{ and } \frac{l}{2} = \frac{2\,rd + d^2}{r}.$$

For the hypocycloid the sign of $\frac{d}{2}$ would be changed in the above equation, thus:

$$r : \left(r - \frac{d}{2}\right) = 2\,d : \frac{l}{2} \text{ and } \frac{l}{2} = \frac{2\,rd - d^2}{r}.$$

For $d = r$, we have $\frac{l}{2} = r$. In this case each point on the circumference of O moves along a diameter of the circle C, and the hypocycloid is a diameter of the circle C.

REMARK. When the radius r is infinity, that is, when the circle C becomes a straight line, the first ratio of the preceding

proportion becomes equal to 1 and therefore also the second, and we have $\frac{l}{2} = 2\,d$ or $l = 4\,d$; the epicycloid has become a cycloid (1249).

1258. The total surface S included by an epicycloid ABA' and its base AA' (Fig. 351), is a fourth proportional to the three quantities: the radius $CA = r$, $3\,CA + 2\,AO$ or $3\,r + d$, and the surface $\frac{\pi d^2}{4}$ of the generating circle (753); thus,

$$r : (3\,r + d) = \frac{\pi d^2}{4} : S \text{ and } S = \frac{3\,\pi r d^2 + \pi d^3}{4\,r}.$$

For the hypocycloid, we have,

$$r : (3\,r - d) = \frac{\pi d^2}{4} : S \text{ and } S = \frac{3\,\pi r d^2 - \pi d^3}{4\,r}.$$

For $d = r$, we have $S = \frac{1}{2}\pi r^2$, that is, in this case the area of the hypocycloid is equal to that of the semicircle C (1257).

Remark. As in the preceding article, when $r = \infty$, dividing the consequents by 3, the first ratio of the preceding proportion becomes equal to 1, and we have,

$$\frac{\pi d^2}{4} = \frac{S}{3} \text{ and } S = \frac{3}{4}\pi d^2;$$

that is, the epicycloid has become a cycloid (1250).

HELIX

1259. The *helix* is a curve generated by a point which moves upon the lateral surface of a cylinder, advancing uniformly in the direction of the axis while revolving at a constant speed about it. That is, it advances an equal amount for each revolution about the axis. The *pitch* is this amount, BK, which the generating point advances in the direction parallel to the axis OA for each revolution about the cylinder (Fig. 353).

That portion of the curve which corresponds to one complete revolution of the generating point is called a *spire*.

1260. From the definition (1259) it follows that the curve $BCDE\ldots$ being a helix traced on a right cylinder, the plan of which is a circle O' and the elevation a rectangle with the axis OA, C and E being any two points on the helix, we have,

$$CM : EN = \text{arc } B'C' : \text{arc } B'E'.$$

$B'C'$ being a unit arc, representing the corresponding constant quantity CM by a, and designating the variable arc $B'E'$ by x and the corresponding value EN by y, we have,

$$a : y = 1 : x, \text{ and } y = ax,$$

which is the equation of a straight line (**1117**), and indicates that if the surface of the cylinder were developed, each spire would develop as a straight line of equation $y = ax$, in which y is any ordinate CM, DO, or EN, etc., and the corresponding x is the development of $B'C'$, $B'D'$, or $B'E'$, etc.

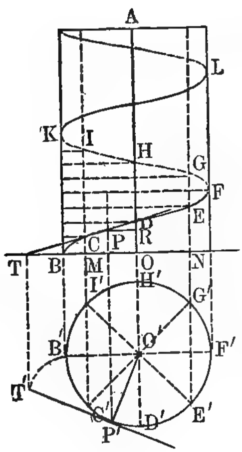

Fig. 353

From this it follows that the developments of thé different spires are equal parallel lines, each the hypotenuse of a right triangle, one side of which is the pitch BK, and the other the development of the base of the cylinder.

1261. *To draw an helix* (Fig. 353). BK being the pitch, divide BK and the base of the cylinder into the same number of equal parts, 8 for example. Drawing the lines $BK, IM, HO \ldots$ through the points of division $B', C', D' \ldots$ of the circumference of the base of the cylinder, and laying off from the base on these successive lines, O, $\frac{1}{8}BK$, $\frac{2}{8}$ BK, $\frac{3}{8}$ BK, etc., the points B, C, D, etc., which are obtained belong to the helix. When, instead of tracing the helix on a cylinder, its projections are traced as indicated in Fig. 353, the perpendiculars drawn to the axis OA of the cylinder through the points of division of the pitch BK, meet the projections IM, HO, etc., of the lines drawn through the points of division of the circumference of the base of the cylinder, on the points C, D, etc., which belong to the vertical projection of the helix, the base of the cylinder being the horizontal projection of the helix.

1262. *To draw a tangent to an helix at the point P'* (Fig. 353). Draw the line PP' parallel to the axis and passing through the point P; at the foot P' of this line draw a tangent $P'T'$ to the base of the cylinder, taking it equal to the development of the arc

$P'B'$; joining the point T' to the point P, the line $T'P$ is the required tangent.

Proof. According to the principle in (1233), the tangent to the point P is the diagonal of a parallelogram, which in this case is a rectangle, having the altitude of P above the base and PT' for its sides, therefore the diagonal coincides with $T'P$.

$P'T'$ is the horizontal projection of the tangent; and taking the vertical projection T of the point T', the line TP is the vertical projection of the tangent.

The vertical projection TP is tangent to the point P at the vertical projection $BCFP$ of the helix. It is to be noted that any tangent coincides with the curve when the latter is developed.

1263. *The normal* at a point P on the helix is the perpendicular dropped from the point P to the axis of the cylinder. It is projected on the horizontal as a radius $O'P'$, and on the vertical as a perpendicular PR to the axis OA.

1264. The length L of an arc BP of an helix is equal to the hypotenuse of a right triangle whose sides are the distance from the point P to the base of the cylinder and the development $P'T'$ of the arc $P'B'$.

Proof. When the cylinder is developed, these three lines become the sides of a right triangle, and we have (730),

$$L = \sqrt{\overline{PP'}^2 + \overline{T'P'}^2}.$$

This same triangle being the surface S included by the arc BCP of the helix, the arc of the circle $B'P'$, and the perpendicular dropped from P to the base, we have (718),

$$S = \frac{PP' \times T'P'}{2}.$$

For a spire, designating the pitch by p, if the cylinder is one of revolution and of radius r, we have (752),

$$L = \sqrt{p^2 + 4\,\pi^2 r^2} \text{ and } S = p\pi r.$$

MISCELLANEOUS CURVES

1265. A curve being given, it is possible that by rigorous geometrical processes tangents and normals may be drawn to it. But when this is not possible, approximate methods must be used.

1st. *To draw a tangent and a normal through a point M taken on any curve RMS.*

First construct an auxiliary curve $B''Mb''$, as follows: from the point M draw secants to each side of the given curve RMS; starting from the curve, lay off on the secants the equal lengths AB, $A'B'$ . . . ab, $a'b'$ . . ., having care to lay off these lengths toward the inside of the curve on one side of the point and away from the curve on the other; then draw a smooth curve through the points thus obtained on the secants, which gives the required auxiliary curve $B''Mb''$. This curve passes through the point M, since evidently there is some secant for which the chord Ma_1

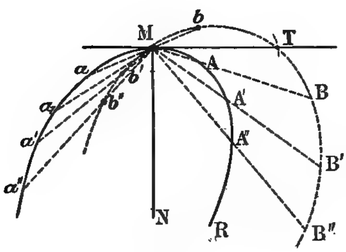

Fig. 354

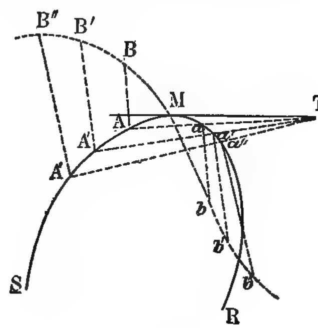

Fig. 355

is equal to the constant AB. Furthermore, if from the point M as center, with the constant AB as radius, an arc is described, this arc will cut the auxiliary curve $B''Mb''$ at the point T, which is on the required tangent; because this tangent may be considered as a secant drawn through the point M of intersection of the curves, and the point T on the tangent giving $MT = AB$, it is seen that T must lie on the curve $B''Mb''$.

To draw a normal to any curve RMS at the point M, draw the tangent MT, and the perpendicular MN to MT at M is the required normal.

2d. *To draw a tangent to any curve RMS from a point T exterior to the curve.*

The tangent MT, may be traced with sufficient accuracy with the aid of a rule. But since the actual point of contact is uncertain, drawing the secants TA, TA' . . ., from the point T, at the extremities of the chords Aa, $A'a'$ . . ., erecting perpendiculars in opposite directions equal in length to the respective chords ($AB = Aa = ab$, $A'B' = a'b' = A'a'$ . . .), and tracing the

curve $B''Mb''$ through the extremities of these perpendiculars, this curve cuts the given curve at the point of contact M. The chords Aa, $A'a'$ . . ., instead of radiating from T may be drawn parallel to MT.

If it is desired to draw a tangent parallel to a given line, with the triangles draw the tangent (948) and then determine the point of contact M by means of an auxiliary curve $B''Mb''$.

Being able to draw a tangent parallel to a given line or making a given angle with a given line (955), we can also draw a normal making any given angle with a given line.

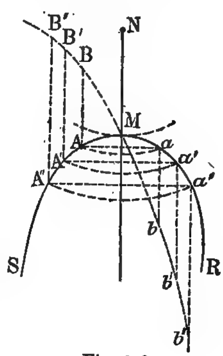

Fig. 356

To draw a normal to the curve from a point N outside of the curve.

From the point N as center, describe a series of arcs Aa, $A'a'$ . . ., at the extremities of each of the chords Aa, $A'a'$ . . ., erect perpendiculars in opposite directions and equal to the respective chords ($AB = ab = Aa$, $A'B' = a'b' = A'a'$ . . .), then the curve $B''Mb''$ drawn through the extremities of these perpendiculars cuts the given curve at the foot of the required normal NM.

Proof. Among the arcs described from N as center there is one which is tangent to the curve, and therefore its point of contact is at the foot of the normal; furthermore, it is on the curve $B''Mb''$, since the chord to this arc and the perpendicular reduce to a single point M. Therefore MN is the required normal.

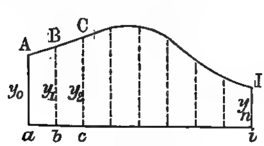

Fig. 357

1266. To obtain *the length of any curve* AI, divide it into parts AB, BC . . ., so small that they may be considered as straight lines; with the aid of the compasses lay off these parts on a straight line (1111), and the length of the line is the approximate length of AI.

1267. *The area S of the plane surface $AIia$ included by any plane curve AI and its projection ai upon the straight line ai* (1039).

Dividing AI into parts AB, BC . . ., so small that they may be considered as straight lines, and drawing the perpendiculars Bd, Cc . . ., the surface $AIia$ is divided into elements $abBA$, $bcCB$ . . ., which may be considered as trapezoids, and we have (723),

$$S = abBA + bcCB + \cdots = ab\frac{aA + bB}{2} + bc\frac{bB + cC}{2} + \cdots \quad (a)$$

Let $ai = E$; $ab = bc = \cdots = \frac{E}{n}$, which assumes the projection ai to be divided into n equal parts, and $aA = y_0$, $bB = y_1$, $cC = y_2 \ldots$, $iI = y_n$ be the different ordinates of the curve.

Substituting these expressions in the equation (a),

$$S = \frac{E}{n} \times \frac{y_0 + y_1}{2} + \frac{E}{n} \times \frac{y_1 + y_2}{2} + \cdots + \frac{E}{n} \times \frac{y_{n-1} + y_n}{2}.$$

Simplifying, we have,

$$S = \frac{E}{n}\left(\frac{y_0}{2} + y_1 + y_2 + \cdots y_{n-1} + \frac{y_n}{2}\right),$$

which is simple and easy to apply.

1268. Thomas Simpson's *formula.* This formula gives the area of a plane curve $AIia$ (Fig. 357) more accurately than the preceding one. The number n of divisions of ai being even, Thomas Simpson has shown that the area S of the curve is given approximately by the following expression:

$$\frac{E}{3\,n}[y_0 + y_n + 4\,(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2\,(y_2 + y_4 + y_6 + \cdots + y_{n-2})].$$

$\frac{E}{n}$ being the distance between two consecutive ordinates, it is seen that *the approximate value of the area S of the curve is equal to the product of a third of the distance $\frac{E}{3\,n}$ between two consecutive ordinates, and the sum $(y_0 + y_n)$ of the two extreme ordinates, plus 4 times the sum of the odd ordinates $(y_1 + y_3 + y_5 + \cdots y_{n-1})$, plus 2 times the sum of the even ordinates $(y_2 + y_4 + y_6 + \cdots y_{n-2})$.*

For $n = 8$, we have,

$$S = \frac{E}{3 \times 8}[y_0 + y_8 + 4\,(y_1 + y_3 + y_5 + y_7) + 2\,(y_2 + y_4 + y_6)].$$

Remark. In case one or both extremities of the curve fall upon the base line ai, the ordinates at those points are put equal to 0 and the above formulas used.

If the curve is closed, draw a line through the middle and find the area on each side of the line. This may be done in one single

operation by taking the ordinates as the sums of the corresponding ordinates of the two parts of the curve and using the above formulas.

Derivation of the preceding formula.

It may be assumed without appreciable error that the arc ABC of the curve (Fig. 357) coincides with the arc of a parabola passing through the three points A, B, and C, and having its

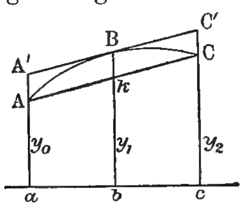

Fig. 358

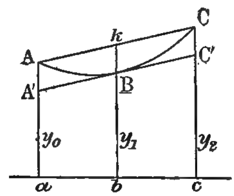

Fig. 359

axis parallel to Aa, Bb is a diameter of the parabola, the $A'C'$ (Fig. 358) drawn through B parallel to AC is tangent to the parabola (1214), and the parabolic segment ACB is $\frac{2}{3}$ of the parallelogram $ACC'A'$ (1221). The portion s of the area bounded by the ordinates y_0 and y_2 is therefore equal to the area of the trapezoid $acCA$ plus $\frac{2}{3}$ the parallelogram $ACC'A'$; and representing $ab = bc$ by δ, we have (721, 723),

$$s = 2\,\delta\left(bk + \frac{2}{3}kB\right),$$

or, noting that $bk = \frac{y_0 + y_2}{2}$ and $kB = bB - bk = y_1 - \frac{y_0 + y_2}{2}$,

$$s = 2\,\delta\left[\frac{y_0 + y_2}{2} + \frac{2}{3}\left(y_1 - \frac{y_0 + y_2}{2}\right)\right] = \frac{\delta}{3}(y_0 + 4\,y_1 + y_2).$$

This expression of the value of s is the same when the arc ABC has its convex side toward ac; because we have,

$$s = 2\,\delta\left(bk - \frac{2}{3}Bk\right),$$

or, noting that $bk = \frac{y_0 + y_2}{2}$ and $Bk = bk - bB = \frac{y_0 + y_2}{2} - y_1$,

$$s = 2\,\delta\left[\frac{y_0 + y_2}{2} - \frac{2}{3}\left(\frac{y_0 + y_2}{2} - y_1\right)\right]$$
$$= \frac{\delta}{3}(y_0 + 4\,y_1 + y_2).$$

The portions of the area included between the ordinates y_2 and y_4, y_4, and y_6, . . ., y_{n-2} and y_n, are respectively,

$$\frac{\delta}{3}(y_2 + 4y_3 + y_4), \frac{\delta}{3}(y_4 + 4y_5 + y_6) \ldots, \frac{\delta}{3}(y_{n-2} + 4y_{n-1} + y_n).$$

Summing all these partial areas, replacing $\frac{\delta}{3}$ by $\frac{E}{3n}$ and simplifying, we have the formula for S as given above.

1269. Poncelet, following a different method, derived the following formula for the area S included by a curve:

$$S = \frac{E}{n}\left[2(y_1 + y_3 + \ldots + y_{n-1}) + \frac{1}{4}(y_0 + y_n) - \frac{1}{4}(y_1 + y_{n-1})\right].$$

This formula, in which n is an even number, shows that *the area S is about equal to the product of the distance $\frac{E}{n}$ between two consecutive ordinates, and twice the sum $(y_1 + y_3 + \cdots y_{n-1})$ of the odd ordinates plus a quarter of the sum $(y_0 + y_n)$ of the ordinates at the extremities, less a quarter of the sum $(y_1 + y_{n-1})$ of the second and the next to the last ordinates.*

The formula of Poncelet gives results oftentimes more accurate than that of Simpson, and has the advantage that all the even ordinates except y_0 and y_n do not enter into consideration.

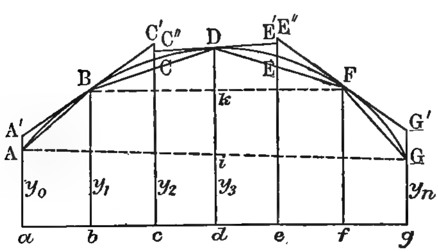

Fig. 360

Derivation of the formula of Poncelet. Join the extremities A and G to the nearest vertices B and F, then join every other one as indicated in the figure. The sum s of the areas of the trapezoids $abBA$, $bdDB$ . . ., $fgGF$, thus formed, is

$$s = \frac{\delta}{2}(y_0 + y_1) + \delta(y_1 + y_3) + \cdots + \delta(y_{n-3} + y_{n-1}) + \frac{\delta}{2}(y_{n-1} + y_n),$$

wherein $$\delta = \frac{E}{n} = ab = bc = \cdots.$$

or, $$s = \delta\left[\frac{1}{2}(y_0 + y_n) + \frac{3}{2}(y_1 + y_{n-1}) + 2(y_3 + y_5 + \cdots + y_{n-3})\right];$$

adding and subtracting in the parenthesis the quantity $\frac{1}{2}(y_1+y_{n-1})$:

$$s=\delta\left[\frac{1}{2}(y_0+y_n)-\frac{1}{2}(y_1+y_{n-1})+2(y_1+y_3+y_5+\cdots+y_{n-1})\right].$$

Now drawing tangents through the extremities B, D . . ., of the odd ordinates, and calling S' the sum of the areas of the trapezoids $acC'A'$, $ceE'C''$ . . ., thus formed, we have,

$$s'=2\delta(y_1+y_3+y_5+\cdots\cdot y_{n-1}).$$

The mean of these two areas s and s', one of which is smaller and the other larger than the required area S, give an approximate value of the latter. Thus,

$$S=\frac{s+s'}{2}=\delta\left[2(y_1+y_3+\ldots+y_{n-1})+\frac{1}{4}(y_0+y_n)-\frac{1}{4}(y_1+y_{n-1})\right].$$

In the above, $\delta=\frac{E}{n}$.

This expression being a mean between s and s', gives the required area S with an error whose upper limit is

$$\frac{s'-s}{2}=\frac{1}{4}\delta\left[(y_1+y_{n-1})-(y_0+y_n)\right],$$

and which is ordinarily much below this limit.

Drawing the chords AG and BF, we have, on the mean ordinate,

$$dk=\frac{1}{2}(y_1+y_{n-1})\quad\text{and}\quad di=\frac{1}{2}(y_0+y_n);$$

therefore, $$\frac{s'-s}{2}=\frac{1}{2}\delta(dk-di)=\frac{1}{2}\delta\times ik.$$

Thus, having to apply the formula, it is an easy matter to determine the maximum error which this formula will give.

A NOTE ON THE POLAR COÖRDINATE SYSTEM

1270. 1st. *The polar equation of a straight line.* If the distance from the line to the pole is designated by p, and the slope of the line with reference to the polar axis by α, the coör-

dinates ρ and ω of any point of this line have the following relation,

$$\frac{p}{\rho} = \sin(\omega - \alpha),$$

from which,
$$\rho = \frac{p}{\sin(\omega - \alpha)}. \tag{1}$$

If the line passes through the pole, we have $p = 0$ and $\omega = \alpha$; the equation (1) becomes indeterminate,

$$\rho = \frac{0}{0};$$

but then the variable ω becomes constant α, for any value of the radius vector. Therefore we have,

$$\omega = \alpha = \text{constant}.$$

2d. *The polar equation of a circle.* If the center of the circle is taken as pole, designating the radius as R, we have,

$$\rho = R = \text{constant}.$$

If the center is not at the pole, as in Fig. 283 (1223), and if the coördinates of the center are designated by β and α, and those of any point M on the circumference by ρ and ω, taking OA as axis, the equation of the circle R is,

$$\rho^2 + \beta^2 - 2\beta\cos(\omega - \alpha)\rho - R^2 = 0. \tag{1}$$

If the pole is placed on the circumference in A, as in Fig. 343 (1229), and if Ax is the polar axis, we have in the preceding equation (1) $\beta = R$ and $\alpha = 0$, and the equation reduces to,

$$\rho = 2R\cos\omega.$$

Putting
$$2R = b = AB,$$

$$\rho = b\cos\omega. \tag{2}$$

Thus we find that which was indicated in the remark concerning the limaçon of Pascal (1229, 4th).

In the equation (2), by varying ω from 0° to 90°, positive values for the radius vector are obtained, which determine the semicircle above the polar axis AB; and then by varying ω from 90° to 180°, the values of $\cos\omega$ are negative, and are plotted below the axis, which gives the semicircle below the axis.

3d. *Another polar equation of the circle.*

If the circle is tangent to the polar axis at the pole B, as in Fig. 244 (1017), the equation is deduced from equation (1) by putting $\beta = R$, and $\alpha = 90°$, which gives,

$$\rho = 2\,R \sin \omega.$$

From this it follows that taking the diameter $AB = b = 1$, a chord such as $BD = \rho$ is the measure of the sine of the angle $DBC = \omega$. This property may be used for constructing a graphical table for giving approximate values of the sines of angles.

4th. *The polar equation of the ellipse, the hyperbola, and the parabola.*

If, for the ellipse and hyperbola, the focus at the right is taken as the pole and in the parabola the focus is taken as pole (Fig. 290, ellipse, Fig. 310, hyperbola, Fig. 322, parabola), the three curves have the common equation,

$$\rho = \frac{p}{1 + e \cos \omega},$$

wherein $$p = \frac{b^2}{a}, \qquad e = \frac{c}{a};$$

a and b are the semi-axes of the ellipse and hyperbola, and $2\,c$ is the focal distance.

The ratio $\frac{c}{a}$ gives the relations,

$$\frac{c}{a} < 1 \text{ for the ellipse,}$$

$$\frac{c}{a} > 1 \text{ for the hyperbola,}$$

$$\frac{c}{a} = 1 \text{ for the parabola.}$$

5th. *Spiral of Archimedes* (see 1230) is represented by the equation,

$$\rho = a\omega + b.$$

(See 1339, rectification of the spiral of Archimedes.)

Logarithmic spiral is represented by the equation,

$$\log \rho = k\omega \quad \text{or} \quad \omega = A^{k\omega}.$$

NOTE. The general equation for its development is,

$$S = k'\rho,$$

that is, it is proportional to the radius vector (see **1339** for its application).

6th. *Parabolic spiral.* The simplest has the following equation:

$$\rho = k\omega^2.$$

The radius vector is proportional to the square of the angular values. (See **1338**, its rectification and its application.)

7th. *Parabolic spirals of different degrees.* The general equation of all the parabolic spirals is,

$$\rho^m = k\omega^n.$$

EXAMPLE. $\rho^2 = k\omega^3, \qquad \rho^3 = k\omega^4$, etc.

8th. *Hyperbolic spiral* is represented by the equation,

$$\rho = \frac{k}{a\omega},$$

from which, $\rho\omega = \frac{k}{a} =$ constant.

The general equation of the hyperbolic spirals is,

$$\rho^m\omega^n a^n = k = \text{constant}.$$

PART VI

ELEMENTS OF CALCULUS

DIFFERENTIAL CALCULUS

INTRODUCTION

1271. *Variable. Constant. Function. Explicit function. Implicit function.*

A *variable* is a quantity which takes successively different values, and a *constant* is one that retains the same value throughout the calculation. The nature of the problem to be solved indicates which are variables and which constants.

Knowing the law according to which a quantity varies, this quantity may be made to take different values, and each of these particular values may be determined. Given the equation,

$$y = ax \tag{1}$$

of a straight line passing through the origin (1117). It is seen immediately that x and y are variables, and that a is a constant. x may be varied from $-\infty$ to $+\infty$, and y will also vary from $-\infty$ to $+\infty$, and giving x a determinate value the preceding equation gives the value of y, or giving a determinate value to y, the corresponding value of x is obtained.

Given the equation,
$$y^2 + x^2 = r^2, \tag{2}$$
or
$$y = \pm \sqrt{r^2 - x^2} \tag{3}$$

of a circle with its center at the origin. Here it is also seen that x and y are variables and r a constant. x may be given all the values from $-r$ to $+r$ (538). For $x = 0$, $y = \pm r$, and for $x = \pm r$, $y = 0$; thus y varies also from $-r$ to $+r$.

From an equation between two variables y and x, such as (1) and (2) for example, by giving any value to one of these variables the corresponding value of the other may be deduced; this is expressed when each variable is said to be a *function* of the other.

However, if the equation is solved for y, as in equation (3) for example, the name *function* applies more particularly to the variable y, and that of *independent variable* to x, to which arbitrary values are given in order to deduce the corresponding values of the function or *dependent variable.*

In general, when an algebraic relation exists between any number of variables x, y, z, solving the equation for one of these variables, x for example, an algebraic expression is obtained which may be represented by

$$x = f(y, z),$$

and is pronounced, *x is a function of y and z,* and signifies that x is dependent upon the variables y and z.

Representing the volume of a rectangular parallelopiped by V, and its dimensions by x, y, and z, we have (887),

$$V = xyz \quad \text{or} \quad V = f(x, y, z);$$

the volume of the parallelopiped is a function of its three dimensions x, y, and z.

When an equation involving several variables is solved with respect to one of these variables, this variable is called an *explicit function;* if the equation is not solved, each variable is an *implicit function* of the others. y is an explicit function of x in equation (3), and an implicit function of x in equation (2). By solving the equation, the implicit function becomes an explicit function.

1272. *Graphic representation of functions.* No matter what the nature of the variable quantities which enter in the algebraic expression may be, when this expression contains only two variables, a curve, whose coördinates represent the two variables to a given scale, can be constructed (1113).

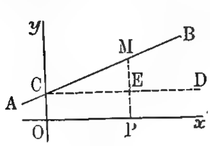

Fig. 361

EXAMPLE 1. Given the equation

$$y = ax + b,$$

in which x and y are the variables, and a and b the constants. As we have seen (1117), this is the equation of a straight line AB, b is the ordinate OC at the origin, and a is the slope. For $x = 0$, $y = b$, and taking $OC = b$, the point C is on the straight line AB; making $x = OP$, and taking $y = ax + b$, the point M belongs also to the line,

which is then determined by the points C and M, and may be indefinitely prolonged.

EXAMPLE 2. Let S be the area of a rectangle, b and h its two dimensions, then (716),

$$S = bh.$$

Supposing the base b constant and the altitude h variable, this equation is one of a straight line passing through the origin. Taking an abscissa equal to a value of h, and erecting an ordinate equal to bh, we have a second point on the line, which is then determined.

Any ordinate of this line represents the area S of this rectangle whose altitude is the corresponding abscissa, that is, that this area S will contain the unit of surface as many times as the ordinate contains the unit of length.

EXAMPLE 3.
$$y = ax^2. \tag{1}$$

y and x being variables and a a constant, this is the equation of a parabola (1197), which is constructed by assuming different values for x and calculating the corresponding values of y.

From equation (1),

$$\frac{y}{a} = x^2,$$

and putting
$$\frac{1}{a} = 2\,p,$$

$$2\,py = x^2.$$

The quantity p is the distance from the focus to the directrix, and $\frac{p}{2}$ is the distance from the vertex of the parabola to the focus and the directrix.

REMARK. The law of falling bodies,

$$h = \frac{1}{2}gt^2,$$

is of the same form as (1).

EXAMPLE 4. The function,

$$y = ax^3,$$

in which a is a constant, is an equation of parabolic curve of the

third degree, which can be constructed by points, giving different values to x and solving for y.

The curve which represents the following equation may be constructed in the same way:

$$V = \frac{4}{3}\pi R^3,$$

V being the volume of a sphere (920).

Any ordinate of the curve would express the volume of the sphere whose radius is the abscissa R; that is, the sphere would contain as many units of volume as the ordinate contained units of length.

The functions,

$$y = x^3 - ax^2 + bx - c \quad \text{and} \quad y = x^5 + ax^4 + x^3 - bx^2 - cx - d,$$

which contain different powers of the independent variable, may also be represented by curves constructed by points (580).

EXAMPLE 5. A variable quantity may be a function of several other variables.

Thus, a being a constant,

$$V = axyz.$$

Such a function may be plotted when the values of y and z, which correspond to different values of x, are known.

If, for example, $\qquad xyz = x',$

then $\qquad V = ax',$

and the values of V are represented by the ordinates of a straight line passing through the origin, the values of x' being the abscissas.

If we had $\qquad y = axz^2,$

according as we put

$$xz^2 = x' \quad \text{or} \quad xz^2 = x'^2,$$

we would have,

$$y = ax' \quad \text{or} \quad y = ax'^2,$$

which are the equation of a straight line and that of a parabola, respectively.

These divers examples show that the value of any function may be represented by the ordinates of a curve, the abscissas of which represent the values of the independent variable.

Conversely, any curve referred to two axes represents the law of the simultaneous variation of two variables x and y.

1273. *The variation of functions. Increasing and decreasing functions.*

EXAMPLE 1. Let an equation of the first degree, involving two variables, that is, an equation of a straight line, be given (1272),

$$y = ax + b. \tag{1}$$

If the independent variable $x = OP$ is increased by a quantity $PP' = \alpha$, the function $y = MP$ becomes $y' = M'P'$, and we have,

$$y' = a(x + \alpha) + b; \tag{2}$$

subtracting (1) from (2),

$$y' - y = a\alpha \quad \text{or} \quad M'Q = a \times PP';$$

dividing both members by α,

$$\frac{y' - y}{\alpha} = a \text{ or } \frac{M'Q}{PP'} = a,$$

which shows that the ratio of the increment $y' - y$ of the function y to that of the increment α of the variable x, is independent of these increments.

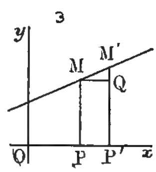

Fig. 362

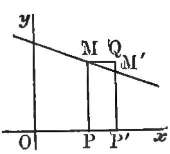

Fig. 363

EXAMPLE 2. Given the function,

$$y = ax^2,$$

x becoming $(x+\alpha)$, and designating the new value of the function by y',

$$y' = a(x + \alpha)^2 = ax^2 + 2a\alpha x + a\alpha^2.$$

The increment of the function is,

$$y' - y = 2a\alpha x + a\alpha^2 \text{ and } \frac{y' - y}{\alpha} = 2ax + a\alpha.$$

The ratio $\dfrac{y' - y}{\alpha}$ is not independent of α, as in the first example; but, according as α decreases, the term $a\alpha$ becomes smaller and smaller, and it is evident that α may become so small that the term $a\alpha$ may be neglected in comparison to the term $2ax$, and at the limit we have,

$$\frac{y' - y}{\alpha} = 2ax.$$

Thus the ratio $\frac{y' - y}{a}$ has a determinate limit $2\,ax$.

This property is general for all algebraic relations involving two variables.

A function $y = f(x)$ is *increasing* or *decreasing* according as y increases or decreases when x increases. Thus, Fig. 362 represents an increasing function, and Fig. 363 a decreasing function. The same function can be alternately increasing and decreasing.

1274. *A differential quantity. Differential coefficient. Derivative. Object of differential calculus.* When the increment a of the abscissa or variable x is small, it is designated by Δx, pronounced *delta x*, and may be considered as a fraction of x; in the same way a small increment $y - y'$ of y is designated by Δy. Thus,

$$\frac{y' - y}{a} = \frac{\Delta y}{\Delta x}.$$

When Δy and Δx decrease and become infinitely small, the limit is represented by dx and dy. In example 2 of the preceding article, the limit of the ratio of the increments of y and x is,

$$\frac{dy}{dx} = 2\,ax \quad \text{and} \quad dy = 2\,axdx,$$

which shows that an infinitely small increment dy of the function or ordinate y is expressed algebraically by the product of the infinitely small increment dx of the variable abscissa x and the variable coefficient $2\,ax$.

The quantities dy and dx, considered as being infinitely small, are called the *differentials of y and x.* The coefficient $2\,ax$ by which the differential dx is multiplied to obtain the differential dy, is called the *differential coefficient.*

The ratio $\frac{dy}{dx}$ is called the *derivative* of y with respect to x, or the derivative of the function y with respect to the variable x; it is equal to the differential coefficient.

In the preceding example the inverse ratio,

$$\frac{dx}{dy} = \frac{1}{2\,ax},$$

is the derivative of x with respect to y; x is then the function,

and y the variable. Ordinarily the derivative $\frac{dy}{dx}$ is designated by y' or $f'(x)$; thus,

$$\frac{dy}{dx} = y' = f'(x);$$

which indicates that the derivative of the function y is taken with respect to the variable x. If the derivative of x with respect to y had been taken, we would have,

$$\frac{dx}{dy} = x' = f'(y).$$

The difference between two quantities must not be confused with the differential of a quantity. Thus, having

$$y' - y = 2\,aax + ax^2,$$

the differential of the function y is

$$dy = 2\,axdx.$$

It is seen that the difference between two quantities, no matter how small, may be expressed in numbers, while the differential dy cannot.

The differential of a quantity must be considered as an algebraic expression or symbol resulting from a calculation; but a derivative $\frac{dy}{dx}$ has a perfectly determinate value, and may be expressed in numbers.

The chief purpose of differential calculus is the determination of the law which governs the increments of a function and those of the variable upon which it depends, that is, the value of the ratio $\frac{dy}{dx}$.

1275. *Geometric interpretation of the derivative of a function.* Let C be any curve referred to two rectangular coördinate axes, whose equation is,

$$y = f(x).$$

$f(x)$ represents the calculations which must be made in constructing the curve by points, by assuming different values of x and calculating the corresponding values of y. Let us consider the points M and M' of the curve C whose coördinates are respectively y_1x and $y'x'$. It is seen that in going from M to M',

the ordinate increases by the amount $M'Q$, which is positive or negative according as the function is increasing (Fig. 364) or decreasing (Fig. 365), and the ratio of the simultaneous increments of the ordinates and abscissas is,

$$\frac{M'Q}{PP'} = \frac{y' - y}{x' - x}.$$

Drawing the secant MM', the ratio $\frac{y' - y}{x' - x}$ is the tangent of the angle α, which is included by MM' and the x-axis; and if the point M' approaches M indefinitely, that is, if the increments

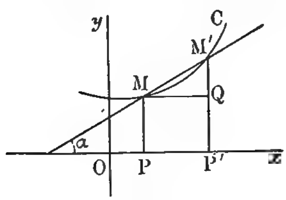

Fig. 364

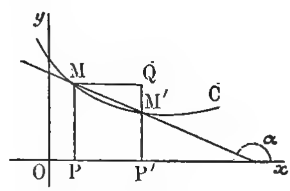

Fig. 365

are indefinitely decreased, the tangent MM' will approach a limit where it is tangent to the curve and M' coincides with M. This corresponds to the limit,

$$y' = \frac{dy}{dx} \text{ from the ratio } \frac{y' - y}{x' - x}.$$

Thus *the limit of the ratio of the simultaneous increments of a function y and the variable x is equal to the tangent of the slope of the curve C which represents the function.* The determination of this limit of the ratio solves the general problem of tangents, which is an important application of differential calculus.

DIFFERENTIALS AND DERIVATIVES OF FUNDAMENTAL FUNCTIONS

$$y = x^m. \quad y = \log x, \quad y = \sin x.$$

1276. *Let it be given to determine the derivative of the differential of*

$$y = x^m. \tag{1}$$

1st. *Assume that m is whole and positive.* Giving x an increment Δx, a corresponding increment Δy follows for y, and equation (1) becomes (564),

$$y + \Delta y = (x + \Delta x)^m = x^m + mx^{m-1}\Delta x + \frac{m(m-1)}{1.2} x^{m-2} (\Delta x)^2 + \cdots \quad (2)$$

Subtracting (1) from (2),

$$\Delta y = mx^{m-1}\Delta x + \frac{m\,(m-1)}{1.2} x^{m-2} (\Delta x)^2 + \cdots$$

Dividing both members by Δx,

$$\frac{\Delta y}{\Delta x} = mx^{m-1} + \frac{m\,(m-1)}{1.2} x^{m-2}\Delta x + \cdots$$

which shows that the value of the ratio $\frac{\Delta y}{\Delta x}$ contains one term mx^{m-1} independent of the increment Δx, but that all the others contain Δx as a factor. Since Δx may be taken infinitely small, the terms which contain Δx as a factor become negligible when Δx and Δy are infinitely small, and we have as a limit,

$$\lim \frac{\Delta y}{\Delta x} \text{ or } \frac{dy}{dx} = mx^{m-1}, \quad (3)$$

or (1274) $\quad y' = f'(x) = mx^{m-1},$

which shows that *to obtain the derivative y' of the function $y = x^m$ it suffices to take the variable x with its exponent m for coefficient, and the same exponent less one $m - 1$ for an exponent.* Thus, for

$$y = x^5, \text{ we have } \frac{dy}{dx} \text{ or } y' = 5\,x^4,$$

and for $y = x$, we have $y' = x^0 = 1$. (553)

From the equation (3),

$$dy = mx^{m-1}dx,$$

which shows that the differential dy of the function y is equal to the derivative of y with respect to x, multiplied by the differential dx of the variable x.

2d. *Case where the exponent of x is a fraction.* Given the function

$$y = x^{\frac{p}{q}},$$

in which p and q are whole positive numbers. Raising both members to the q power, we have (555),

$$y^q = x^p.$$

Taking the differential of each member (1st),

$$qy^{q-1}dy = px^{p-1}dx.$$

Transposing, $$\frac{dy}{dx} = \frac{p}{q}\,\frac{x^{p-1}}{y^{q-1}}.$$

Having $$x^{p-1} = \frac{x^p}{x} \text{ and } y^{q-1} = \frac{y^q}{y}, \tag{555}$$

we have $$\frac{dy}{dx} = \frac{p}{q}\,\frac{x^p}{x}\,\frac{y}{y^q};$$

and since $$y^q = x^p,$$

$$\frac{dy}{dx} = \frac{p}{q}\,\frac{y}{x}.$$

Substituting $$x^{\frac{p}{q}} \text{ for } y,$$

$$y' = f'\,(x) \text{ or } \frac{dy}{dx} = \frac{p}{q}\,x^{\frac{p}{q}-1}.$$

Thus the 1st rule applies in this case. From this equation, we have the equation of the differential, thus;

$$dy = \frac{p}{q}\,x^{\frac{p}{q}-1}\,dx.$$

EXAMPLE. $$y = \sqrt{x} = x^{\frac{1}{2}},$$

$$y' = \frac{dy}{dx} = \frac{1}{2}\,x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}, \tag{554}$$

and $$dy = \frac{dx}{2\sqrt{x}}.$$

3d. *Negative exponent.* Given

$$y = x^{-m},$$

in which m is a whole number. This may be written (**518**),

$$y = \frac{1}{x^m}, \tag{1}$$

or $$\frac{1}{y} = x^m. \tag{2}$$

Increasing x by Δx and y by Δy, the equation (2) becomes,

$$\frac{1}{y+\Delta y} = (x+\Delta x)^m. \tag{3}$$

Subtracting (2) from (3),

$$\frac{1}{y} - \frac{1}{y+\Delta y} = x^m - (x+\Delta x)^m,$$

$$\frac{y+\Delta y - y}{y^2 + y\Delta y} \quad \text{or} \quad \frac{\Delta y}{y^2+y\Delta y} \quad - = [\,(x+\Delta x)^m - x^m],$$

$$\frac{\Delta y}{\Delta x}\frac{1}{y^2+y\Delta y} = \frac{-\,[(x+\Delta x)^m - x^m]}{\Delta x}.$$

Making the increments infinitely small and approaching the limit, $y\Delta y$ is negligible, and from (1st) we have,

$$\frac{dy}{dx}\frac{1}{y^2} = -\ mx^{m-1};$$

and

$$y' = \frac{dy}{dx} = -\ y^2mx^{m-1}. \tag{4}$$

Since the equation (1) gives $y^2 = \frac{1}{x^{2m}}$, by replacing y^2 by its value in (4), we have,

$$y' = \frac{dy}{dx} = -\ \frac{1}{x^{2m}}mx^{m-1} = -\ mx^{m-1-2m} = -\ mx^{-m-1}. \tag{5}$$

Thus the rule given in 1st applies when the exponent of the variable x is negative. The relation (5) shows that the derivative is negative. This is as it should be, because, according to equation (1), an increment Δx of the variable x corresponds to a diminution of y, that is, a negative increment of the function.

The differential is deduced from (5), thus:

$$dy \text{ or } dx^{-m} = -\ mx^{-m-1}dx.$$

1277. *Derivative and differential of*

$$y = \log x. \tag{1}$$

x increasing by Δx, y increases by a corresponding quantity Δy, and we have,

$$y + \Delta y = \log(x+\Delta x). \tag{2}$$

Subtracting (1) from (2),

$$\Delta y = \log(x + \Delta x) - \log x = \log \frac{x + \Delta x}{x} = \log\left(1 + \frac{\Delta x}{x}\right), \quad (396)$$

$$\frac{\Delta y}{\Delta x} = \frac{\log\left(1 + \frac{\Delta x}{x}\right)}{\Delta x}. \qquad (3)$$

Putting

$$\Delta x = \frac{x}{m} \quad \text{or} \quad \frac{\Delta x}{x} = \frac{1}{m},$$

expression (3) becomes:

$$\frac{\Delta y}{\Delta x} = \frac{\log\left(1 + \frac{1}{m}\right)}{\frac{x}{m}} = \frac{m}{x}\log\left(1 + \frac{1}{m}\right) = \frac{\log\left(1 + \frac{1}{m}\right)^m}{x}.$$

Taking the limit dx of Δx, which corresponds to $m = \infty$, and representing the limiting value of $\left(1 + \frac{1}{m}\right)^m$ by e, we have

$$y' = \frac{dy}{dx} = \frac{\log e}{x}.$$

To obtain the value of e, expand $\left(1 + \frac{1}{m}\right)^m$ by the binomial theorem of Newton (564):

$$\left(1 + \frac{1}{m}\right)^m = 1 + m\frac{1}{m} + \frac{m(m-1)}{1 \cdot 2} + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}\frac{1}{m^3} + \cdots +$$
$$+ \frac{m(m-1)(m-2)\cdots(m-n+1)}{1 \cdot 2 \cdot 3 \cdots n}\frac{1}{m^n} + \cdots + \frac{1}{m^m};$$

canceling the common factors in each term and dividing by m,

$$\left(1+\frac{1}{m}\right)^m = 1+1+\frac{1}{1 \cdot 2}\left(1-\frac{1}{m}\right) + \frac{1}{1 \cdot 2 \cdot 3}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right) + \cdots +$$
$$+\frac{1}{1 \cdot 2 \cdot 3 \cdots n}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdots\left(1-\frac{n+1}{m}\right) + \cdots + \frac{1}{m^m};$$

and if $m = \infty$, the terms $\frac{1}{m}, \frac{2}{m} \ldots$ become zero:

$$e = \left(1 + \frac{1}{m}\right)^m = 1 + 1 + \frac{1}{1.2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots + \frac{1}{1 \cdot 2 \cdot 3 \ldots n}$$
$$+ \frac{1}{1 \cdot 2 \cdot 3 \cdots n(n+1)} + \frac{1}{1 \cdot 2 \cdot 3 \cdots n(n+1)(n+2)} + \cdots.$$

The terms containing n, having the common factors $\frac{1}{1.2\cdot 3\cdot\cdots n}$ the limit of their sum, less the first term, is

$$\frac{1}{1\cdot 2\cdot 3\cdots n}\lim\left[\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\cdots\right]. \quad (4)$$

The sum of the fractions placed in parentheses being smaller than the sum of the terms of the descending geometrical progression,

$$\frac{1}{n+1}:\frac{1}{(n+1)^2}:\frac{1}{(n+1)^3}\cdots,$$

of which the first term and the constant multiplier are $\frac{1}{n+1}$, this sum having $\frac{1}{n}$ for its limit, the sum of the terms within the parentheses of expression (4) is smaller than $\frac{1}{n}\cdot$ Therefore, the value of e may be calculated with any desired degree of approximation, and it is found that

$$e = 2.7182818 \text{ and } \log e = 0.4342945. \quad \mathbf{(407)}$$

The derivative of $y = \log x$ is, therefore,

$$y' = \frac{dy}{dx} = f'(x) = \frac{\log e}{x} = \frac{0.4342945}{x},$$

and the differential is

$$dy = \log e\frac{dx}{x}\cdot$$

1278. *Derivative and differential of*

$$y = \sin x. \quad (1)$$

Giving the increment Δx to the arc x, the function y or the sin takes the corresponding increment Δy, and (1) becomes,

$$y + \Delta y = \sin(x + \Delta x). \quad (2)$$

Subtracting (1) from (2),

$$\Delta y = \sin(x + \Delta x) - \sin x. \quad (3)$$

Since (**1276**)

$$\sin p - \sin q = 2\cos\frac{1}{2}(p + q)\sin\frac{1}{2}(p - q), \quad (4)$$

putting

$$p = x + \Delta x \quad \text{and} \quad q = x,$$

we have,

$$p+q=2x+\Delta x,\ p-q=\Delta x, \quad \text{or} \quad \frac{1}{2}(p+q)=x+\frac{\Delta x}{2},\ \frac{1}{2}(p-q)=\frac{\Delta x}{2}.$$

The relation (4) applied to the difference (3) gives

$$\Delta y = 2 \cos\left(x+\frac{\Delta x}{2}\right) \sin\left(\frac{\Delta x}{2}\right).$$

Dividing both members by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{2 \cos\left(x+\frac{\Delta x}{2}\right) \sin\left(\frac{\Delta x}{2}\right)}{\Delta x}.$$

Dividing both terms of the fraction in the second member by 2,

$$\frac{\Delta y}{\Delta x} = \frac{\cos\left(x+\frac{\Delta x}{2}\right) \sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}.$$

The ratio of the $\sin\left(\frac{\Delta x}{2}\right)$ to $\frac{\Delta x}{2}$ having 1 for its limit (1277), we have,

$$\frac{dy}{dx} = \cos\left(x+\frac{dx}{2}\right).$$

$\frac{dx}{2}$ being negligible, we have,

$$y' = \frac{dy}{dx} = \cos x;$$

and the differential is,

$$dy = \cos x dx.$$

THEOREMS OF DIFFERENTIATION

1279. *The derivative and differential of a constant quantity are zero.*

Given the functions,

$$y = F(x), \tag{1}$$
$$y = F(x) + C, \tag{2}$$

which differ only by the constant C.

From (1), $y + \Delta y = F(x + \Delta x);$
from (2), $y + \Delta y = F(x + \Delta x) + C;$

both of these expressions give the same value for the increment Δy of the function

$$\Delta y = F(x + \Delta x) - F(x).$$

Therefore both give,

$$\frac{\Delta y}{\Delta x} = \frac{F(x + \Delta x) - F(x)}{\Delta x},$$

and $$y' = \frac{dy}{dx} = \lim \frac{F(x + \Delta x) - F(x)}{\Delta x} = F'(x).$$

Thus the derivatives of the functions (1) and (2) are the same, as are also their differentials; thus both give,

$$dy = F'(x)\,dx.$$

The constant C disappears in the process of differentiation.

1280. *The derivative and differential of the sum of several functions are respectively the sum of the derivatives and the sum of the differentials of the functions.*

Given the sum

$$y = F(x) + F_1(x) + F_2(x) + \ldots \qquad (1)$$

in which $F(x)$, $F_1(x)$, $F_2(x)$ ..., designate different algebraic quantities expressed in terms of x; for example,

$$F(x) = \log x,\ F_1(x) = \sin x,\ F_2(x) = x^m \ldots$$

If x is increased by the increment Δx, the quantity y increases by a corresponding increment Δy, and relation (1) becomes,

$$y + \Delta y = F(x + \Delta x) + F_1(x + \Delta x) + F_2(x + \Delta x) + \cdots \qquad (2)$$

Subtracting (1) from (2), we have,

$$\Delta y = [F(x + \Delta x) - F(x)] + [F_1(x + \Delta x) - F_1(x)] + [F_2(x + \Delta x) - F_2(x)] + \cdots$$

dividing both members by Δx and equating the limits,

$$\frac{dy}{dx} = \lim \frac{F(x + \Delta x) - F(x)}{\Delta x} + \lim \frac{F_1(x + \Delta x) - F_1(x)}{\Delta x} + \cdots$$

or

$$y' = F'(x) + F'_1(x) + F'_2(x) + \cdots = \frac{\log e}{x} + \cos x + mx^{m-1} + \cdots$$

which was to be proved. In the same manner the differential is obtained,

$$dy = F'(x)\,dx + F'_1(x)\,dx + F'_2(x)\,dx + \cdots$$
$$= \frac{\log e}{x}\,dx + \cos x dx + m x^{m-1}\,dx + \cdots$$

1281. *The derivative of the product of several functions or variables is equal to the sum of the products which are obtained by multiplying the derivative of each function by the product of the other variables.*

1st. Given, the function

$$y = uv; \tag{1}$$

deducing the derivative,

$$y' = vu' + uv', \tag{A}$$

in which the variables u and v are the functions of x, such that, for example,

$$u = \log x, \quad v = \sin x.$$

Increasing x by the increment Δx, u, v, and y take the corresponding increments Δu, Δv, and Δy, and relation (1) becomes,

$$y + \Delta y = (u + \Delta u)(v + \Delta v) = ux + v\Delta u + u\Delta v + \Delta u\Delta v. \tag{2}$$

Subtracting (1) from (2),

$$\Delta y = v\Delta u + u\Delta v + \Delta u \Delta v;$$

dividing by Δx,

$$\frac{\Delta y}{\Delta x} = v\frac{\Delta u}{\Delta x} + u\frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta x}\Delta v;$$

and equating the limits,

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = vu' + uv', \tag{3}$$

or

$$y' = vu' + uv'.$$

The limit $\frac{du}{dx}\,dv$ of $\frac{\Delta u}{\Delta x}\,\Delta v$ is negligible, since the factor dv is an infinitesimal. u' and v' designate the derivatives of u and v with respect to x, and the relation (3) is the required derivative. For $u = \log x$, and $v = \sin x$, we have (1277, 1278),

$$\frac{dy}{dx} = \sin x\,\frac{\log e}{x} + \log x \cos x.$$

From the relation (3) the differential is deduced,

$$dy = vu'dx + uv'dx = vdu + udv;$$

which gives, in this case,

$$dy = \sin x \frac{\log e}{x} dx + \log x \cos x dx, \tag{4}$$

and $$y = \sin x \frac{\log e}{x} + \log x \cos x.$$

Thus, *the derivative of the product of two variables is equal to the sum of the products obtained by multiplying each variable by the derivative of the other.*

2d. Given the product, $y = stv,$ (5)

of three variables which are functions of x; we have, for example,

$$s = x^m, \quad t = \log x, \quad v = \sin x.$$

Putting $st = u$, $du = sdt + tds$, the relation (5) becomes,

$$y = uv;$$

and from (1st) its derivative is,

$$y' = \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Substituting for u and du,

$$\frac{dy}{dx} = st\frac{dv}{dx} + \frac{v}{dx}(sdt + tds) = st\frac{dv}{dx} + sv\frac{dt}{dx} + tv\frac{ds}{dx},$$

Designating $\frac{dv}{dx}$, $\frac{dt}{dx}$ and $\frac{ds}{dx}$ respectively by v', t', and s',

$$y' = \frac{dy}{dx} stv' + svt' + tvs' \tag{6}$$

which gives that which was to be proved. Applying the formula (6) to the given example, we have,

$$y' = \frac{dy}{dx} = x^m \log x \cos x + x^m \sin x \frac{\log e}{x} + \log x \sin x m x^{m-1}.$$

The differential of y is deduced from (6),

$$dy = stv'dx + svt'dx + tvs'dx,$$

and for the given example, we have,

$$dy = x^m \log x \cos x dx + x^m \sin x \frac{\log e}{x} dx + \log x \sin x m x^{m-1} dx.$$

In the same manner it may be shown that this theorem applies to any number of factors.

3d. *Special case where one of the factors is constant.*

Given the product,

$$y = ax^m,$$

in which the factor a is a constant.

Applying the general rule for the differentiation of two factors (1st),

$$\frac{dy}{dx} = amx^{m-1} + 0 = amx^{m-1};$$

and the differential is,

$$dy = amx^{m-1}dx.$$

Thus in the differentiation of a product, all constant factors enter both the derivative and the differential as coefficient.

1282. *Derivative and differential of a quotient or a fraction.*

Given the function,

$$y = \frac{u}{v}, \tag{1}$$

in which u and v are functions of the same variable x, we have, for example,

$$u = x^m \quad \text{and} \quad v = \log x.$$

From relation (1) we deduce (482),

$$y = uv^{-1}.$$

Applying the rule for the differentiation of the product of two factors (1281, 1st) and taking the differentials,

$$\boxed{dy = v^{-1}du - uv^{-2}\,dv = \frac{vdu}{v^2} - \frac{udv}{v^2} = \frac{vdu - udv}{v^2}}. \tag{2}$$

To obtain the derivative of relation (1),

$$u = yv.$$

Taking the derivative of both members with respect to x, we have (1281, 1st),

$$\frac{du}{dx} = y\frac{dv}{dx} + v\frac{dy}{dx},$$

and

$$\frac{dy}{dx} = \frac{du}{vdx} - \frac{y}{v}\frac{dv}{dx}. \tag{3}$$

Substituting $\frac{u}{v}$ for y,

$$\frac{dy}{dx} = \frac{du}{vdx} - \frac{u}{v^2}\frac{dv}{dx},$$

or designating the derivatives of y, u, and v, with respect to x, by y', u', and v',

$$y' = \frac{u'}{v} - \frac{uv'}{v^2} = \frac{vu' - uv'}{v^2}.$$

which shows that *the derivative of a quotient is equal to the product of the denominator by the derivative of the numerator less the product of the numerator by the derivative of the denominator, all being divided by the square of the denominator.*

Comparing the relations (2) and (4), it is seen that by replacing the word *derivative* by that of *differential* in the last rule, the rule for the differential of a quotient is obtained.

1283. *Derivatives of a function of a function.*

When a function is not expressed directly by the independent variable x, as in the examples

$$y = \log(\sin x), \quad y = \log(x^m), \quad y = \sin(mx + c),$$

it is said to be a *function of a function.* Such relations are written thus:

$$y = Ff(x).$$

In these examples the quantity within the parenthesis is itself a function of x; representing it by u, the preceding expressions may be written:

$$y = \log u \quad \text{or} \quad u = \sin x;$$

$$y = \log u \quad \text{or} \quad u = x^m;$$

$$y = \sin u \quad \text{or} \quad u = mx + c.$$

The quantity y is called the *principal function,* u the *subordinate function,* and x the *independent variable.*

It is easy to find an algebraic relation between these different quantities. Writing the identity

$$\frac{\Delta y}{\Delta x} \equiv \frac{\Delta y}{\Delta u} \times \frac{\Delta u}{\Delta x},$$

which is true no matter what the simultaneous increments Δx, Δu and Δy of the variables x, u and y may be.

Equating the limits

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}, \tag{a}$$

$\frac{dy}{du}$ being the derivative of y with respect to u, and $\frac{du}{dx}$ that of u with respect to x, it is seen that *the derivative of a function of a function is equal to the product of the derivatives of the simple functions which compose it.*

EXAMPLE 1. Find the derivative of

$$y = \log (\sin x). \tag{1}$$

Putting

$$u = \sin x, \tag{2}$$

the relation (1) becomes

$$y = \log u;$$

and taking the derivative (1755),

$$\frac{dy}{du} = \frac{\log e}{u}.$$

Taking the derivative of u with respect to x (1278), the relation (2) gives

$$\frac{du}{dx} = \cos x.$$

Substituting for $\frac{dy}{du}$ and $\frac{du}{dx}$ in relation (a),

$$\frac{dy}{dx} = \frac{\log e}{u} \cos x;$$

then substituting for u,

$$y' = \frac{dy}{dx} = \frac{\log e}{\sin x} \cos x = \frac{\log e}{\tan x}. \tag{1041}$$

Taking the differential,

$$dy = \frac{\log e}{\tan x} dx.$$

EXAMPLE 2. Find the derivative of

$$y = \cos x; \tag{3}$$

from (1053)

$$y = \sin (90° - x). \tag{4}$$

Putting

$$u = 90° - x, \tag{5}$$

and substituting in (4),

$$y = \sin u.$$

Taking the derivative of y with respect to the subordinate function u (1278),

$$\frac{dy}{du} = \cos u = \cos (90° - x) = \sin x.$$

From the relation (5), taking the derivative of u with respect to x (1276, 1279, 1288),

$$\frac{du}{dx} = -1.$$

Substituting for $\frac{dy}{du}$ and $\frac{du}{dx}$ in relation (a), we have

$$y' = \frac{dy}{dx} = -\sin x,$$

and the differential

$$dy = -\sin x dx.$$

EXAMPLE 3. *Derivative of a radical of the second degree.*

$$y = \sqrt{a^2 - x^2}. \tag{6}$$

Squaring, $$y^2 = a^2 - x^2.$$

Differentiating both members (1276, 1279, 1280),

$$2\,ydy = -2\,xdx.$$

Simplifying and transposing,

$$dy = \frac{-xdx}{y} = -\frac{xdx}{\sqrt{a^2 - x^2}},$$

and $$y' = \frac{dy}{dx} = -\frac{x}{\sqrt{a^2 - x^2}},$$

which may be written

$$y' = \frac{dy}{dx} = \frac{-2\,x}{2\sqrt{a^2 - x^2}};$$

that is, *the derivative of a radical of the second degree is obtained by dividing the derivative of the quantity under the radical by twice the radical.*

The same problem may be solved by aid of the theorem of a function of a function. Putting

$$u = a^2 - x^2, \quad du = -2\,xdx \quad \text{and} \quad \frac{du}{dx} = -2\,x.$$

The relation (6) may be written

$$y = \sqrt{u} = u^{\frac{1}{2}};$$

from (1276)

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\,u^{\frac{1}{2}}} = \frac{1}{2\sqrt{a^2 - x^2}}. \tag{553}$$

Substituting for $\frac{dy}{du}$ and $\frac{du}{dx}$ in (a)

$$\frac{dy}{dx} = \frac{1}{2\sqrt{a^2 - x^2}} \times -2x = -\frac{x}{\sqrt{a^2 - x^2}},$$

and
$$dy = -\frac{xdx}{\sqrt{a^2 - x^2}}.$$

EXAMPLE 4. Find the derivative of

$$y = \sqrt[3]{a^2 - x^2}.$$

Putting
$$u = a^2 - x^2$$
$$y = u^{\frac{1}{3}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{3}u^{\frac{1}{3}-1} \times (-2x) = \frac{-2x}{3(a^2 - x^2)^{\frac{2}{3}}}.$$

1284. *Generalization of the theorem of a function of a function.* Having

$$y = F(u), \quad u = F(v), \quad v = F(z), \quad z = F(x),$$

to determine the derivative $\frac{dy}{dx}$ of y with respect to x, proceed as follows: Giving an increment Δx to x, we have the corresponding increments Δz, Δv, Δu and Δy of the other variables, and we may write the identity

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \times \frac{\Delta u}{\Delta v} \times \frac{\Delta v}{\Delta z} \times \frac{\Delta z}{\Delta x}.$$

Equating the limits,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dz} \times \frac{dz}{dx}, \qquad (a)$$

which shows that *the derivative of a function of any number of functions of a variable x is equal to the product of the derivatives of the different functions.*

EXAMPLE 1. Find the derivative of

$$y = [\log \sin (x + a)]^3. \qquad (1)$$

Putting successively

$$x + a = z \qquad (2)$$
$$\sin z = v \quad \text{where} \quad v = \sin (x + a) \qquad (3)$$
$$\log v = u \quad \text{"} \quad u = \log \sin (x + a) \qquad (4)$$
$$u^3 = y \quad \text{"} \quad y = [\log \sin (x + a)]^3 \qquad (5)$$

and taking the derivatives of these successive functions (5), (4), (3) and (2), we have

$$\frac{dy}{du} = 3\,u^2 = 3\,[\log \sin\,(x + a)]^2 \tag{1276}$$

$$\frac{du}{dv} = \frac{\log e}{v} = \frac{\log e}{\sin\,z} = \frac{\log e}{\sin\,(x + a)} \tag{1277}$$

$$\frac{dv}{dz} = \cos\,z = \cos\,(x + a) \tag{1278}$$

$$\frac{dz}{dx} = 1. \tag{1279, 1280}$$

Substituting these values in (a),

$$y' = \frac{dy}{dx} = 3\,[\log\,\sin\,(x + a)]^2 \frac{\log e}{\sin\,(x + a)} \cos\,(x + a).$$

Multiplying both members by dx, the differential dy is obtained.

EXAMPLE 2. Find the derivative of

$$y = \log \sqrt{x + \sqrt{1 + x^2}}. \tag{1}$$

Putting $$u = \sqrt{x + \sqrt{1 + x^2}} \tag{2}$$

and $$z = x + \sqrt{1 + x^2}, \tag{3}$$

we have $$y = \log u, \tag{4}$$

$$u = \sqrt{z}. \tag{5}$$

From the theorem (a) and the derivatives of (4), (5) and (3),

$$y' = \frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dz}\right)\left(\frac{dz}{dx}\right),$$

$$y' = \left(\frac{\log e}{u}\right)\left(\frac{1}{2\sqrt{z}}\right)\left(1 + \frac{2\,x}{2\sqrt{1 + x^2}}\right),$$

or $$y' = \frac{\log e}{\sqrt{x + \sqrt{1 + x^2}}} \times \frac{1}{2\sqrt{x + \sqrt{1 + x^2}}} \times \left(\frac{\sqrt{1 + x^2} + x}{\sqrt{1 + x^2}}\right)$$

$$y' = \frac{\log e}{\sqrt{1 + x^2}}.$$

1285. *Derivatives and differentials of exponential functions;* that is, functions of the form

$$y = A^x, \tag{1}$$

in which y and x are variables and A a constant,

Taking the logarithms of both members of the equation (1),

$$\log y = x \log A, \tag{556}$$

in which $\log A$ is a constant quantity.

Taking the differentials of both members,

$$\frac{(\log e)\,dy}{y} = \log A dx, \qquad \text{(1277, 3d, 1281)}$$

and $$y' = \frac{dy}{dx} = \frac{\log Ay}{\log e} = \frac{(\log A)\,A^x}{\log e};$$

then $$dy = \frac{(\log A)\,A^x}{\log e} dx.$$

In the Napierian system, we have (407)

$$\frac{dy}{dx} = (\log_e A)\,A^x \text{ and } dy = (\log_e A)\,A^x dx.$$

Special Cases. If the constant A is equal to the base of e of the Napierian system, that is, if

1st. $$y = e^x$$

the derivative becomes

$$\frac{dy}{dx} = \frac{(\log e)\,e^x}{\log e} = e^x.$$

2d. For $y = e^{e^x}$, put $e^x = z$, then $y = e^z$. Therefore, the theorem

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

may be applied, which gives

$$y' = \frac{dy}{dx} = e^z e^x = e^{e^x} e^x.$$

3d. If the given function were

$$y = e^{-x},$$

we would have successively

$$\log y = -\, x \log e,$$

$$\frac{(\log e)\,dy}{y} = -\,(\log e)\,dx,$$

$$y' = \frac{dy}{dx} = -\frac{\log e}{\log e} y = e^{-x}.$$

1286. *Derivatives and differentials of the trigonometric functions* (1024). Such as

Direct Trigonometric Functions.	Inverse Trigonometric Functions.
1 $y = \sin x,$	5 $y = \text{arc}\,(\sin x),$
2 $y = \cos x,$	6 $y = \text{arc}\,(\cos x),$
3 $y = \tan x,$	7 $y = \text{arc}\,(\tan x),$
4 $y = \cot x,$	8 $y = \text{arc}\,(\cot x).$

1st. For $y = \sin x$, we have (1278)

$$y' = \frac{dy}{dx} \text{ or } f'(x) = \cos x \text{ and } dy = \cos x dx.$$

2d. For $y = \cos x$, we have (1283, EXAMPLE 2)

$$y' = \frac{dy}{dx} \text{ or } f'(x) = -\sin x \text{ and } dy = -\sin x dx.$$

3d. For $y = \tan x$, we may write

$$y = \frac{\sin x}{\cos x},$$

from (1282)

$$\frac{dy}{dx} = \frac{\cos x \cos x - (\sin x \times -\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.$$

Having $\cos^2 x + \sin^2 x = 1$ and $\cos^2 x = \dfrac{1}{1 + \tan^2 x}$, (1041)

we have $$y' = \frac{dy}{dx} = \frac{1}{\cos^2 x} = 1 + \tan^2 x,$$

and $$dy = (1 + \tan^2 x)\, dx.$$

4th. For $$y = \cot x$$
write (1041) $$y = \frac{\cos x}{\sin x},$$

and from (1282)

$$\frac{dy}{dx} = \frac{(\sin x \times -\sin x) - \cos x \cos x}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x}$$
$$= -(1 + \cot^2 x),$$

therefore $$y' = -(1 + \cot^2 x),$$

and $$dy = -\frac{dx}{\sin^2 x} = -(1 + \cot^2 x)\, dx.$$

Derivatives and differentials of inverse trigonometric functions.

5th. For $$y = \sin^{-1} x,$$

which indicates that y is the arc or angle whose tangent is equal to x, and we may write

$$x = \sin y,$$

which gives (1278) $$\frac{dx}{dy} = \cos y;$$

and $$y' = \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}};$$

then $$dy = \frac{dx}{\sqrt{1 - x^2}}.$$

6th. For $$y = \cos^{-1} x,$$

write $$x = \cos y.$$

From (1283, Example 2)

$$\frac{dx}{dy} = -\sin y,$$

then $$y' = \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}},$$

and $$dy = \frac{-dx}{\sqrt{1-x^2}}.$$

7th. For $$y = \tan^{-1} x,$$

write $$x = \tan y.$$

From (3) $$\frac{dx}{dy} = 1 + \tan^2 y;$$

then $$y' = \frac{dy}{dx} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2},$$

and $$dy = \frac{dx}{1+x^2}.$$

8th. For $$y = \cot^{-1} x,$$

write $$x = \cot y.$$

Taking the derivative

$$\frac{dx}{dy} = -(1+\cot^2 y) = -(1+x^2),$$

then $$\frac{dy}{dx} = \frac{-1}{1+x^2}.$$

1287. *Examples of derivatives of trigonometric functions.*

Example 1. Find the derivative of

$$y = \sin^{-1}\frac{\sqrt{2Rx - x^2}}{R}. \tag{1}$$

Putting $$z = \sqrt{2Rx - x^2}, \tag{2}$$

$$z^2 = 2Rx - x^2. \tag{3}$$

The relation (1) becomes

$$y = \sin^{-1}\frac{z}{R}. \tag{4}$$

From (1283) $$y' = \frac{dy}{dz} \times \frac{dz}{dx}. \tag{A}$$

From (4) we deduce

$$\frac{dy}{dz} = \frac{1}{\sqrt{1 - \frac{z^2}{R^2}}} = \frac{R}{R - x},$$

and from (2) (**1284**, EXAMPLE 3),

$$\frac{dz}{dx} = \frac{2\,(R - x)}{2\,\sqrt{Rx - x^2}} = \frac{R - x}{\sqrt{Rx - x^2}}.$$

Therefore the required derivative (A) is

$$y' = \frac{dy}{dx} = \frac{R}{R - x} \times \frac{R - x}{\sqrt{Rx - x^2}} = \frac{R}{\sqrt{Rx - x^2}}.$$

EXAMPLE 2. Find the derivative of

$$u = \sin^{-1}\frac{\sqrt{2\,Ry - y^2}}{R}. \tag{1}$$

Let
$$y = F\,(x) = 2\,ax \tag{2}$$

be given to find the derivative $\frac{du}{dx}$ of the function u with respect to x.

Putting
$$z = \sqrt{2\,Ry - y^2}, \tag{3}$$
$$z^2 = 2\,Ry - y^2.$$

The relation (1) becomes

$$u = \sin^{-1}\frac{z}{R}. \tag{4}$$

The theorem of a function of a function (**1284**):

$$\frac{du}{dx} = \frac{du}{dz} \times \frac{dz}{dy} \times \frac{dy}{dx}. \tag{5}$$

The relations (4), (3) and (2) give the derivatives:

$$\frac{du}{dz} = \frac{R}{R - y},$$
$$\frac{dz}{dy} = \frac{R - y}{\sqrt{2\,Ry - y^2}},$$
$$\frac{dy}{dx} = 2\,a.$$

Therefore the relation (5) gives the required derivative:

$$\frac{du}{dx} = \frac{2\,aR}{\sqrt{2\,Ry - y^2}}.$$

1288. *Derivatives and differentials of implicit functions.*

To apply the foregoing rules to the determination of the derivative $\frac{dy}{dx}$, commence by solving the equations for y, that is, reducing them to the form

$$y = f(x).$$

But often this method is laborious, and it may be simpler to have recourse to a general theorem which does not require the solution of the equation with respect to one of the variables.

Let us assume that all the terms of an equation have been transposed to one side, and reduced to the form

$$f(x, y) = 0, \tag{1}$$

which indicates that a relation exists between the two variables x and y such that the simultaneous values of the two written in one member make that member equal to zero.

Giving x an increment Δx, y takes a corresponding increment Δy, and the relation (1) becomes

$$f(x + \Delta x, y + \Delta y) = 0. \tag{2}$$

Subtracting (1) from (2),

$$f(x + \Delta x, y + \Delta y) - f(x, y) = 0.$$

Subtracting and adding the function

$$f(x + \Delta x, y),$$

in which y is considered as a constant and x as a variable, we have

$$f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) + f(x + \Delta x, y) - f(x, y) = 0.$$

Dividing all the terms by Δx,

$$\frac{f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)}{\Delta x} + \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = 0.$$

Multiplying and dividing the first term by Δy, we have

$$\frac{f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)}{\Delta y}\frac{\Delta y}{\Delta x} + \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = 0, \tag{3}$$

which is true, no matter what the simultaneous increments Δx and Δy may be

In taking the limits, it is to be noted:

1st. That

$$\lim \frac{f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)}{\Delta y} = f'_y(x + \Delta x, y),$$

representing by $f'_y(x + \Delta x, y)$ the derivative with respect to y of the function $f(x + \Delta x, y)$, in which $x + \Delta x$ is considered as a constant and y as a variable (1274);

2d. That

$$\lim \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = f'_x(x, y),$$

that is, the derivative with respect to x of the given function $f(x, y)$ in which y is considered as a constant and x as a variable.

Then the limit of the relation (3) is

$$f'_y(x, y)\frac{dy}{dx} + f'_x(x, y) = 0,$$

and the required derivative

$$y' = \frac{dy}{dx} = \frac{-f'_x(x, y)}{f'_y(x, y)}. \tag{4}$$

Thus *the derivative of an implicit function involving two variables is equal to at least the derivative of the given function taken with respect to x, considering x as variable and y as constant, divided by the derivative of the same function with respect to y, considering x as constant and y as variable.*

REMARK. The quantities f'_x and f'_y are called *partial derivatives* of the function $f(x, y)$.

EXAMPLE 1. Find the derivative of the implicit function (1131)

$$a^2y^2 + b^2x^2 - a^2b^2 = 0. \tag{5}$$

We have

$$-f'_x(x, y) = -2b^2x \quad \text{and} \quad f'_y(x, y) = 2a^2y;$$

therefore $$y' = \frac{dy}{dx} = \frac{-2b^2x}{2a^2y} = \frac{-b^2x}{a^2y}.$$

REMARK. The same result is obtained by taking the differentials of the different terms of the relation (5). Thus, we have

$$2a^2ydy + 2b^2xdx = 0;$$

transposing, $$y' = \frac{dy}{dx} = \frac{-b^2x}{a^2y}.$$

EXAMPLE 2. Find the derivative of the function

$$y^2 = 2\,px.$$

Write $$f(x, y) = y^2 - 2\,px = 0,$$

then $$-f'_x(x, y) = 2\,p \quad \text{and} \quad f'_y(x, y) = 2\,y,$$

and $$y' = \frac{dy}{dx} = \frac{2\,p}{2\,y} = \frac{p}{y}.$$

EXAMPLE 3. Find the derivative of

$$(y - q)^2 + (x - p)^2 = r^2.$$

Having $f(x, y) = (y - q)^2 + (x - p)^2 - r^2 = 0,$
we have

$$f'_x(x, y) = f'_x(x - p)^2 = f'_x(x^2 - 2\,px + p^2) = 2\,x - 2\,p = 2\,(x - p),$$

$$f'_y(x, y) = f'_y(y - q)^2 = 2\,(y - q),$$

and

$$y' = \frac{dy}{dx} = \frac{-f'_x(x, y)}{f'_y(x, y)} = \frac{-2\,(x - p)}{2\,(y - q)} = \frac{-(x - p)}{y - q}.$$

1289. *Compound functions.*

Let us consider a function of two variables u and v which we will designate by

$$y = F(u, v). \tag{1}$$

The quantities u and v being the functions of x, it is required to find the derivative $y' = \frac{dy}{dx}$.

Giving x an increment Δx, the other variables, u, v and y, take the corresponding increments Δu, Δv and Δy, and the relation (1) becomes

$$y + \Delta y = F(u + \Delta u, v + \Delta v). \tag{2}$$

Subtracting (1) from (2),

$$\Delta y = F(u + \Delta u, v + \Delta v) - F(u, v). \tag{3}$$

Adding and subtracting the following mixed function in the second member of (3),

$$F(u, v + \Delta v),$$

we obtain

$$\Delta y = \begin{cases} F(u + \Delta u, v + \Delta v) - F(u, v + \Delta v), \\ \qquad + F(u, v + \Delta v) - F(u, v). \end{cases}$$

With reference to the mixed function, it must be observed that u is to be considered as a constant and v as a variable. This

being true, if all the terms of the last relation are divided by Δx, we have

$$\frac{\Delta y}{\Delta x} = \frac{F(u + \Delta u, v + \Delta v) - F(u, v + \Delta v)}{\Delta x} + \frac{F(u, v + \Delta v) - F(u, v)}{\Delta x}.$$

Finally, if the common factors Δu and Δv are introduced into the two general terms of the last expression, we have

$$\frac{\Delta y}{\Delta x} = \frac{F(u + \Delta u, v + \Delta v) - F(u, v + \Delta v)}{\Delta u}\frac{\Delta u}{\Delta x} + \frac{F(u, v + \Delta v) - F(u, v)}{\Delta v}\frac{\Delta v}{\Delta x}.$$

The limits of these ratios are

$$\frac{\Delta y}{\Delta x} = y', \quad \frac{\Delta u}{\Delta x} = u', \quad \frac{\Delta v}{\Delta x} = v';$$

which may be written

$$y' = F'_u(u, v)\, u' + F'_v(u, v)\, v', \tag{5}$$

designating the derivative of $F(u, v + \Delta v)$ by F'_u, neglecting Δv and considering v as a constant and u as a variable; likewise the derivative of $F(u, v)$ is F'_v when v is the variable, and the relation (5) may be written

$$y' = F'_u u' + F'_v v, \tag{6}$$

Thus, *the derivative of a compound function of two variables u and v is equal to the sum of the products obtained by multiplying each partial derivative by the derivative of the corresponding variable taken with respect to the independent variable x.*

REMARK. This theorem is of general application. Thus the function

$$y = F(u, v, z)$$

gives

$$y' = F'_u u' + F'_v v' + F'_z z'.$$

EXAMPLE 1. Find the derivative of

$$y = x^{\sin x}.$$

Putting $x = u$ and $\sin x = v$, we have

$$y = u^v.$$

Applying theorem (6), that is,

$$y' = F'_u u' + F'_v v',$$

we have successively

$$y' = vu^{v-1}\,u' + u^v \frac{\log u}{\log e} v',$$

$$y' = \sin x\, x^{\sin x - 1} + x^{\sin x} \frac{\log x}{\log e} \cos x.$$

EXAMPLE 2. Find the derivative of

$$y = x^x, \tag{a}$$

which may be written in the form

$$y = u^v$$

by putting $\qquad u = x \quad \text{and} \quad v = {}^x.$

Applying theorem (6),

$$y' = vu^{v-1} + \frac{u^v \log u}{\log e} = u^v\left(1 + \frac{\log u}{\log e}\right),$$

or
$$y' = x^x\left(1 + \frac{\log x}{\log e}\right).$$

REMARK. This derivative may also be found as follows. From the given function (a), taking the logarithms, we have

$$\log y = x \log x,$$

and the derivative gives

$$\frac{\log e}{y} y' = x \frac{\log e}{x} + \log x = \log e + \log x,$$

from which $\quad y' = y\left(\dfrac{\log e + \log x}{\log e}\right) = x^x\left(1 + \dfrac{\log x}{\log e}\right).$

EXAMPLE 3. Find the derivative of the compound function

$$y = uv$$

which is
$$y' = F'_u u' - F'_v v'. \tag{A}$$

As a special case, take

$$y = x \log x.$$

Putting $u = x$ and $v = \log x$, the theorem (A) gives

$$y' = \log x + \frac{\log e}{x} x = \log e + \log x.$$

This result may also be obtained by applying the theorem relative to the product of two functions (1281).

EXAMPLE 4. *Application of the theorem of compound functions to the determination of an implicit function.*

The theorem of (1288) may be deduced from the general theorem of compound functions.

Let the implicit function

$$F(x, y) = 0 \qquad (A)$$

be given. Comparing this with

$$y = F(u, v)$$

and putting $\qquad u = x$ and $v = y$,

the latter gives $\qquad y' = F'_u u' + F'_v v'. \qquad (B)$

From the relation $F(x, y) = 0$, it is seen that the derivative of the two members should be zero. Then (B) gives

$$0 = F'_x x' + F'_y y';$$

but the derivative x' is equal to one, and the above expression reduces to

$$0 = F'_x + F'_y y',$$

from which $\qquad y' = \frac{-F'_x}{F'_y}.$

TANGENTS.

1290. We saw in article (1275) that the limit $\frac{dy}{dx}$ of the ratio of the increment of the function y to that of the variable x, was equal to the slope of the tangent to the curve which represents the function. From this property it is easy to deduce a method of drawing a tangent to a curve whose equation is given and determine the equation of the tangent. y' and x' being the coördinates of the point of contact of a tangent to any curve, the equation of any line which passes through this point is (1118)

$$y - y' = a(x - x').$$

In order that this line be tangent to the curve, the coefficient a must be equal to the derivative $\frac{dy}{dx}$ of the equation of the curve taken at the point of contact. From this it follows that the general equation of a tangent to any curve is

$$y - y' = \frac{dy}{dx}(x - x'). \qquad (a)$$

We will now apply this equation in some examples.

1291. *Tangent to a circle.*

The equation of a circle referred to its center being (1123)

$$y^2 + x^2 = r^2,$$

applying the rule for implicit functions (1288) we have

$$\frac{dy}{dx} = \frac{-x}{y}.$$

For the point of contact (x', y'), which is given, the derivative is

$$\frac{dy}{dx} = \frac{-x'}{y'}.$$

Therefore the equation of a tangent to the circle at this point, upon substituting for $\frac{dy}{dx}$ in (a) of the preceding article, becomes

$$y - y' = \frac{-x'}{y'}(x - x').$$

Eliminating the denominator and reducing,

$$yy' - y'^2 = -xx' + x'^2$$

or

$$yy' + xx' = y'^2 + x'^2 = r^2.$$

Thus the sum $\quad yy' + xx' = r^2 = \text{constant}.$

1292. *Tangent to an ellipse.*

The equation of an ellipse referred to its principal axes being (1131)

$$a^2y^2 + b^2x^2 = x^2b^2,$$

from (1288)

$$\frac{dy}{dx} = \frac{-b^2x}{a^2y},$$

and for the point of contact

$$\frac{dy}{dx} = \frac{-b^2x'}{a^2y'};$$

therefore the equation of the tangent is (1290, Equation (a))

$$y - y' = \frac{-b^2x'}{a^2y'}(x - x').$$

1293. *Tangent to an hyperbola.*

The equation of an hyperbola is

$$a^2y^2 - b^2x^2 = -a^2b^2.$$

From (1288)

$$\frac{dy}{dx} = \frac{b^2x}{a^2y},$$

and therefore the equation of the tangent is (1290, 1291)

$$y - y' = \frac{b^2x'}{a^2y'}(x - x').$$

1294. *Tangent to a parabola.*

The equation of a parabola referred to its principal axis and vertex being (1197)

$$y^2 = 2\,px,$$

we have
$$\frac{dy}{dx} = \frac{p}{y}.$$

Therefore the equation of the tangent is (1290, 1291)

$$y - y' = \frac{p}{y'}(x - x').$$

For the vertex $x' = 0\ y' = 0$, and

$$\frac{dy}{dx} = \frac{p}{0} = \infty.$$

This indicates that the tangent is perpendicular to the x-axis and coincides with the y-axis.

1295. *Tangent to a logarithmic curve.*

The equation of the curve being

$$y = \log x, \tag{a}$$

from (1277)
$$\frac{dy}{dx} = \frac{\log e}{x} = \frac{0.4342945}{x},$$

consequently the tangent to the curve at the point $(x', y',)$ is represented by the equation (1290, 1291)

$$y - y' = \frac{0.4342945}{x}(x - x'),$$

Special Cases.

1st. For $x' = 0$ the equation (a) becomes

$$y' = \log x' = -\infty,$$

and
$$\frac{dy}{dx} = \frac{\log e}{x'} = \frac{\log e}{0} = \infty.$$

This shows that the y-axis is an asymptote of the curve on the negative side.

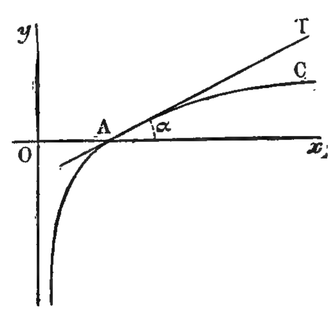

Fig. 366

2d. For $x' = 1$, we have

$$y' = \log x' = 0$$

and
$$\frac{dy}{dx} = \frac{\log e}{x'} = \log e.$$

Thus at the point A where the curve meets the x-axis, we have

$$\tan \alpha = 0.4342945.$$

3d. For $x' = \infty$, we have

$$y' = \log x' = \infty, \text{ and } \frac{dy}{dx} = \frac{\log e}{x'} = 0.$$

Thus the curve goes constantly away from the x-axis, and at infinity the tangent is parallel to the x-axis.

1296. *Tangent to a sine wave.*

The equation of a sine wave is

$$y = \sin x$$

from (1278) $$\frac{dy}{dx} = \cos x.$$

Consequently the equation of tangent at the point $(x'y')$ is represented by the equation (1290, 1291)

$$y - y' = \cos x' (x - x').$$

Special Cases.

1st. For $x' = 0$, we have (1027)

$$y' = \sin x' = 0, \text{ and } \frac{dy}{dx} = \cos x' = \cos 0° = 1.$$

Thus the curve passes through the origin, and at this point $\tan \alpha = 1$, and, therefore, $\alpha = 45°$. These same values are ob-

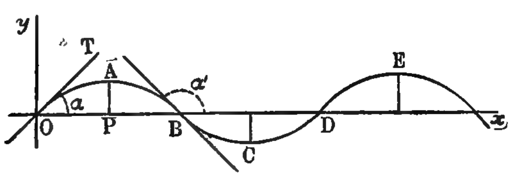

Fig. 367

tained for the point D, which gives $x' = 2\pi$, and so on for the successive values 4π, 6π . . . . of x'.

2d. For $x' = \pi = 180°$, we have.

$$y' = \sin x' = 0, \text{ and } \frac{dy}{dx} = \cos x' = \cos \pi = -1.$$

Thus, at the point B, where $x' = \pi$, we have $\tan \alpha = -1$, and consequently $\alpha = 135°$. The same values are obtained for $x' = 3\pi$, $x' = 5\pi$ . . . .

3d. For $\quad x' = \frac{\pi}{2}, \quad x' = \frac{3}{2}\pi, \quad x' = \frac{5}{2}\pi \ldots,$

we have $\cos x' = 0$ and $\tan \alpha = 0$, which indicates that the tangents to the curve at the points $A, C, E, \ldots$, are parallel to the x-axis.

1297. *Tangent to a cycloid* (see Fig. 350, **1247**).
If the radius of the generating circle of a cycloid is represented by R, and the point A is taken as origin, the equation of the cycloid is

$$x = \sin^{-1}\frac{\sqrt{2Ry - y^2}}{R} - \sqrt{2Ry - y^2}. \tag{1}$$

The equation of a tangent at the point M is

$$y - \beta = m(x - \alpha), \tag{A}$$

α and β being the coördinates of the point of contact and in the slope of the tangent. We know that $m = \frac{dy}{dx}$ is the derivative of equation (1) of the curve. To find this derivative, put

$$z = \sqrt{2Ry - y^2}, \quad \text{then} \quad z^2 = 2Ry - y^2, \tag{2}$$

and equation (1) may be written

$$x = \sin^{-1}\frac{z}{R} - z. \tag{3}$$

Taking the derivatives of all the terms with respect to the independent variable x (1298)

$$1 = \frac{z'}{\sqrt{1 - \frac{z^2}{R^2}}} - z' = \frac{z'}{\sqrt{\frac{R^2 - z^2}{R^2}}} - z';$$

substituting for z^2, $\quad 1 = z'\left(\frac{y}{R - y}\right),$

and $\quad z' = \frac{R - y}{y}. \tag{4}$

In equation (2), taking the derivative with respect to x,

$$z' = \frac{R - y}{\sqrt{2Ry - y^2}}y'. \tag{5}$$

The relations (4) and (5) give

$$\frac{R - y}{y} = \frac{R - y}{\sqrt{2R - y^2}}y',$$

and $\quad y' = \sqrt{\frac{2R - y}{y}} = m.$

Therefore the equation (A) of the tangent to a cycloid is

$$y - \beta = \sqrt{\frac{2R - y}{y}}(x - \alpha). \tag{6}$$

For the highest point or the vertex of the cycloid, we have $y = 2R$, and the value of the coefficient m is

$$m = \sqrt{\frac{2R - 2R}{2R}} = 0.$$

Thus the tangent is parallel to the x-axis or the base of the cycloid.

REMARK. If the point of contact is placed at the height of the center of the generating circle, we have $y = R$, and the coefficient becomes

$$m = \sqrt{\frac{2R - R}{R}} = 1,$$

which shows that the angle between the tangent and the x-axis is 45°. At the origin and at the end of the cycloid, we have $y = 0$, and the coefficient for each of these values is

$$m = \sqrt{\frac{R}{0}} = \infty.$$

Therefore the tangents at these points are perpendicular to the x-axis.

1298. *Tangents to curves referred to polar coördinates.*

Let the equation of the curve be

$$\rho = F(\omega). \tag{1}$$

The expression $$\frac{\rho d\omega}{d\rho} = \tan(T\rho) = \tan\theta \tag{2}$$

is the coefficient or the slope of the tangent T with respect to the radius vector ρ drawn to the point of contact.

EXAMPLE 1. *Tangent to the spiral of Archimedes* (1230 *and* 1270).

1st. *The curve starting from the pole, its equation is of the form*

$$\rho = K\omega. \tag{a}$$

If for $\omega = 2\pi$ we have $\rho = a$, the preceding equation (a) gives

$$a = K\,2\pi,$$

and $$K = \frac{a}{2\pi}.$$

Equation (a) becomes $$\rho = \frac{a}{2\pi}\omega. \tag{b}$$

The general expression of the slope of the tangent with respect to the radius vector, as given by equation (2), has the value

$$\tan\theta = \rho\frac{d\omega}{d\rho} = \rho\frac{2\pi}{a} = \frac{2\pi\rho}{a}.$$

This value is for the curve traced in (1233). For $\rho = 0$, $\tan\theta = 0$; therefore, at the origin the spiral is tangent to the polar axis.

2d. *The spiral not starting at the pole has the equation of the form.*

$$\rho = b + K\omega. \qquad (A)$$

For $\omega = 0$, $\rho = b$. If for each revolution of the spiral the radius vector increases by an amount a, the above equation will hold for $\omega = 2\pi$ and $\rho = b + a$, and we have

$$b + a = b + K2\pi$$

and

$$K = \frac{a}{2\pi}.$$

Then the equation of the spiral is

$$\rho = b + \frac{a}{2\pi}\omega,$$

and we have, as in the first example,

$$\tan\theta = \rho\frac{d\omega}{d\rho} = \rho\frac{2\pi}{a}.$$

For $\rho = b$ we have $\omega = 0$ and

$$\tan\theta = \frac{b}{a}2\pi.$$

Thus the first element of the spiral is no longer tangent to the polar axis as in the preceding case. If we make $b = 0$, the spiral passes through the pole, and we have

$$\tan\theta = 0.$$

EXAMPLE 2. *Tangent to a logarithmic spiral* (1270).

The equation of the logarithmic spiral is

$$\log\rho = A\omega \quad \text{or} \quad \rho = b^{A\omega}$$

Taking the differentials,

$$\frac{\log e}{\rho}d\rho = Ad\omega,$$

and therefore the slope of the tangent to the curve with respect to the radius vector is

$$\tan\theta = \rho\frac{d\omega}{d\rho} = \frac{\log e}{A}.$$

This quantity is constant. Thus the tangent to the logarithmic spiral makes a constant angle with the radius vector.

EXAMPLE 3. *A circle tangent to the polar axis at the pole.*

The equation in polar coördinates (1270):

then $$\rho = 2R \sin \omega,$$

and $$\frac{d\rho}{d\omega} = 2R \cos \omega,$$

$$\rho \frac{d\omega}{d\rho} = \frac{\rho}{2R \cos \omega}.$$

For $\rho = 2R$, we have $\omega = 90°$, $\cos \omega = 0$; then

$$\tan \theta = \rho \frac{d\omega}{d\rho} = \frac{2R}{2R \times 0} = \infty;$$

which indicates that the tangent to the circle at the point farthest from the polar axis is perpendicular to the radius vector $2R$, to the point of contact, and parallel to the polar axis.

SUCCESSIVE DERIVATIVES

1299. We have seen (1275, 1290) that the relation

$$y = f(x)$$

is the equation of a curve, the tangent to which makes an angle with the x-axis whose trigonometric tangent is the derivative of y with respect to x. This derivative is represented by $\frac{dy}{dx}$, $f'(x)$ or y', and is called a *derivative of the first order*, or *first derivative.*

The relation $$y' = f'(x)$$

being a new function of x, it is possible to find the derivative of y' with respect to x, in the same manner as the derivative of y with respect to x was found; and if this derivative is designated by $f''(x)$ or y'', we have

$$y'' = \frac{dy'}{dx} = \frac{df'(x)}{dx} = f''(x).$$

This new derivative is called a *derivative of the second order*, or a *second derivative*, and is also represented by the notation

$$\frac{d^2y}{dx^2},$$

the figure 2 indicating the order of the derivative.

The relation
$$y'' = f''(x)$$
being also a new function of x, the derivative of y'' with respect to x gives the *third derivative*, which is represented thus:
$$y''' = f'''(x) = \frac{d^3y}{dx^3}.$$

Continuing thus, we may obtain the *fourth, fifth*, etc., derivatives, which are given in a table below

$$\begin{array}{lll}
y = f(x) & & \text{original function,} \\
y' = f'(x) = \dfrac{dy}{dx} & & \text{1st derivative,} \\
y'' = f''(x) = \dfrac{d^2y}{dx^2} & & \text{2d} \quad " \\
y''' = f'''(x) = \dfrac{d^3y}{dx^3} & & \text{3d} \quad " \\
y^{\text{IV}} = f^{\text{IV}}(x) = \dfrac{d^4y}{dx^4} & & \text{4th} \quad "
\end{array}$$

EXAMPLE. The successive derivatives of the function
$$y = x^m$$
are given below (1276):

$$\begin{array}{ll}
y' = mx^{m-1} & \text{1st derivative} \\
y'' = m(m-1)x^{m-2} & \text{2d derivative} \\
y''' = m(m-1)(m-2)x^{m-3} & \text{3d derivative} \\
y^{\text{IV}} = m(m-1)(m-2)(m-3)x^{m-4} & \text{4th derivative} \\
\cdots\cdots\cdots & \cdots\cdots\cdots
\end{array}$$

1300. *Geometrical interpretation of successive derivatives.*

Given the function
$$y = A + Bx + Cx^2 + Dx^3. \tag{1}$$
Taking the successive derivatives (1299)
$$\frac{dy}{dx} \text{ or } y' = B + 2Cx + 3Dx^2, \tag{2}$$
$$\frac{dy'}{dx} \text{ or } y'' = 2C + 6Dx, \tag{3}$$
$$\frac{dy''}{dx} \text{ or } y''' = 6D. \tag{4}$$

This shows that the given function (1) being of the 3d degree, the *third derivative is a constant.*

In the same way, mth *derivative* of function of the mth *degree*

$$y = x^m$$

is a *constant.*

Let us interpret geometrically the equations (1), (2), (3) and (4).

Refer the equations (1), (2), (3) and (4) respectively to the coördinate systems Ox and Oy, Ox_1 and O_1y, Ox_2 and O_2y, etc., taking the axes Ox, Ox_1, Ox_2, etc., parallel to each other and the y-axes coinciding with the same line.

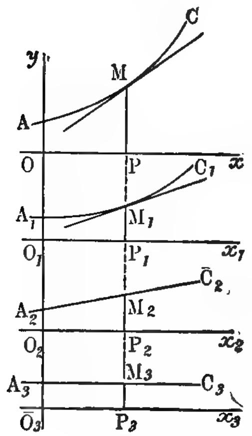

Fig. 368

Now construct by points, the curves C, C_1, C_2 and C_3, representing the functions y, y', y'' and y'''. Thus making $b = OP$, the relation (1) gives $y = MP$; the relation (2) gives the slope $\frac{dy}{dx}$ of the tangent to the curve C at M so that MT may be drawn (1290), and since this angular coefficient is nothing other than $M_1P_1 = y'$, of the curve C_1, the point M_1 of the curve C_1 is obtained. The abscissa at this point is x. The relation (3) in the same way, gives $\frac{dy'}{dx}$ or y'', that is, the tangent to the curve C_1 at M_1, and the point M_2 of the curve C_2 and so on. . Giving x different values as many points on the curves C, C_1, C_2, ... may be determined as one wishes and the curves traced, then with the aid of the successive derivatives the tangents may be drawn.

In the above example (1) the curves C and C_1 are parabolic; C_2 is a straight line whose slope $\frac{dy''}{dx}$ is $6\,D$; and C_3 is a straight line parallel to the x-axis, therefore its slope $\frac{dy'''}{dx} = 0$; it is the line representing the constant function $6\,D$. From the successive derivatives and their geometrical interpretation, the following important theorems are deduced.

CONCAVITY AND CONVEXITY. — DIRECTION OF BENDING.

1301. A curve is concave or convex at a point M with respect to a line Ox, for example, according as the neighboring elements

to the point M are situated within the acute angle α or the obtuse angle α', which the tangent MT to the curve at that point M

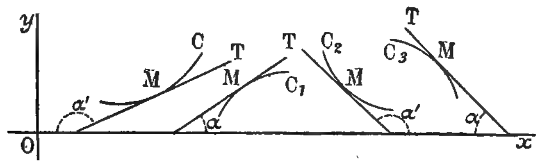

Fig. 369

makes with the axis Ox. Thus the curves C_1 and C_3 concave at M with respect to Ox and C and C_2 are convex to the same line Ox.

The concavity and convexity constitute the direction of bending of a curve. Let us express analytically the distinctive character of the direction of bending with respect to the x-axis.

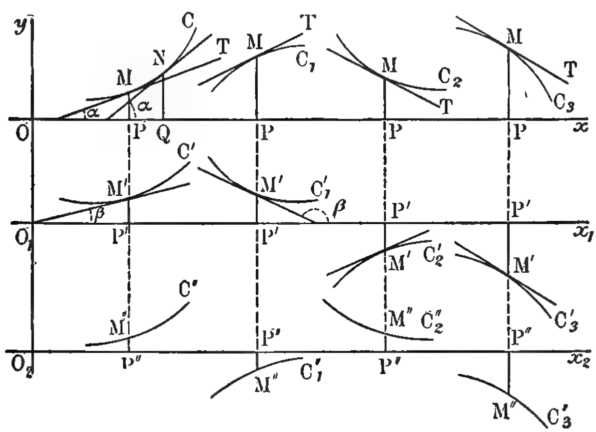

Fig. 370

For the curves C and C_1, the function

$$y = f(x)$$

being increasing (1273), their tangents make acute angles with the x-axis and their slopes or angular coefficients are positive. Constructing the curves C and C', representing their first derivatives

$$y' = \frac{dy}{dx} = f'(x),$$

the ordinates of both of these curves will be positive, but they will have a characteristic difference due to the opposite direc-

tions of bending of the curves C and C_1; thus the ordinates of the curve C' will be increasing the same as the corresponding function, while the ordinates of C_1' will be decreasing.

It is seen, in fact, that x increasing the tangent makes greater and greater acute angles with the x-axis, the slopes increase, and the function $\frac{dy}{dx} = f'(x) = y'$, which is represented by the curve C_1, is also increasing. In the same way it is seen that increasing x, the tangent to the curve C_1 makes smaller and smaller acute angles with the x-axis; therefore the slopes diminish, and the function $\frac{dy}{dx} = f'(x) = y'$, which is represented by the curve C_1', is decreasing.

Now constructing the curves C'' and C_1'', representing the second derivatives of the original functions $y = f(x)$, we have curves whose equations have the form

$$\frac{dy'}{dx} = f''(x) = y'',$$

that is, the ordinates y'' of which are equal to the slopes of the tangents to the curves C' and C'_1, it is easily seen that $f''(x)$ is positive and increasing in the case of the curve C'.

Thus the curve C which is convex to the x-axis corresponds to the curve C'' whose ordinates are positive, and the curve C_1 concave to the x-axis corresponds to the curve C_1'' whose ordinates are negative. As is shown in Fig. 370, this property applies also to the curves C_2 and C_3; and in general, we may say *that any curve whose equation is of the form*

$$y = f(x)$$

is convex or concave to the x-axis according as $y'' = f''(x)$ *is positive or negative.*

POINT OF INFLECTION.

1302. *In general, the second derivative of a curve for the point of inflection is zero or equal to* $\pm\infty$.

1st. *General Case.* When a curve AMB changes its direction of bending, the point M where this change takes place is called a *point of inflection.* Drawing a tangent to the curve at the point M, the two elements Mm and Mn which are situated just

before and just after the point of inflection lie on opposite sides of the tangent MT;

$$y = f(x),\ y' = f'(x) = \frac{dy}{dx},$$

$$y'' = f''(x) = \frac{dy'}{dx}$$

being respectively the equations of the required curve AMB,

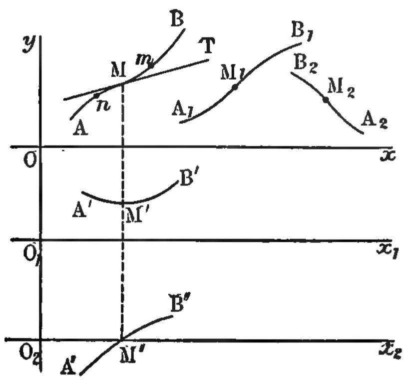

Fig. 371

and of the first and second derivative functions; if there is a point of inflection M, we obtain for this point

$$y'' = f''(x) = 0;$$

which indicates that the point M'' of the second derivative curve is on the x-axis.

This is evident *a priori*, because, the portion AM being concave to Ox, the corresponding curve $A''M''$ of the second derivative has negative ordinates (1301), and the portion MB being convex to Ox, the corresponding curve $M''B''$ of its second derivative has positive ordinates; from this it follows that the continuous curve $A''M''B''$ must cut the axis at M''. The same is true of the curves $A_1M_1B_1$ and $A_2M_2B_2$.

2d. *Special Case.* Given, two curves AMB and $A_1M_1B_1$ whose points of inflection M and M_1 correspond to the tangents MT and M_1T_1 which are parallel to the y-axis. Constructing the first and second derivative curves, it is easily seen that the points M'' and M_1'', which correspond to the points of inflection M

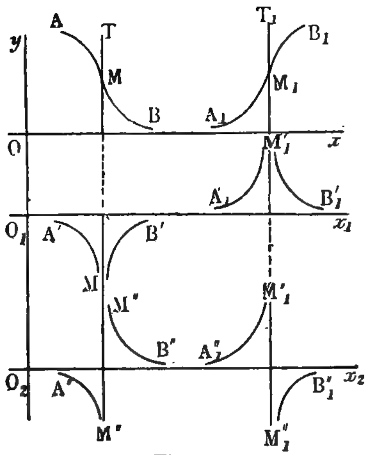

Fig. 372

and M_1, are situated at infinity; that is, the second derivatives for the points M and M_1 are

$$y'' = f''(x) = \pm\infty.$$

Thus, for the points of inflection of a curve whose equation is

$$y = f(x),$$

we have

$$y'' = f''(x) = 0,$$

or $$y'' = f''(x) = \pm\infty.$$

Exception. It is possible for a curve whose equation is

$$y = f(x)$$

to give $$y'' = f''(x) = \pm\infty$$

without having a point of inflection.

For example, the equation of a circle is

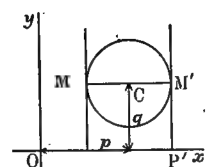

Fig. 373

$$(y - q)^2 + (x - p)^2 - r^2 = 0,$$

or $$f(x, y) = 0.$$

From (1291, Example 3),

$$\frac{dy}{dx} = \frac{p - x}{y - q},$$

and therefore (1286, 1299),

$$y'' = f''(x) = \frac{-(y - q) - (p - x)}{(y - q)^2}.$$

For $x = OP = p - r$ and $y = q$, that is, for the point M, we have

$$y'' = \frac{-r}{0} = -\infty,$$

and for $x = p + r$ and $y = q$, that is, for M', we have

$$y'' = \frac{r}{0} = +\infty.$$

Thus the second derivatives for the points M and M' are $-\infty$ and $+\infty$; nevertheless, they are not points of inflection, but there is a change in the direction of bending with respect to the x-axis.

EXAMPLE. Given a sine curve whose equation is

$$y = \sin x.$$

From (1282, 1287)

$$y' = f'(x) = \cos x,$$

$$y'' = f''(x) = -\sin x.$$

The value

$$y'' = f''(x) = 0$$

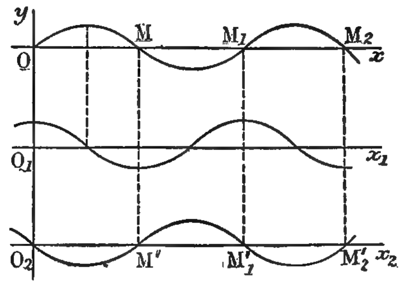

Fig. 374

corresponding to $x = 0, \pi, 2\pi, 3\pi, \ldots \; n\pi$, since for these values of x we have $y = 0$, it follows that all these points of inflection $O, M, M_1, M_2, \ldots\ldots$ are situated on the x-axis, and furthermore, the corresponding points $O_2, M'', M_1'', M_2'' \ldots\ldots$ on the curve representing the function $y'' = f''(x)$ are also on the axis.

TAYLOR'S THEOREM

1303. *Preliminary theorem.*

If in a function

$$y = f(x), \tag{1}$$

x is replaced by $x + h$, it follows that y takes the value y' and relation (1) becomes

$$y' = f(x + h). \tag{2}$$

The first derivative $\frac{dy'}{dx}$ of y' with respect to x, considering x as a variable and h as a constant, is equal to the first derivative $\frac{dy'}{dh}$ of y' with respect to h, considering h as a variable and x as a constant. Thus, we have

$$\frac{dy'}{dx} = \frac{dy'}{dh}.$$

In fact, putting $x + h = x'$, relation (2) becomes

$$y' = f(x'),$$

and

$$\frac{dy'}{dx'} = f'(x'),$$

or

$$\frac{dy'}{d(x+h)} = f'(x + h). \tag{3}$$

Assuming h constant and x variable,

$$d(x + h) = dx,$$

and expression (3) may be written

$$\frac{dy'}{dx} = f'(x + h). \tag{4}$$

Now supposing x constant and h variable,

$$d(x + h) = dh,$$

and relation (3) becomes

$$\frac{dy'}{dx} = f'(x + h). \tag{5}$$

Equating expressions (4) and (5),

$$\frac{dy'}{dx} = \frac{dy'}{dh}.$$

1304. *Taylor's theorem.*

Suppose that the expansion of the function

$$y' = f(x + h) \tag{1}$$

with respect to the successive powers of h be given,

$$y' = y + Ah + Bh^2 + Ch^3 + Dh^4 + \cdots \text{ 2)}$$

It is evident that the polynomial which expresses the value of y' contains an infinite number of terms, in which the exponent of h increases indefinitely from the first term where it is zero.

The coefficients $A, B, C, D, \ldots$, are unknown functions of the variable x, which are to be determined.

Taking the derivative of y' with respect to h in equation (2), we have (1276)

$$\frac{dy'}{dh} = A + 2\,Bh + 3\,Ch^2 + 4\,Dh^3 + \cdots \tag{3}$$

In the same equation (2) the derivative of y' with respect to x considering h constant, is

$$\frac{dy'}{dx} = \frac{dy}{dx} + \frac{dA}{dx}h + \frac{dB}{dx}h^2 + \frac{dC}{dx}h^3 + \cdots \tag{4}$$

The first members of equations (3) and (4) being equal (1303), equating the second members, we have

$$A + 2Bh + 3Ch^2 + 4Dh^3 + \cdots = \frac{dy}{dx} + \frac{dA}{dx}h + \frac{dB}{dx}h^2 + \frac{dC}{dx}h^3 + \cdots \tag{5}$$

Putting the terms of the same order equal to each other, we have

$$A = \frac{dy}{dx}, \quad B = \frac{dA}{2\,dx}, \quad C = \frac{dB}{3\,dx}, \quad D = \frac{dC}{4\,dx} \cdots$$

Replacing A by its value in the expression of B, then B by its new value in C, etc., we have

$$A = \frac{dy}{dx},$$

$$B = \frac{d\frac{dy}{dx}}{2\,dx} = \frac{d^2y}{dx^2}\,\frac{1}{1 \cdot 2},$$

$$C = \frac{d\frac{d^2y}{dx^2}}{3\,dx}\,\frac{1}{1 \cdot 2} = \frac{d^3y}{dx^3}\,\frac{1}{1 \cdot 2 \cdot 3},$$

$$D = \frac{d\frac{d^3y}{dx^3}}{4\,dx}\,\frac{1}{1 \cdot 2 \cdot 3} = \frac{d^4y}{dx^4}\,\frac{1}{1 \cdot 2 \cdot 3 \cdot 4},$$

$$\cdots\cdots\cdots\cdots$$

Substituting these values of $A, B, C, D, \ldots$, in the series (2), we have

$$y' = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\,\frac{h^2}{1 \cdot 2} + \frac{d^3y}{dx^3}\,\frac{h^3}{1 \cdot 2 \cdot 3} + \frac{d^4y}{dx^4}\,\frac{h^4}{1 \cdot 2 \cdot 3 \cdot 4} \cdots$$

which may be written in the form

$$f(x + h) = f(x) + f'(x)h + f''(x)\frac{h^2}{1 \cdot 2} + f'''(x)\frac{h^3}{1 \cdot 2 \cdot 3} + f^{\mathrm{IV}}(x)\frac{h^4}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots \tag{6}$$

which is Taylor's theorem for expanding a function with the aid of its successive derivatives.

1305. *Maclaurin's theorem or a special case of Taylor's theorem.* If in the function

$$y' = f(x + h) \tag{A}$$

and in its expansion (1304)

$$y' = f(x) + f'(x)h + f''(x)\frac{h^2}{1 \cdot 2} + f'''(x)\frac{h^3}{1 \cdot 2 \cdot 3} + \cdots \tag{1}$$

x is made equal to 0 and $h = x$, the function (A) becomes (designating y' by y)

$$y = f(x),$$

and its expansion takes the form

$$y = f(x) = f(0) + f'(0)\,x + f''(0)\frac{x^2}{1 \cdot 2} + f'''(0)\frac{x^3}{1 \cdot 2 \cdot 3} + \cdots \tag{2}$$

which is known as Maclaurin's theorem, and in which $f(0)$, $f'(0)$, $f''(0)$, ... , are values of the function y and its successive derivatives when $x = 0$.

1306. *Application of Taylor's and Maclaurin's theorems to the expansion of the sine and cosine in terms of the arc.*

1st. Expand

$$y' = (x + a)^m.$$

From this relation we deduce successively (1276, 1305)

$$\begin{aligned} f(x) &= y = x^m, \\ f'(x) &= mx^{m-1}, \\ f''(x) &= m(m-1)x^{m-2}, \\ f'''(x) &= m(m-1)(m-2)x^{m-3}, \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

Substituting these values of $f(x)$, $f'(x)$, $f''(x)$, ... in formula (6) (1304), and noting that h is replaced by a, we have

$$(x+a)^m = x^m + max^{m-1} + \frac{m(m-1)}{1 \cdot 2} a^2 x^{m-2} + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} a^3 x^{m-3} + \cdots$$

which is nothing other than Newton's binomial theorem (564).

2d. *Expansion of sine x as a function of arc x.*

From the function

$$y = \sin x$$

we deduce successively (1278, 1283)

$$\begin{aligned} f(x) &= \sin x, & f^{\text{IV}}(x) &= \sin x \\ f'(x) &= \cos x, & f^{\text{V}}(x) &= \cos x, \\ f''(x) &= -\sin x, & f^{\text{VI}}(x) &= -\sin x, \\ f'''(x) &= -\cos x, & f^{\text{VII}}(x) &= -\cos x, \\ &\ldots\ldots & &\ldots\ldots \end{aligned}$$

Making arc $x = 0°$ in these expressions, and using the notation of Maclaurin's theorem (1305), we have

$$\begin{aligned} f(x) &= f(0) = \sin x = \sin 0° = 0, \\ f'(x) &= f'(0) = \cos x = \cos 0° = 1, \\ f''(x) &= f''(0) = -\sin x = -\sin 0° = 0, \\ f'''(x) &= f'''(0) = -\cos x = -\cos 0° = -1, \\ f^{\text{IV}}(x) &= f^{\text{IV}}(0) = \sin x = \sin 0° = 0, \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

Substituting these values of $f(x)$, $f'(x)$, $f''(x)$ . . . in formula (2) of (1305), and noting that the odd terms are equal to zero, we have

$$\sin x = x - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{x^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \cdots$$

3d. *Expansion of the cos x as a function of the arc x.*

From the function

$$y = \cos x$$

we deduce successively

$$\begin{array}{llll} f(x) = \cos x, & & f^{\text{IV}}(x) = \cos x, \\ f'(x) = -\sin x, & & f^{\text{V}}(x) = -\sin x, \\ f''(x) = -\cos x, & & f^{\text{VI}}(x) = -\cos x, \\ f'''(x) = \sin x, & & f^{\text{VII}}(x) = \sin x, \\ \ldots & & \ldots \end{array}$$

Making arc $x = 0°$, and using the notation of Maclaurin's theorem, these expressions become (1305)

$$\begin{aligned} f(x) &= f(0) = \cos x = \cos 0° = 1, \\ f'(x) &= f'(0) = -\sin x = -\sin 0° = -0, \\ f''(x) &= f''(0) = -\cos x = -\cos 0° = -1, \\ f'''(x) &= f'''(0) = \sin x = \sin 0° = 0, \\ f^{\text{IV}}(x) &= f^{\text{IV}}(0) = \cos x = \cos 0° = 1, \\ &\ldots \end{aligned}$$

Substituting these values of $f(x)$, $f'(x)$, $f''(x)$ . . . in Maclaurin's formula (1305), and noting that the even terms equal zero, we have:

$$\cos x = 1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{x^8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \cdots$$

MAXIMA AND MINIMA

1307. *Maxima and minima of functions.*

Let the curve C represent the function

$$y = f(x).$$

If for a value $OP = x$ of the abscissa, the corresponding value $MP = y$ of the ordinate is greater than the values of the ordinates $m'p'$ and $m''p''$, corresponding to the abscissas Op' and Op'' one of which comes just before and the other just after $OP = x$, the function or the ordinate $y = MP$ is said to be a maximum.

In the same way the ordinate M_1P_1 being smaller than the ones infinitely near it, the ordinate or the function y which it represents, is said to be a minimum. Thus, in general, a function is a maximum or a minimum according as a particular value is greater or smaller than the values infinitely near the point in question.

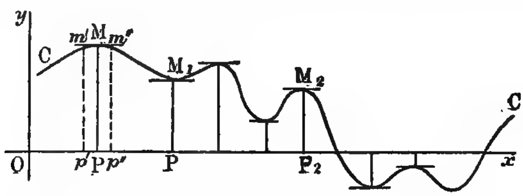

Fig. 375

As shown in Fig. 375: 1st. A function may have several maximum values and several minimum values; 2d. A minimum M_1P_1 may be greater than a maximum M_2P_2; 3d. A maximum or a minimum may be positive or negative. A function may have relative maximum and minimum values, and at the same time have an *absolute maximum* and an *absolute minimum value.*

In order to obtain a clear conception of the behavior of a function when it passes through maximum and minimum values, construct the curves C, C_1, and C_2, representing the given function

$$y = f(x),$$

and its first and second derivatives (1299),

$$y' = f'(x) \text{ and } y'' = f''(x).$$

At first the function $\quad y = f(x)$

is increasing, that is, when the abscissa Op' is increased, the ordinate $m'p'$ increases also, and this is true until the point M is reached, where the function takes a maximum value $y = MP$. Up to this point the slope remained positive, that is,

$$y' = f'(x) = \frac{dy}{dx}$$

remains positive, but diminishes continuously until at M it is equal to zero. The tangent to the curve C at M is parallel to the x-axis.

Starting at M the function y becomes decreasing, that is, when the abscissa Op'', for example, is increased, the ordinate $m''p''$

decreases; this goes on until at M_1 the function reaches a minimum. From M to M_1 the slope or first derivative is negative. It goes on increasing up to the point of inflection between M and M_1, and from this point it decreases continuously until it reaches M_1, where it becomes zero, since the tangent to the curve at M_1 is parallel to the x-axis.

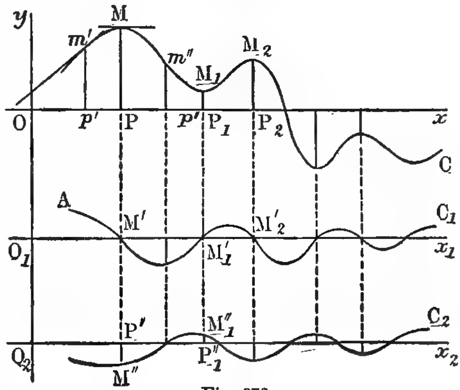

Fig. 376

In the same way, between M_1 and M_2, the function is increasing, and the first derivative is positive, becoming zero at M_2, which is another maximum.

Thus for all maximum or minimum values of the function

$$y = f(x),$$

the first derivative is zero,

$$y' = f'(x) = \frac{dy}{dx} = 0;$$

that is, the points $M', M_1', M_2' \ldots$ which correspond to the points $M, M_1, M_2, \ldots$ are situated on the axis O_1x_1.

To distinguish a maximum from a minimum we have recourse to the curve C_2 which represents the second derivative. It is seen that the ordinate of the curve C_2 or the second derivative, which corresponds to the maximum MP, is negative, while the second derivative, which corresponds to the minimum M_1P_1, is positive.

It may be demonstrated that this is always the case. Thus, when the function,

$$y = f(x)$$

is increasing, the first derivative for the part m', for example, is positive, and at M is equal to zero. Since a quantity which is positive tends towards zero, it is decreasing, as is indicated by the portion AM' of the curve C_1, and therefore,

$$y' = f'(x) = \frac{dy}{dx}$$

is a decreasing function. This established, as we see in Fig. 376, when a function is decreasing, the derivative of this function is negative; therefore, the second derivative $M''P''$ is negative when the original function reaches a maximum value.

In the same manner it may be demonstrated that the second derivative of a function corresponding to a minimum value of that function, is positive.

Since it is simply the sign of the second derivative which distinguishes between maximum and minimum values of a given function, if it happens that the second derivative is zero, it can have no sign, and could not indicate whether the corresponding value of the function were a maximum or a minimum.

In this case it is necessary to have recourse to the 3d and 4th derivatives, as shown below.

We have seen (1304) that a function

$$y = f(x + h)$$

may be written in the form,

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{1 \cdot 2} + f'''(x)\frac{h^3}{1 \cdot 2 \cdot 3} + f^{\text{IV}}(x)\frac{h^4}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$$

The increment of the function may be written:

$$f(x+h) - f(x) = f'(x)h + f''(x)\frac{h^2}{1 \cdot 2} + f'''(x)\frac{h^3}{1 \cdot 2 \cdot 3} + f^{\text{IV}}(x)\frac{h^4}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$$

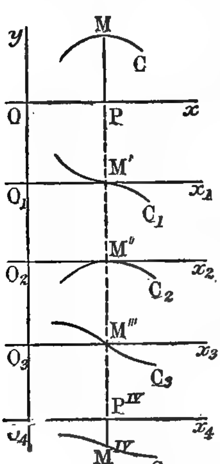

Fig. 377

If for a certain value of x the functions $f'(x)$ and $f''(x)$ are zero at the same time (Fig. 377), this last relation is reduced to

$$f(x + h) - f(x) = f'''(x)\frac{h^3}{1 \cdot 2 \cdot 3} + f^{\text{IV}}(x)\frac{h^4}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$$

and since when the increment h of the variable x is very small, the terms of the second member which follow the first term are neeligible in comparison with it, and we have,

$$f(x + h) - f(x) = f'''(x)\frac{h^3}{1 \cdot 2 \cdot 3}. \qquad (1)$$

Therefore, if the increment $f(x + h) - f(x)$ of the function is zero, which corresponds to a maximum or a minimum, we have,

$$f'''(x)\frac{h^3}{1 \cdot 2 \cdot 3} = 0,$$

which requires that $$f'''(x) = 0;$$

since the increment h of the abscissa, although very small, is not zero.

Thus we see that the maximum or minimum of a function corresponds to

$$f'''(x) = 0.$$

It now remains to determine when we have a maximum and when a minimum. Noting that before a maximum the increment $f(x + h) - f(x)$ is positive and before a minimum it is negative, from the relation (1) $f'''(x)$ has the same sign as this increment, since h and therefore h^3 is always positive. Since a positive function $f'''(x)$ which approaches zero is decreasing, and the derivative of a decreasing function is negative, it follows that $f^{\text{IV}}(x)$ is negative for a maximum value of the function (Fig. 377).

For the same reason, if the increment $f(x + h) - f(x)$ is negative, $f'''(x)\dfrac{h^3}{1 \cdot 2 \cdot 3}$ will be negative, and therefore $f'''(x)$ will be negative. Since a negative function which approaches zero is increasing, and the derivative of an increasing function is positive, it follows that $f^{\text{IV}}(x)$ is positive for a minimum value of the function.

There is a maximum or a minimum when the third derivative $f'''(x)$ is zero, and it is a maximum or a minimum according as the fourth derivative $f^{\text{IV}}(x)$ is negative or positive.

In general, when several successive derivatives are equal to zero, there is neither maximum nor minimum if the first derivative after the one which is not equal to zero is of an odd order; but if it is of an even order, there is a maximum or minimum, according as it is negative or positive.

1308. A function y of a single variable x being given in the form

$$y = f(x), \tag{1}$$

to find the maximum or minimum of this function, take the first derivative of y with respect to x and put it equal to zero, thus:

$$\frac{dy}{dr} = f'(x) = 0. \tag{2}$$

This equation solved for x gives the value of x corresponding to the maximum or minimum. Then find the second derivative,

$$y'' = f''(x), \tag{3}$$

and according as this derivative is negative or positive, there is a maximum or a minimum. The value of x deduced from equation (2), substituted in equation (1), gives a maximum or minimum value of y.

If the second derivative y'' is zero, take the third and fourth derivatives,

$$y''' = f'''(x),\ (4) \qquad y^{\text{IV}} = f^{\text{IV}}(x); \qquad (5)$$

put $f'''(x)=0$, and solve for x and substitute in (1), which will give the maximum or minimum value of y according as y^{IV} is negative or positive.

If the fourth derivative were also zero, we would take the fifth and sixth, and so on.

1309. *Applications of the preceding rule.*

Example 1. *The product y of two variables x and z, whose sum c is constant, is a maximum when the two factors are equal* (**583**).

Accordingly, we have,

$$x + z = c \ (a) \qquad y = xz \quad (b)$$

From (a)

$$z = c - x.$$

Substituting this value in (b),

$$y = cx - x^2. \tag{1}$$

Taking the first derivative and putting it equal to zero (**1276**, **1280**),

$$\frac{dy}{dx} = f'(x) = c - 2x = 0. \tag{2}$$

Solving for x, we obtain the value corresponding to the maximum or minimum,

$$x = \frac{c}{2}.$$

Taking the second derivative (**1279**),

$$\frac{d^2y}{dx^2} = f''(x) = -2.$$

This derivative being negative, $x = \frac{c}{2}$ corresponds to a maximum and not to a minimum. Substituting this value in (a), we find

$$z = \frac{c}{2}.$$

Thus we have a maximum when the two factors are equal,

$$x = z = \frac{c}{2}.$$

EXAMPLE 2. *Of all cylinders having the same volume V, determine which has the minimum total surface S.*

r being the radius of the base and h the altitude of the cylinder, we have,

$$S = 2\pi r^2 + 2\pi rh, \tag{a}$$

and

$$V = \pi r^2 h, \quad h = \frac{V}{\pi r^2}. \tag{b}$$

Substituting this value of h in (a), we obtain an expression involving only two variables S and r,

$$S = 2\pi r^2 + \frac{2V}{r} = 2\pi r^2 + 2Vr^{-1}. \tag{1}$$

Taking the first derivative and putting it equal to zero,

$$\frac{dS}{dr} = f'(r) = 4\pi r - 2Vr^{-2} = 0. \tag{2}$$

Solving for r, we obtain the value of r corresponding to the maximum or minimum,

$$4\pi r = \frac{2V}{r^2}, \quad r = \sqrt[3]{\frac{V}{2\pi}}. \tag{3}$$

Taking the second derivative,

$$\frac{d^2S}{dr^2} = f''(r) = 4\pi + 4Vr^{-3} = 4\pi + \frac{4V}{r^3}.$$

This derivative being positive, $r = \sqrt[3]{\frac{V}{2\pi}}$ corresponds to a minimum and not to a maximum. Substituting this value of r in (1), we obtain the minimum value of S in terms of V; but the dimension h being of more importance, substituting in (3) the value of V given in (b), we have,

$$r = \sqrt[3]{\frac{\pi r^2 h}{2\pi}} \text{ or } r^3 = \frac{r^2 h}{2} \text{ and } h = 2r = 2\sqrt[3]{\frac{V}{2\pi}}.$$

Thus S is a minimum when the altitude of the cylinder is twice the radius of the base, and we have

$$V = 2\pi r^3 = \frac{\pi h^3}{4}.$$

EXAMPLE 3. *The mean temperature in a chimney corresponding to the maximum draft,* according to the old theory of Péclet, is expressed by the formula

$$Q_1 = 1.3\,D^2\sqrt{\frac{Ha}{M}\times\frac{t'-t}{(1+at')^2}},$$

wherein

Q_1 is the weight of air passed through the chimney per second;
1.3 is the weight of a cubic meter of air at 0° and 860 millimeter pressure;
D is one side of the minimum interior section, taken as square;
$a = 0.00367$ is the temperature coefficient of air;
M is a constant for any one class of chimneys;
t' is the mean temperature of the air in the chimney;
t is the temperature of the outside air.

$$1.3\,D^2\sqrt{\frac{Ha}{M}}$$

being a constant quantity for any one chimney, Q_1 will be a maximum when $1.3\,D^2\sqrt{\dfrac{t'-t}{(1+at')^2}}$ or $\sqrt{\dfrac{t'-t}{(1+at')^2}}$ is a maximum.

Representing this radical by y and the variable t' by x, we have,

$$y^2 = \frac{x-t}{(1+ax)^2}, \tag{1}$$

or $$y^2 + 2\,axy^2 + a^2x^2y^2 - x + t = 0.$$

Taking the first derivative (1288) and putting it equal to zero,

$$\frac{dy}{dx} = \frac{-2\,ay^2 - 2\,a^2y^2x + 1}{2\,y + 4\,axy + 2\,a^2x^2y} = 0.$$

This being true only when

$$-2\,ay^2 - 2\,a^2y^2x + 1 = 0, \text{ or } -2\,ay^2(1+ax) + 1 = 0.$$

Substituting the value of y^2 given in (1), we have

$$-2\,a\frac{x-t}{(1+ax)^2}(1+ax) + 1 = 0;$$

from which we deduce successively,

$$2a\frac{x-t}{1+ax}=1,$$

$$2ax-2at=1+ax,$$

$$ax=1+2at,$$

$$x=\frac{1}{a}+2t.$$

If we assume the temperature t of the outside air to be zero, we have

$$x \text{ or } t'=\frac{1}{a}=\frac{1}{0.00367}=272.47^{\circ}.$$

1310. *Special cases of maxima and minima.*

1st. When a function has a value equal to infinity or zero, this value cannot properly be considered as a maximum or a minimum. The parabola whose equation is (1197)

$$y^2=2px,$$

giving $y=0$ for $x=0$, and $y=\pm\infty$ for $x=\infty$, the function varies continuously from $+\infty$ to $-\infty$, and has neither maximum nor minimum.

The derivative of the preceding function being

$$\frac{dy}{dx}=f'(x)=\frac{p}{y},$$

putting it equal to zero,

$$f'(x)=\frac{p}{y}=0,$$

we have $y=\pm\infty$, values which correspond to $x=\infty$. Thus the points at which the tangents are parallel to the x-axis are at infinity. For $x=0$, we have $y=0$, and therefore,

$$f'(x)=\frac{p}{y}=\infty.$$

Thus the y-axis is tangent to the curve.

If the logarithmic curve,

$$y=\log x,$$

is given:

Taking the derivative (1281),

$$\frac{dy}{dx}=f'(x)=\frac{\log e}{x}=\frac{0.4342945}{x};$$

putting this derivative equal to zero,

$$f'(x) = \frac{0.4342945}{x} = 0;$$

from this $x = \infty$, and therefore, $y = \log x = \infty$; moreover, since for $x = 0$, we have $y = \log 0 = -\infty$, the function varies continuously from $+\infty$ to $-\infty$, and nevertheless has no maximum nor minimum.

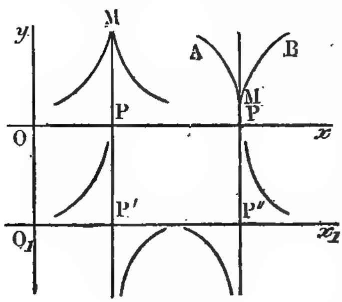

Fig. 378

2d. *Another peculiarity of maxima and minima. Point of retrogression.* When a curve has two branches AM and MB, having a common tangent parallel to the y-axis (Fig. 378), the point M necessarily corresponds to a maximum or a minimum. At this point M the slope of the tangent is

$$\frac{dy}{dx} = f'(x) = \pm\infty.$$

The point M is called *the point of retrogression.*

A point of retrogression M (Fig. 379) may correspond to a tangent whose slope is not parallel to the y-axis, that is, a value of $\frac{dy}{dx}$ which is not zero.

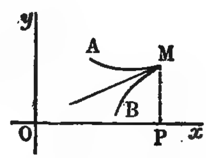

Fig. 379

3d. A curve may give a value of zero for the first derivative, and still have neither maximum nor minimum. This is the case when the curve (Fig. 380) has a tangent at a point of inflection which is parallel to the x-axis; because for this point,

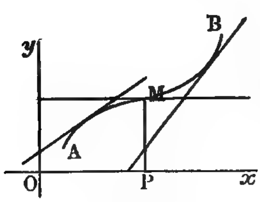

Fig. 380

$$f'(x) = 0.$$

This case may be recognized from the fact that, starting from the point M, the curve is convex or concave to the x-axis, according as the second derivative is positive or negative (1301). It may also be noted that in the case where M is a point of inflection the first derivative does not change its sign, since the tangent to the curve at

that point and beyond does not change the direction of its slope with reference to the x-axis; except that it is zero at the point of inflection.

Example of curves which have a maximum, a minimum, and a point of inflection.

Given the equation

$$y = x^3 - 3x + 1 \qquad (1)$$

of a curve referred to a system of coördinate axes Ox and Oy. Taking the first and second derivative, we have,

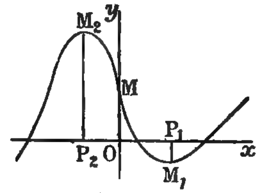

Fig. 381

$$y' = 3x^2 - 3,$$
$$y'' = 6x.$$

For the point of inflection M the second derivative is equal to zero (1302).

$$y'' = 6x = 0, \text{ and } x = 0.$$

It is seen that the point of inflection is situated on the y-axis. To determine the ordinate, make $x = 0$ in equation (1), which gives $y = 1$.

To obtain the coördinates of the points M_1 and M_2 corresponding to the minimum and maximum, put the first derivative equal to zero,

$$3x^2 - 3 = 0;$$

then,

$$x = \pm 1.$$

Therefore, equation (1) gives,

$$y = 1 - 3 + 1 = -1,$$

$$y = -1 + 3 + 1 = +3.$$

Thus the points M_1 and M_2 have the coördinates

$$M_1 \begin{cases} y = -1 \\ x = +1 \end{cases} \qquad M_2 \begin{cases} y = +3 \\ x = -1 \end{cases}$$

1311. *A study of quantities which have an indeterminate form.*

Let us consider a quotient of two functions of the same variable x,

$$y = \frac{F(x)}{\phi(x)}. \qquad (1)$$

Giving x the value a, we have,

$$y = \frac{0}{0}.$$

Putting
$$u = F(x), \tag{2}$$
$$v = \phi(x). \tag{3}$$

The relation (1) may be written,
$$y = \frac{u}{v}. \tag{4}$$

Giving an increment Δx to the variable x, the variables u, v and y take corresponding increments, and relation (4) becomes

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v};$$

dividing both terms of the fraction by Δx,

$$y + \Delta y = \frac{\dfrac{u + \Delta u}{\Delta x}}{\dfrac{v + \Delta v}{\Delta x}}. \tag{5}$$

If for the value $x = a$, the functions (2) and (3) become zero, it follows that relation (5) has the limit

$$y = \frac{\dfrac{\Delta u}{\Delta x}}{\dfrac{\Delta v}{\Delta x}} = \frac{F'u}{F'v};$$

that is, the value of the given quotient will be given by the quotient of the derivatives of both the terms, in which $x = a$.

EXAMPLE 1. Find the value of

$$y = \frac{x^n - 1}{x - 1},$$

for $x = 1$. The direct calculation gives the indeterminate form,

$$y = \frac{0}{0}.$$

To make certain that the value is really indeterminate, replace the two terms by their derivatives, and in the new quotient put $x = 1$.

$$y = \frac{nx^{n-1}}{1} = n,$$

which is the required value.

EXAMPLE 2. Calculate

$$y = \frac{ax^3 - ab^6}{ax - ab^2}$$

for the particular value $x = b^2$. The direct calculation gives,

$$y = \frac{0}{0}.$$

Taking the derivatives of both the terms, and putting $x = b^2$, we obtain the real value,

$$y = \frac{3\,ax^2}{a} = \frac{3\,ab^4}{a} = 3\,b^4.$$

EXAMPLE 3. Calculate the following expression for $x = 30°$:

$$y = \frac{\frac{1}{2} - \sin x}{\sin x - \frac{1}{2}}. \qquad (A)$$

Sin $30° = \frac{1}{2}$, consequently the value of the expression takes the indeterminate form,

$$y = \frac{0}{0}.$$

Taking the derivatives of both terms of (A),

$$y = \frac{-\cos x}{\cos x} = -1.$$

It may be noted that the given expression reduces to the constant value -1 for all values of x. Thus,

$$y = \frac{\frac{1}{2} - \sin x}{\sin x - \frac{1}{2}} = \frac{-\left(\sin x - \frac{1}{2}\right)}{\sin x - \frac{1}{2}} = -1.$$

EXAMPLE 4. Referring to the form $\frac{\infty}{\infty}$, let the function

$$y = \frac{u}{v} \qquad (a)$$

be given, u and v being functions of x. It is required to calculate the value of y where a particular value given to x gives $u = \infty$ and $v = \infty$; such that

$$y = \frac{\infty}{\infty}.$$

The relation (a) may be written

$$y = \frac{\frac{1}{v}}{\frac{1}{u}}. \tag{b}$$

Since v and u become infinite for a particular value $x = a$, the reciprocals $\frac{1}{u}$ and $\frac{1}{v}$ are equal to zero. Therefore, we may consider y in relation (b) as having the form $y = \frac{0}{0}$ for the particular value $x = a$; and applying the above rule, that is, substituting the first derivatives for the terms of the quotient (b), the required value is obtained,

$$y = \frac{-\frac{1}{v^2}v'}{-\frac{1}{u^2}u'} = \frac{u^2}{v^2}\,\frac{v'}{u'},$$

or

$$\frac{u}{v} = \frac{u^2}{v^2}\,\frac{v'}{u'}.$$

Cancelling the common factor $\frac{u}{v}$, we have,

$$\lim \frac{u}{v} = \frac{u'}{v'}.$$

Thus we calculate the value of $y = \frac{u}{v}$ as in the first example, by substituting the derivatives of the terms in the given expression and putting $x = a$.

Example 5. Find the value of the function

$$y = \frac{\log x}{x}$$

for $x = \infty$. The direct calculation gives

$$y = \frac{\infty}{\infty}.$$

Taking the derivatives of the terms of the fraction separately, and making $x = \infty$, we obtain the real value,

$$y = \frac{\log e}{x} = \frac{\log e}{\infty} = 0.$$

If, giving $$y = \frac{x}{\log x},$$

the value for $x = \infty$ is desired, replacing both terms by their derivatives and putting $x = \infty$, the real value is obtained,

$$y = \frac{1}{\frac{\log e}{x}} = \frac{x}{\log e} = \frac{\infty}{\log e} = \infty.$$

EXAMPLE 6. Find the value of

$$y = \tan x - \frac{1}{\cos x} \tag{a}$$

for $x = 90°$. The direct calculation gives

$$y = \infty - \frac{1}{0} = \infty - \infty.$$

The relation (a) may be written

$$y = \frac{\sin x}{\cos x} - \frac{1}{\cos x} = \frac{\sin x - 1}{\cos x}. \tag{b}$$

For $x = 90°$, this becomes

$$y = \frac{1-1}{0} = \frac{0}{0}.$$

Substituting the derivatives for the terms of the fraction (b), and making $x = 90°$, we have,

$$y = \frac{\cos x}{-\sin 90°} = \frac{\cos 90°}{-1} = \frac{0}{-1} = 0.$$

REMARK. This value, $x = 90$, corresponds to a maximum of the given function.

$$y = \frac{\sin x - 1}{\cos x}.$$

Thus taking the derivative,

$$y' = \frac{\cos^2 x - (\sin x - 1)(-\sin x)}{\cos^2 x},$$

or
$$y' = \frac{\cos^2 x + \sin^2 x - \sin x}{\cos^2 x} = \frac{1 - \sin x}{\cos^2 x}.$$

The maximum corresponds to

$$1 - \sin x = 0;$$

then
$$\sin x = 1, \text{ and } x = 90°.$$

For all other values of x the function y is negative.

RADII OF CURVATURE

1312. The equation of a curve $MM'D$ of the form

$$y = f(x)$$

being given to find the value of the radius of curvature (1239).

Let M and M' be two points on the curve, MA and $M'B$ the tangents to the curve at these points, and MC and $M'C$ the normals at the same points. Decreasing the arc MM', at the limit the chord MM' coincides with the tangent to the curve at M; and the triangle MCM', whose vertex C is the center of curvature, is a right triangle, and we have

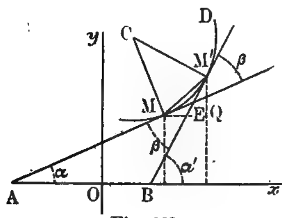

Fig. 382

$$\tan C = \frac{MM'}{MC}, \quad \text{and} \quad MC = \frac{MM'}{\tan C}.$$

The angle C included by the two normals, and the angle β included by the tangents, are equal, having their sides perpendicular to each other; and we have $\tan C = \tan \beta$, and therefore,

$$MC = \frac{MM'}{\tan \beta}. \tag{1}$$

The angle α' being an exterior angle of the triangle AEB, we have $\beta = \alpha' - \alpha$, and (1046)

$$\tan \beta = \frac{\tan \alpha' - \tan \alpha}{1 + \tan \alpha \tan \alpha'} = \frac{i' - i}{1 + ii'}, \tag{2}$$

designating the trigonometric tangents by i and i'. Since at the limit the slope of the tangents differs only by a differential di, we have,

$$i' = i + di;$$

and substituting this value in (2),

$$\tan \beta = \frac{i + di - i}{1 + i(i + di)} = \frac{di}{1 + i^2 + idi}. \tag{3}$$

Furthermore, the right triangle $MM'Q$ gives

$$MM' = \sqrt{\overline{MQ}^2 + \overline{M'Q}^2} = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{4}$$

Substituting the values (3) and (4) for $\tan \beta$ and MM' in (1), we have,

$$MC = \frac{dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}(1+i^2+idi)}{di}.$$

Noting that idi in the numerator may be neglected in comparison with $1 + i^2$, dividing both terms of the fraction by dx and designating the radius of curvature MC by ρ, we have

$$\rho = \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,(1+i^2)}{\frac{di}{dx}}.$$

Having $\quad i = \tan \alpha = \dfrac{dy}{dx} = f'(x)$ and $\dfrac{di}{dx} = \dfrac{d^2y}{dx^2} = f''(x)$,

the above relation may be written,

$$\rho = \frac{(1+[f'(x)]^2)^{\frac{1}{2}}(1+[f'(x)]^2)}{f''(x)} = \frac{(1+[f'(x)]^2)^{\frac{3}{2}}}{f''(x)}. \qquad (5)$$

If the sign of the numerator is always taken as plus +, ρ will have the same sign as $f''(x)$, and consequently will be positive or negative according as the curve is concave to the positive y-ordinates or the negative y-ordinates.

1. *Application to the parabola.* The equation of curvature being (1197)

$$y^2 = 2\,px,$$

we have successively,

$$i = \frac{dy}{dx} = f'(x) = \frac{p}{y},$$

$$i^2 = [f'(x)]^2 = \frac{p^2}{y^2},$$

$$y\frac{dy}{dx} \quad \text{or} \quad yi = p.$$

Differentiating this last relation (1281),

$$y\frac{di}{dx} + i\frac{dy}{dx} = 0,$$

or

$$y\frac{di}{dx} + i^2 = 0;$$

and

$$\frac{di}{dx} \text{ or } f''(x) = \frac{-i^2}{y} = \frac{-p^2}{y^3}.$$

These values substituted in formula (5) for the radius of curvature give,

$$\rho = \frac{\left(1+\frac{p^2}{y^2}\right)^{\frac{3}{2}}}{-\frac{p^2}{y^3}} = \frac{-y^3(y^2+p^2)^{\frac{3}{2}}}{p^2(y^2)^{\frac{3}{2}}} = \mp\frac{(y^2+p^2)^{\frac{3}{2}}}{p^2};$$

$\mp$ indicates that ρ has a sign opposite to that of y.

For $y = 0$,

$$\rho = \frac{(p^2)^{\frac{3}{2}}}{p^2} = \frac{p^{\frac{6}{2}}}{p^2} = p.$$

Thus at the vertex of the parabola the radius of curvature is twice the distance from the vertex to the focus (1195).

2. *Application to the circle.* From the equation of the circle (1123)

$$y^2 + x^2 = r^2,$$

we deduce successively (1288),

$$i = \frac{dy}{dx} = f'(x) = \frac{-x}{y},$$

$$i^2 = \frac{x^2}{y^2},$$

$$-x = yi,$$

$$-dx = idy + ydi,$$

$$\frac{di}{dx} = \frac{1}{y}\left(-1 - i\frac{dy}{dx}\right) = \frac{-(1+i^2)}{y},$$

or
$$f''(x) = \frac{-\left(1+\frac{x^2}{y^2}\right)}{y} = \frac{-(y^2+x^2)}{y^3}.$$

Substituting these values of $f'(x)$ and $f''(x)$ in the general formula (5), we have,

$$\rho = \frac{\left(1+\frac{x^2}{y^2}\right)^{\frac{3}{2}} y^3}{-(y^2+x^2)} = \frac{(y^2+x^2)^{\frac{3}{2}} y^3}{-(y^2+x^2)(y^2)^{\frac{3}{2}}} = \mp(y^2+x^2)^{\frac{1}{2}} = \mp\sqrt{y^2+x^2} = \mp r.$$

Thus the radius of curvature is constant and equal to the radius of a circle.

3. *Application to the sine wave* (1296, Fig. 367). The equation of the curve is

$$y = \sin x, \text{ or } y = R\sin x,$$

and $$f'(x) = R \cos x, \; f''(x) = -R \sin x.$$

The formula (5) for the radius of curvature gives,

$$\rho = \frac{(1 + R^2 \cos^2 x)^{\frac{3}{2}}}{-R \sin x}.$$

For $$x = 0, \quad \pi \text{ or } 180°, \quad 2\pi \text{ or } 360°,$$

$$\rho = \frac{(1 + R^2)^{\frac{3}{2}}}{0} = \infty;$$

that is, at the points $O, B, D \ldots$, there is an inflection or change in curvature.

For $x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ the radius of curvature has the value $\rho = \frac{1}{\mp R} = \mp \frac{1}{R}$, which is the radius of curvature in $A, C, \ldots$

4. *Application to the ellipse.* From the equation of the ellipse,

$$a^2y^2 + b^2x^2 = a^2b^2,$$

we deduce successively,

$$f'(x) = y' = \frac{-b^2x}{a^2y}, \qquad y'^2 = \frac{b^4x^2}{a^4y^2},$$

$$f''(x) = y'' = \frac{-a^2b^2y + a^2b^2xy'}{a^4y^2} = \frac{-b^2}{a^2y} + \frac{y'^2}{y},$$

or $$y'' = \frac{-b^2}{a^2y} + \frac{b^4x^2}{a^4y^3}.$$

Substituting these values of y' and y'' in the general formula (5) for the radius of curvature, we obtain,

$$\rho = \frac{\left(1 + \frac{b^4x^2}{a^4y^2}\right)^{\frac{3}{2}}}{-\left(\frac{b^2}{a^2y} - \frac{b^4x^2}{a^4y^3}\right)} = \frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^2b^2(b^2x^2 - a^2y^2)}.$$

For $a = b = r$, the formula gives $\rho = r$, which is as it should be, since the curve is then a circle.

For $x = 0$ and $y = b$, $\rho = \frac{a^2}{b}$, which is the radius of curvature of the minor vertices of the axis. For $y = 0$ and $x = a$, $\rho = \frac{b^2}{a}$, which is the radius of curvature of the vertices of the major axis.

INTEGRAL CALCULUS

INTRODUCTION

1313. *The object of integral calculus. Integration. Integral.*

Integral calculus can be used to find a function

$$y = f(x)$$

whose derivative $$y' = f'(x)$$

is given; or to find a function

$$y = f(x)$$

whose differential or differential coefficient

$$dy = f'(x)\,dx$$

is given.

As is seen, integral calculus is the inverse of differential calculus.

Thus the fundamental functions (1276, 1277, 1278, 1283)

$$y = x^m, \qquad y = \log x, \qquad y = \sin x, \qquad y = \cos x,$$

having respectively the derivatives and differentials

$$y' = mx^{m-1},\ y' = \frac{\log e}{x}, \qquad y' = \cos x, \qquad y' = -\sin x;$$

$$dy = mx^{m-1}\,dx,\ dy = \frac{\log e}{x}\,dx, \quad dy = \cos x dx, \quad dy = -\sin x dx,$$

if one of these derivatives or differentials is given, the above table gives the fundamental function from which it is derived.

However, since the same derivative, for example,

$$y' = mx^{m-1},$$

or the same differential,

$$dy = mx^{m-1}dx,$$

corresponds to two functions, namely,

$$y = f(x) = x^m$$

and $$y = f(x) + C \tag{1}$$

C being a constant (1279), which can be determined, the result of an integration is always written in the form

$$y = f(x) + C,$$

which signifies that if the curve C (Fig. 383), whose equation is

$$y = f(x),$$

satisfies the conditions, the same will be true of all other curves C', whose ordinate at any point A gives,

$$AP = MP \pm MA.$$

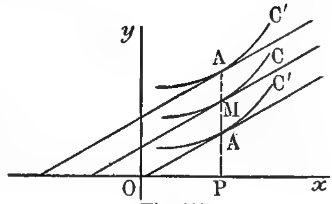

Fig. 383

The length MA is the constant C in relation (1).

It is to be noted that the three curves have the same slope at the points A, M, and A, since $f'(x)$ is the same for each; that is, the tangents at these points are parallels.

In practice, the constant C ceases to be arbitrary as soon as one point on the curve is known, or, which is the same thing, as soon as a system of values of x and y are known; because, substituting these values in equation (1) we may solve for C.

The process of finding the function

$$y = f(x) + C$$

of a differential equation

$$dy = f'(x)\,dx$$

is called *integration*, and the function is the *integral* of the differential dy.

1314. *Geometrical interpretation of an integral. Sign of integration. Limits of an integral. Definite integral. Indefinite integral.*

The first derivative, $y' = f'(x) = \dfrac{dy}{dx}$,

being given, we have $dy = f'(x)\,dx$,

and wish to find the original function

$$y = f(x) + C.$$

Suppose the problem to be solved, and let the curve $AMM'B$ represent the function.

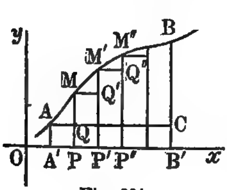

Fig. 384

Considering the two points M and M', which approach infinitely near each other; at the limit, the increment $M'Q'$ of the ordinate MP is the differential dy of this ordinate $MP = y$; and the increment PP' of the abscissa OP is the differential dx of

the abscissa $OP = x$; and it is seen that in order to pass from the ordinate of a point A on the curve to another point B, the sum of a certain number of increments $M'Q'$, $M''Q''$, . . . must be added to the ordinate at the point A.

Since at the limit the arc MM' coincides with the chord MM' or with tangent to the curve at M, the figure $MM'Q'$ is a right triangle, and we have,

$$M'Q' = MQ' \tan (M'MQ'),$$

or

$$dy = dx\frac{dy}{dx} = dxf'(x) = y'dx.$$

The element $M'M''$ gives,

$$M''Q'' = dy_1 = dx_1 \frac{dy_1}{dx_1};$$

and since we have the same for each element of the curve AB, it is seen that the quantity BC which is to be added to the ordinate at the point A in order to obtain that at the point B, is equal to the sum of the differentials dy, dy_1, . . . that is,

$$\Sigma dy = \Sigma y'dx,$$

wherein Σdy represents the sum of all the quantities analogous to dy' and $\Sigma y'dx$ the sum of all the products analogous to $y'dx$.

This sum is the required integral of dy, and is written

$$\int dy = \int y'dx,$$

which is read, *integral of dy equal to integral of y'dx.*

To indicate that this sum or integral is to be taken from the point A to the point B, designating the abscissa at A by a and that at B by b, we write,

$$\int_a^b dy = \int_a^b y'dx,$$

which is read, *integral between the limits a and b of dy equal to the integral between the limits a and b of y'dx,* and signifies that the integral of the differential quantity of the form

$$dy = f'(x)\,dx$$

is the sum of the increments dy of the function y, made between the *limits* a and b corresponding to two ordinates or particular finite values of the function y. One of these limits can be zero

or negative; that is what happens when the point A is on the y-axis or at the left of it; in each case the integral is written,

$$\int_0^b dy = \int_0^b y'dx, \text{ and } \int_{-a}^b dy = \int_{-a}^b y'dx.$$

The limit a being negative, the limit b can also be zero or negative.

An integral taken between two limits is called a *definite integral,* and an integral under the general form $\int dy$ is called an *indefinite integral.*

1315. *The calculation of a definite integral whose limits are given.*

Let
$$y = \int x^2 dx \tag{1}$$
be given. Then from (1276, 1313),
$$y = \frac{x^3}{3} + C. \tag{2}$$

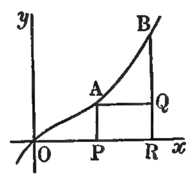

Fig. 385

Now let it be required to calculate this integral between the limits corresponding to the points A and B, whose coördinates are

$$A\begin{cases} x = a = OP \\ y = a' = AP \end{cases}, \qquad B\begin{cases} x = b = OR \\ y = b' = BR \end{cases}.$$

To calculate the integral $\int x^2 dx$ between the limits corresponding to the points A and B, amounts to finding the length BQ which must be added to AP in order to obtain BR. From the relation (2) we have,

$$AP = y = \frac{a^3}{3} + C \quad \text{and} \quad BR = y = \frac{b^3}{3} + C$$

and
$$BR - AP = \frac{b^3}{3} - \frac{a^3}{3} = \int_a^b x^2 dx.$$

Thus *the required result is obtained by substituting successively in the indefinite integral* (1) *the values of x which correspond to the limits of the integral and taking the algebraic difference of these two results.*

1316. *A definite integral may be represented geometrically by the area of a curve.*

Constructing the curves C and C' which represent respectively the function and its first derivative,

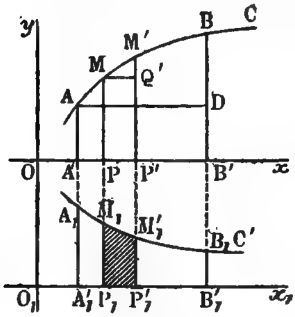

Fig. 386

$$y = f(x) \tag{1}$$

and $$y' = \frac{dy}{dx} = f'(x),$$

from which

$$dy = f'(x)\,dx = y'dx.$$

Since in integrating this last expression we obtain the original function (1), we have,

$$\int dy = \int y'dx \quad \text{or} \quad y = \int y'dx. \tag{2}$$

The infinitesimal increment dx of the variable x being represented geometrically by $PP' = P_1P_1'$, and y' by the ordinate M_1P_1, the product $y'dx$ is represented by the trapezoid $M_1P_1P_1'M_1'$, since at the limit $M_1P_1 = M_1'P_1'$, and it follows that the increment $dy = M'Q'$ of the ordinate $y = MP$ of the curve C is represented by the area $M_1P_1P_1'M_1'$. Since any other increment of the ordinate is likewise represented by a corresponding area, it follows that in passing from the ordinate at the point A to the ordinate at the point B, sum-total BD of all the increments of y will be represented by the sum of the corresponding areas, that is, by the area $A_1A_1'B_1'B_1$. Thus,

$$\int_a^b dy = \int_a^b y'dx = A_1A_1'B_1'B_1,$$

wherein a and b are the limits of the integral, that is, they determine the ordinates which bound the area.

Summing up, it is seen that the calculation of a definite integral may always be reduced to the determination of the area of a curve included between two ordinates which correspond to the limits of the integral, thus representing the first derivative of the required function

$$y = f(x) = \int y'dx.$$

RULES FOR INTEGRATION

1317. *Integrals of simple functions.*

There is no general method of integration. Analogy serves as the rule. Thus the function

$$y = x^m$$

having the derivative (1280), (1)

$$\frac{dy}{dx} = y' = mx^{m-1}, \tag{2}$$

and the differential, $dy = mx^{m-1}dx,$ (3)

if one of the expressions (2) or (3) were given to find the original function, the answer would be,

$$y = f(x) + C,$$

and we would write,

$$\int dy = \int mx^{m-1}\,dx = x^m + C,$$

that is, the exponent $m - 1$ is increased by one unit and the quantity divided by the new exponent and dx; thus,

$$\int dy = \int mx^{m-1}dx \quad \text{or} \quad y = \frac{mx^{m-1+1}}{m} = x^m;$$

then the arbitrary constant C is added so as to obtain a general expression of the function whose derivative is mx^{m-1}.

Therefore we have,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

This rule does not apply in the case where $n = -1$.

Thus we would have,

$$\int x^{-1}dx = \int \frac{dx}{x} = \frac{x^{-1+1}}{-1+1} + C = \frac{x^0}{0} + C = \frac{1}{0} + C = \infty + C,$$

or, if we had

$$dy = \frac{dx}{x},$$

by analogy (1281),

$$\int dy = \int \frac{dx}{x} = \frac{\log x}{\log e} + C.$$

Table of Integrals and Their Corresponding Differentials

$dx^{n+1} = (n+1)\,x^n dx,$	(1280)	$\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + C.$	(1)
$d\log x = \dfrac{\log e}{x}\,dx,$	(1281)	$\displaystyle\int \frac{\log e}{x}\,dx = \log x + C.$	(2)
$d\,\dfrac{\log x}{\log e} = \dfrac{dx}{x},$		$\displaystyle\int \frac{dx}{x} = \frac{\log x}{\log e} + C.$	(3)
$da^x = \dfrac{\log a}{\log e}\,a^x dx,$	(1289)	$\displaystyle\int a^x dx = \frac{\log e}{\log a}\,a^x + C.$	(4)
$d\sin x = \cos x\,dx,$	(1282)	$\displaystyle\int \cos x\,dx = \sin x + C.$	(5)
$d\cos x = -\sin x\,dx,$	(1287)	$\displaystyle\int \sin x\,dx = -\cos x + C.$	(6)
$d\tan x = \dfrac{dx}{\cos^2 x} = (1+\tan^2 x)\,dx,$	(1290)	$\displaystyle\int \frac{dx}{\cos^2 x} = \tan x + C.$	(7)
$d\cot x = \dfrac{-dx}{\sin^2 x},$	(1290)	$\displaystyle\int \frac{dx}{\sin^2 x} = -\cot x + C.$	(8)
$d\sec x = \dfrac{\sin x}{\cos^2 x}\,dx,$		$\displaystyle\int \frac{\sin x}{\cos^2 x}\,dx = \sec x + C.$	(9)
$d\csc x = -\dfrac{\cos x}{\sin^2 x},$		$\displaystyle\int \frac{-\cos x}{\sin^2 x}\,dx = \csc x + C.$	(10)
$d\sin^{-1} x = \dfrac{dx}{\sqrt{1-x^2}},$	(1290)	$\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C.$	(11)
$d\cos^{-1} x = \dfrac{-dx}{\sqrt{1-x^2}},$	(1290)	$\displaystyle\int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1} x + C.$	(12)
$d\tan^{-1} x = \dfrac{dx}{1+x^2},$	(1290)	$\displaystyle\int \frac{dx}{1+x^2} = \tan^{-1} x + C.$	(13)
$d\cot^{-1} x = \dfrac{-dx}{1+x^2},$	(1290)	$\displaystyle\int \frac{-dx}{1+x^2} = \cot^{-1} x + C.$	(14)
$d\sec^{-1} x = \dfrac{dx}{x\sqrt{x^2-1}},$		$\displaystyle\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C.$	(15)
$d\csc^{-1} x = \dfrac{-dx}{x\sqrt{x^2-1}},$		$\displaystyle\int \frac{-dx}{x\sqrt{x^2-1}} = \csc^{-1} x + C.$	(16)
$d\,\dfrac{1}{x} = \dfrac{-dx}{x^2},$	(1280)	$\displaystyle\int \frac{-dx}{x^2} = +\frac{1}{x} + C.$	(17)
$d\sqrt{x} = \dfrac{dx}{2\sqrt{x}}$	(1280)	$\displaystyle\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C.$	(18)
$d\sqrt{F(x)} = \dfrac{F'(x)\,dx}{2\sqrt{F(x)}},$	(1287)	$\displaystyle\int \frac{F'(x)\,dx}{\sqrt{F(x)}} = 2\sqrt{F(x)} + C.$	(19)

1318. *The integral of the sum of several differentials of the same variable x is equal to the sum of the integrals which compose this sum.* Thus, the algebraic sum,

$$y = u + v - z, \tag{1}$$

in which u, v and z are any functions of the same variable x, giving (1284),

$$d(u + v - z) = du + dv - dz.$$

Integrating both members, we have,

$$\int d(u + v - z) = \int du + \int dv - \int dz + C,$$

or

$$y = u + v - z + C,$$

C being the sum of the constants which must be added to each particular integral.

EXAMPLE 1. Integrating the differential expression,

$$dy = x^m dx + x^n dx - x^p dx,$$

we obtain (1317),

$$y = \frac{x^{m+1}}{m+1} + \frac{x^{n+1}}{n+1} - \frac{x^{p+1}}{p+1} + C.$$

EXAMPLE 2. Integrating,

$$dy = \frac{\log e}{x} dx + \cos x dx,$$

we obtain (1317),

$$y = \int \frac{\log e}{x} dx + \int \cos x dx = \log x + \sin x + C.$$

1319. *All constant factors in a differential expression appear in the coefficient of the integral of this expression.* Thus, the function,

$$y = af(x),$$

in which a is a constant, giving (1285, 3d)

$$dy = af'(x) dx.$$

Integrating this function, we have

$$\int af'(x) dx = af(x) + C.$$

As example we have (1317)

$$y = \int 5x^2 dx = \frac{5x^3}{3} + C.$$

PRINCIPAL THEOREMS OF INTEGRATION

1320. *Considering the constant coefficient, the integrals of certain functions* (1317) *may be deduced directly by making these constants appear as multipliers or divisors.*

EXAMPLE 1. The differentials

$$dy = \frac{dx}{x} \text{ and } dy = \frac{\log e}{x}dx,$$

differing only by the constant coefficient $\log e$, their integrals differ also by this same coefficient; thus (1317),

$$\int \frac{\log e}{x}dx = \log x + C,$$

$$\int \frac{dx}{x} = \frac{\log x}{\log e} + C.$$

REMARK. If the logarithms are taken in the Napierian system (408), since $\log_e e = 1$, we would have,

$$\int \frac{dx}{x} = \log_e x + C.$$

EXAMPLE 2. a and b being constant coefficients, we have (479, 1317, 1318),

$$\int (ax + bx^2)^2\, dx = \int a^2x^2dx + \int 2\,abx^3dx + \int b^2x^4dx$$

$$= \frac{a^2x^3}{3} + \frac{2\,abx^4}{4} + \frac{b^2x^5}{5} + C.$$

1321. *Integration by changing the independent variable or by substitution.*

A differential function which is not immediately integrable sometimes becomes so by changing the independent variable.

EXAMPLE 1. Let it be required to integrate

$$dy = (ax + bx)^m dx. \qquad (1)$$

The second member may be expanded by Newton's binomial theorem (530), and each term separately integrated; but it is simpler to operate in the following manner:

Putting $\qquad ax + bx = z, \text{ or } (a + b)x = z,$

we have $\qquad x = \dfrac{z}{a+b} \text{ and } dx = \dfrac{1}{a+b}dz.$

Substituting these values of $ax + bx$ and dx in relation (1), we have

$$dy = \frac{1}{a+b} z^m dz;$$

and integrating both members (1317, 1319),

$$y = \frac{1}{a+b} \frac{z^{m+1}}{m+1} + C;$$

then substituting $ax + bx$ for z, we have,

$$y = \frac{1}{a+b} \frac{(ax+bx)^{m+1}}{m+1} + C.$$

Example 2. Find the integral

$$y = \int \frac{a^2}{\sqrt{a^2 - x^2}} dx = \int \frac{a^2}{a\sqrt{1 - \frac{x^2}{a^2}}} dx = \int \frac{a}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} dx. \quad (1')$$

Putting $\frac{x}{a} = z$, then $dx = a dz$, and $\left(\frac{x}{a}\right)^2 = z^2$;
and substituting in (1′),

$$y = \int \frac{a^2}{\sqrt{1-z^2}} dz = a^2 \int \frac{dz}{\sqrt{1-z^2}} = a^2 \sin^{-1} z + C = a^2 \sin^{-1} \frac{x}{a} + C. \quad (1317)$$

Example 3. Find the integral

$$y = \int \tan x\, dx = \int \frac{\sin x}{\cos x} dx. \quad (1'')$$

Putting

$$\cos x = z, \text{ then } dz = -\sin x\, dx \text{ or } \sin x\, dx = -dz,$$

and substituting in (1″),

$$y = \int \frac{-dz}{z} = \frac{-\log z}{\log e} + C = \frac{-\log \cos x}{\log e} + C.$$

Taking the logarithms in the Napierian system (408), $\log_e e = 1$, and therefore

$$y = -\log_e \cos x + C.$$

Example 4. A being a constant, integrate

$$dy = \frac{Ax^2 dx}{(ax+b)^3}. \quad (1''')$$

Putting

$$ax + b = z, \quad x = \frac{z-b}{a} \text{ and } dx = \frac{dz}{a};$$

and substituting in (1‴),

$$dy = \frac{A\,(z-b)^2dz}{a^3z^3} = \frac{A}{a^3}\left(\frac{z^2dz}{z^3} - \frac{2\,bz\,dz}{z^3} + \frac{b^2dz}{z^3}\right)$$

or

$$dy = \frac{A}{a^3}\left(\frac{dz}{z} - 2\,bz^{-2}dz + b^2z^{-3}dz\right);$$

then integrating both members (1317, 1318, 1320),

$$y = \frac{A}{a^3}\left(\frac{\log z}{\log e} - \frac{2\,bz^{-1}}{-1} + \frac{b^2z^{-2}}{-2}\right) + C = \frac{A}{a^3}\left(\frac{\log z}{\log e} + \frac{2\,b}{z} - \frac{b^2}{2\,z^2}\right) + C;$$

and replacing z by its value $ax + b$,

$$y = \frac{A}{a^3}\left(\frac{\log\,(ax+b)}{\log e} + \frac{2\,b}{ax+b} - \frac{b^2}{2\,(ax+b)^2}\right) + C.$$

EXAMPLE 5. Find the integral

$$y = \int \sqrt{a^2 - x^2}\,dx. \qquad (a)$$

z being taken as the first auxiliary variable, put

$$x = a \sin z; \qquad (a')$$

from (1756)

$$dx = a \cos z\,dz \quad \text{and} \quad x^2 = a^2 \sin^2 z,$$

and therefore

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 z} = a\sqrt{1 - \sin^2 z} = a \cos z. \qquad (1041)$$

Substituting these values in (a),

$$y = \int a^2 \cos^2 z\,dz = a^2 \int \cos^2 z\,dz. \qquad (b)$$

Having (1047)

$$\cos 2\,z = 2 \cos^2 z - 1, \quad \text{and} \quad \cos^2 z = \frac{1 + \cos 2z}{2},$$

the relation (b) may be written

$$y = a^2 \int \frac{1 + \cos 2z}{2}\,dz = a^2 \int \frac{dz}{2} + a^2 \int \frac{\cos 2\,z}{2}\,dz,$$

or

$$y = \frac{a^2z}{2} + a^2 \int \frac{\cos 2\,z}{2}\,dz.$$

In order to integrate the second term of this last relation, put

$$2\,z = u, \text{ then } z = \frac{u}{2} \text{ and } dz = \frac{du}{2},$$

and then we have

$$y = \frac{a^2z}{2} + a^2\int\frac{\cos u}{2}\,\frac{du}{2} = \frac{a^2z}{2} + \frac{a^2}{4}\sin u = \frac{a^2z}{2} + \frac{a^2}{4}\sin 2z.$$

Since the relation (a') gives

$$\sin z = \frac{x}{a} \quad\text{and}\quad z = \sin^{-1}\frac{x}{a},$$

and from (**1041**, **1047**) we have

$$\sin 2z = 2\sin z\cos z,$$

and $$\cos z = \sqrt{1 - \sin^2 z} = \sqrt{1 - \frac{x^2}{a^2}} = \frac{\sqrt{a^2 - x^2}}{a},$$

now substituting these values in the last expression for y,

$$y = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{a^2}{4}\,2\,\frac{x}{a}\,\frac{\sqrt{a^2 - x^2}}{a};$$

simplifying and adding the constant C, we have

$$y = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C.$$

This formula finds application in (**1328**) for determining the area of the circle and the ellipse.

Example 6. Find the integral

$$y = \int\sqrt{p^2 + x^2}\,dx, \tag{a}$$

wherein p is a constant.

Putting $$\sqrt{p^2 + x^2} = z - x, \tag{b}$$

wherein z is an auxiliary variable, the relation (a) becomes

$$y = \int(z - x)\,dx = \int z\,dx - \int x\,dx = \int z\,dx - \frac{x^2}{2}. \tag{a'}$$

From the relation (b) we deduce successively,

$$p^2 + x^2 = z^2 - 2zx + x^2,$$
$$p^2 = z^2 - 2zx, \tag{c}$$
$$x = \frac{z^2 - p^2}{2z},$$
$$z = x + \sqrt{p^2 + x^2}, \tag{572}$$
$$z^2 = 2x^2 + p^2 + 2x\sqrt{p^2 + x^2}.$$

Differentiating the equation (c), we obtain (1276, 1279, 1280, 1281)

$$0 = 2\,zdz - 2\,zdx - 2\,xdz,$$

from which

$$dx = \frac{(z-x)\,dz}{z} = \frac{\left(z - \dfrac{z^2-p^2}{2\,z}\right)dz}{z} = \frac{(z^2+p^2)\,dz}{2\,z^2}.$$

Substituting this value of dx in $\int z\,dx$ of relation (a'), we have

$$\int z\,dx = \int\frac{(z^2+p^2)\,dz}{2\,z} = \int\frac{z\,dz}{2} + \int\frac{p^2}{2}\frac{dz}{z} = \frac{z^2}{4} + \frac{p^2}{2}\frac{\log z}{\log e}. \quad (1320)$$

Now substituting for z and z^2,

$$\int z\,dx = \frac{x^2}{2} + \frac{p^2}{4} + \frac{x}{2}\sqrt{p^2+x^2} + \frac{p^2}{2}\frac{\log\left(x+\sqrt{p^2+x^2}\right)}{\log e}.$$

This value of $\int z\,dx$ substituted in relation (a') gives the integral upon adding the constant C; thus,

$$y = \int\sqrt{p^2+x^2}\,dx = \frac{p^2}{4} + \frac{x}{2}\sqrt{p^2+x^2} + \frac{p^2}{2}\frac{\log\left(x+\sqrt{p^2+x^2}\right)}{\log e} + C. \quad (d)$$

This formula will be used in (1338) for the rectification of a parabola, and in (1339) for the rectification of the spiral of Archimedes.

1322. *Integration by parts.*

Integrating the expression

$$dy = u\,dv,$$

in which u and v are functions of x, we obtain,

$$y = \int u\,dv = uv - \int v\,du.$$

In fact, differentiating the expression

$$y = uv,$$

we have (1281) $\quad dy = d\,(uv) = v\,du + u\,dv,$

from which, $\quad u\,dv = d\,(uv) - v\,du;$

and integrating both members,

$$y = \int u\,dv = uv - \int v\,du. \quad (A)$$

Thus the integral of the product $u dv$ is transformed to an algebraic difference one term of which is the product uv of the variables (functions of x), and the other $\int v du$, although of the same form as the given integral, may be simpler.

EXAMPLE 1. Find the integral

$$y = \int \log x dx.$$

Putting $\log x = u$, we have

$$du = \frac{\log e\, dx}{x}; \tag{1277}$$

and putting $dx = dv$, we have $x = v$.

Then from formula (A),

$$y = \int \log x dx = x \log x - \int x \frac{\log e dx}{x} = x \log x - \int - x \log e,$$

or $$\int \log x dx = x (\log x - \log e) + C = x \log \frac{x}{e} + C. \tag{396}$$

EXAMPLE 2. Find the integral

$$y = \int x \sin x dx.$$

Putting $\qquad x = u, \quad dx = du,$

and $\qquad \sin x\, dx = dv,\ v = \int \sin x\, dx = - \cos x. \qquad$ (1317)

Then from formula (A),

$$\int u dv = uv - \int v du,$$

$$y = \int x \sin x dx = - x \cos x - \int - \cos x dx = - x \cos x + \sin x + C.$$

EXAMPLE 3. Find the integral

$$y = \int x^2 a^x dx.$$

Putting $\qquad x^2 = u$ and $a^x = v,$

we have $\qquad 2 x dx = du$ and $\frac{\log a}{\log e} a^x dx = dv. \qquad$ (1285)

Then from formula (A),

$$\int u\,dv = uv - \int v\,du,$$

$$y = \int x^2 a^x\,dx = x^2 a^x - \int a^x 2\,x\,dx. \qquad (B)$$

To calculate $\int a^x 2\,x\,dx,$

put $2\,x = u,$ then $2\,dx = du$

and $a^x dx = dv,$ then $\frac{\log\,e}{\log\,a}a^x = v.$ **(1317)**

Substituting once more in formula (A),

$$\int a^x 2\,x dx = 2\,x\frac{\log\,e}{\log\,a}a^x - \int 2\frac{\log\,e}{\log\,a}a^x dx$$
$$= 2\,x\frac{\log\,e}{\log\,a}a^x - 2\frac{\log\,e}{\log\,a}\frac{\log\,e}{\log\,a}a^x = 2\frac{\log\,e}{\log\,a}a^x\left(x - \frac{\log\,e}{\log\,a}\right).$$

Now substituting this integral in formula (B),

$$y = \int x^2 a^x dx = x^2 a^x - 2\frac{\log\,e}{\log\,a}a^x\left(x - \frac{\log\,e}{\log\,a}\right) + C.$$

Example 4. Find the integral **(1321)**

$$y = \int \sqrt{a^2 - x^2}\,dx.$$

Putting $u = \sqrt{a^2 - x^2}$ and $x = v$

and differentiating, these relations give **(1283)**

$$du = -\frac{x}{\sqrt{a^2 - x^2}}dx, \text{ and } dx = dv.$$

Therefore, from formula (A),

$$\int u\,dv = uv - \int v\,du,$$

$$y = \int \sqrt{a^2 - x^2}\,dx = x\sqrt{a^2 - x^2} - \int -\frac{x^2}{\sqrt{a^2 - x^2}}dx. \qquad (a)$$

Multiplying and dividing the first member of this equation by $\sqrt{a^2 - x^2}$,

$$\int \sqrt{a^2 - x^2}\,dx = \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}}\,dx = \int \frac{a^2}{\sqrt{a^2 - x^2}}\,dx - \int \frac{x^2}{\sqrt{a^2 - x^2}}\,dx,$$

or, from (1794, EXAMPLE 2),

$$\int \frac{a^2}{\sqrt{a^2 - x^2}} dx = a^2 \sin^{-1} \frac{x}{a},$$

$$\int \sqrt{a^2 - x^2} dx = a^2 \sin^{-1} \frac{x}{a} - \int \frac{x^2}{\sqrt{a^2 - x^2}} dx. \qquad (b)$$

Adding the equations (a) and (b), we have,

$$2 \int \sqrt{a^2 - x^2} dx = a^2 \sin^{-1} \frac{x}{a} + x \sqrt{a^2 - x^2};$$

then the required integral is (1321)

$$y = \int \sqrt{a^2 - x^2} dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C.$$

1323. *Examples of integrals involving logarithmic functions.*

EXAMPLE 1. Find the integral

$$y = \int \frac{dx}{a^2 - x^2}. \qquad (1)$$

Replacing $\frac{1}{a^2 - x^2}$ by the sum of two fractions; thus, putting

$$\frac{1}{a^2 - x^2} = \frac{A}{a + x} + \frac{B}{a - x} \qquad (2)$$

and reducing to a common denominator,

$$\frac{1}{a^2 - x^2} = \frac{x(B - A) + a(A + B)}{a^2 - x^2}. \qquad (3)$$

The quantities A and B in the preceding relations are *indeterminate quantities*, to which values may be assigned such that the two numerators of relation (3) be equal. Thus, putting

$$A = B, \ a(A + B) = 1,$$

$$A = B = \frac{1}{2a}.$$

Substituting these values of A and B in expression (2), we have

$$\frac{1}{a^2 - x^2} = \frac{1}{2a}\left(\frac{1}{a + x} + \frac{1}{a - x}\right),$$

and the given integral (1) becomes

$$y = \int \frac{dx}{a^2 - x^2} = \int \frac{1}{2a}\left(\frac{dx}{a + x} + \frac{dx}{a - x}\right),$$

or

$$y = \int \frac{dx}{2a(a + x)} + \int \frac{dx}{2a(a - x)}. \qquad (4)$$

Putting $$a + x = u \text{ and } a - x = v, \tag{5}$$

we have $$dx = du \text{ and } -dx = +dv.$$

Relation (4) becomes,

$$y = \int \frac{du}{2\,au} + \int \frac{-dv}{2\,av} = \frac{\log u}{2\,a \log e} - \frac{\log v}{2\,a \log e}.$$

Now replacing u and v by their values (5), and observing that the difference of two logarithms is equal to the logarithm of a quotient,

$$y = \frac{1}{2\,a \log e} \log \left(\frac{a+x}{a-x}\right) + C.$$

EXAMPLE 2. Find the integral

$$y = \int \frac{dx}{x^2 - a^2}.$$

Following the same method as in the first example, we obtain

$$y = \frac{1}{2\,a \log e} \log \left(\frac{x-a}{x+a}\right) + C.$$

EXAMPLE 3. Find the integral

$$y = \int \frac{dz}{a + \frac{\log e}{z}}. \tag{1}$$

Put $$a + \frac{\log e}{z} = \frac{x}{z}, \tag{2}$$

x being an auxiliary variable.

From (2) $$az + \log e = x \tag{3}$$

$$dz = \frac{dx}{a} \text{ and } z = \frac{x - \log e}{a}. \tag{4}$$

Relation (1) may be written

$$y = \int \frac{\frac{dx}{a}}{\frac{x}{z}} = \int \frac{dx}{ax} z = \int \frac{dx}{ax}\left(\frac{x - \log e}{a}\right),$$

or $$y = \int \frac{dx}{a^2} - \int \frac{\log e}{a^2} \frac{dx}{x} = \frac{x}{a^2} - \frac{\log x}{a^2}. \tag{5}$$

Finally, by replacing x by its value (3), we obtain the required integral,

$$y = \frac{1}{a^2}[(az + \log e) - \log (az + \log e)] + C.$$

EXAMPLE 4. Find the integral

$$y = \int \frac{dz}{a - \dfrac{\log e}{z}}.$$

Following the same method as in the third example, we find

$$y = \frac{1}{a^2}[(az - \log e) + \log (az - \log e)] + C.$$

EXAMPLE 5. Find the integral

$$y = \int \frac{dz}{1 - \left(\dfrac{\log e}{z}\right)^2}. \qquad (A)$$

Referring to first example (1323), make the following substitutions in relation (1):

$$a = 1 \text{ and } x = \frac{\log e}{z}, \qquad (B)$$

then, the above relation (A) may be written,

$$y = \int \frac{dz}{1 - x^2} = \int dz \frac{1}{1 - x^2}. \qquad (C)$$

Proceeding as in the first example in article (1323), we have,

$$\frac{1}{1 - x^2} = \frac{1}{2}\left(\frac{1}{1 + x} + \frac{1}{1 - x}\right);$$

and replacing x by its value (B),

$$\frac{1}{1 - x^2} = \frac{1}{2}\left(\frac{1}{1 + \dfrac{\log e}{z}} + \frac{1}{1 - \dfrac{\log e}{z}}\right);$$

and substituting in (C),

$$y = \int \frac{dz}{2\left(1 + \dfrac{\log e}{z}\right)} + \int \frac{dz}{2\left(1 - \dfrac{\log e}{z}\right)}.$$

These integrals are the same as those in the third and fourth examples, considering $a = 1$, and we can write the result in the form

$$y = \frac{1}{2}[(z + \log e) - \log (z + \log e)]$$
$$+ \frac{1}{2}[(z - \log e) + \log (z - \log e)] + C.$$

Simplifying, $y = z - \frac{1}{2}\log (z + \log e) + \frac{1}{2}(z - \log e) + C.$

1324. *Integrals of trigonometric functions obtained in the form of logarithmic functions.*

Example 1. Find the integral

$$y = \int \frac{dx}{\sin x}. \tag{1}$$

Putting $\cos x = z,$

we have $\sin x = \sqrt{1 - \cos^2 x} = \sqrt{1 - z^2}.$

Taking the derivatives (1283, 3d),

$$\cos x \, dx = \frac{-2z\,dz}{2\sqrt{1 - z^2}}.$$

$$dx = \frac{-z\,dz}{\cos x \sqrt{1 - z^2}} = \frac{-dz}{\sqrt{1 - z^2}}.$$

Substituting in (1) the values of dx and $\sin x$ in terms of z,

$$y = \int \frac{-dz}{(1 - z^2)} = -\int \frac{dz}{1 - z^2}.$$

Referring to the first example (1323), and considering $a = 1$ and $x = z$, we obtain

$$\int \frac{dz}{1 - z^2} = \frac{1}{2 \log e}[\log (1 + z) - \log (1 - z)].$$

Changing the signs,

$$y = -\int \frac{dz}{1 - z^2} = \frac{1}{2 \log e}[\log (1 - z) - \log (1 + z)],$$

or

$$y = \frac{1}{2 \log e} \log \left(\frac{1 - z}{1 + z}\right) + C.$$

Replacing z by its value $\cos x$,

$$y = \frac{1}{2 \log e} \log \left(\frac{1 - \cos x}{1 + \cos x}\right) + C. \tag{2}$$

From (1048, 3d),

$$\tan \frac{1}{2} x = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$

then

$$\log \tan \frac{1}{2} x = \frac{1}{2} \log \left(\frac{1 - \cos x}{1 + \cos x}\right),$$

therefore, (2) may be written,

$$y = \frac{1}{\log e} \log \tan \frac{1}{2} x + C.$$

EXAMPLE 2. Find the integral

$$y = \int \frac{dx}{\cos x}.$$

Putting $\sin x = z$, and following the same course as in the preceding example,

$$y = \int \frac{dx}{\cos x} = \frac{1}{2 \log e} \log \left(\frac{1 + \sin x}{1 - \sin x}\right) + C.$$

REMARK. *Generalization of the two preceding examples.* The two following general integrals may be solved with the aid of the two preceding examples.

$$\int \frac{dx}{\sin^m x} = \frac{-\cos x}{(m-1)\sin^{m-1} x} + \frac{m-2}{m-1}\int \frac{dx}{\sin^{m-2} x}, \tag{A}$$

$$\int \frac{dx}{\cos^m x} = \frac{\sin x}{(m-1)\cos^{m-1} x} + \frac{m-2}{m-1}\int \frac{dx}{\cos^{m-2} x}. \tag{B}$$

For $m = 2$, the latter gives

$$\int \frac{dx}{\cos^2 x} = \frac{\sin x}{\cos x} = \tan x,$$

which conforms with the result given in the table (1317).

For $m = 3$, formula (B) gives

$$\int \frac{dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\int \frac{dx}{\cos x} + C.$$

Substituting the value found in the second example for $\int \frac{dx}{\cos x}$,

$$y = \int \frac{dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} + \frac{1}{4 \log e} \log \left(\frac{1 + \sin x}{1 - \sin x}\right) + C.$$

EXAMPLE 3. Find the integral

$$y = \int \frac{dx}{\tan x}. \tag{1}$$

This may be written $$y = \int \frac{dx \cos x}{\sin x}. \tag{2}$$

Putting $$\sin x = z,$$

we have, $\cos x = \sqrt{1 - \sin^2 x}$ or $\cos x = \sqrt{1 - z^2}$.

Taking the differentials,

$$d \sin x = dz \text{ or } \cos x\, dx = dz$$

and $$dx = \frac{dz}{\cos x} = \frac{dz}{\sqrt{1 - z^2}}.$$

Substituting in relation (2),

$$y = \int \frac{dz\sqrt{1-z^2}}{(\sqrt{1-z^2})z} = \int \frac{dz}{z} = \frac{\log z}{\log e};$$

therefore relation (1) gives

$$y = \int \frac{dx}{\tan x} = \frac{\log \sin x}{\log e} + C.$$

EXAMPLE 4. Find the integral

$$y = \int \frac{dx}{\cot x}.$$

Writing $\cot x = \dfrac{\cos x}{\sin x}$ and putting $\cos x = z$, and following a course analogous to that in the third example, we obtain

$$y = \int \frac{dx}{\cot x} = -\frac{\log \cos x}{\log e} + C.$$

EXAMPLE 5. Find the integral

$$y = \int \frac{dx}{\sin x \cos x}. \qquad (1)$$

This may be written (1069)

$$y = \int \frac{2\,dx}{2\sin x \cos x} = \int \frac{2\,dx}{\sin 2x}. \qquad (2)$$

Putting $\qquad 2x = z, \quad x = \dfrac{z}{2},$

and $\qquad 2\,dx = dz.$

Substituting in (2) the values of $2x$ and $2\,dx$ in terms of z, we obtain (1324, EXAMPLE 1)

$$y = \int \frac{dz}{\sin z} = \log \tan \frac{z}{2} = \log \tan x,$$

therefore $\qquad y = \displaystyle\int \frac{dx}{\sin x \cos x} = \log \tan x + C.$

INTEGRATION BY SERIES

EXAMPLE 1. Find the integral

$$y = \int \frac{dx}{1 + x^2}.$$

Referring to the table (1317, 13), we should write

$$y = \tan^{-1}x.$$

Expanding $(1 + x^2)^{-1}$ according to the binomial theorem,

$$\frac{dx}{1+x^2} = dx\,(1 + x^2)^{-1} = dx\,(1 - x^2 + x^4 - x^6 + \cdots\cdots).$$

Integrating these different terms,

$$y = \tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\cdots$$

EXAMPLE 2. In the same way for

$$y = \int \frac{dx}{\sqrt{1-x^2}},$$

we should write,

$$y = \sin^{-1}x.$$

Expanding,

$$(1 - x^2)^{-\frac{1}{2}} = 1 + \frac{x_2}{2} + \frac{1\cdot 3\cdot x^4}{2\cdot 4} + \frac{1\cdot 3\cdot 5\,x^6}{2\cdot 4\cdot 6}.$$

Multiplying by dx and integrating,

$$\sin^{-1}x = x + \frac{x^3}{2\cdot 3} + \frac{3\,x^5}{2\cdot 4\cdot 5} + \frac{3\cdot 5\,x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots\cdots$$

EXAMPLE 3. Given

$$y = dx\sqrt{\cos^2 x + 1}.$$

Expanding by the binomial theorem,

$$\sqrt{\cos^2 x + 1} = (\cos^2 x + 1)^{\frac{1}{2}}$$

$$= \cos x + \frac{1}{2\cos x} - \frac{1}{8}\frac{1}{\cos^3 x} + \frac{1}{16}\frac{1}{\cos^5 x} - \frac{5}{128}\frac{1}{\cos^7 x} + \cdots\cdots$$

Multiplying all the terms of the second member by dx, and integrating each term, we obtain,

$$y = \int \cos x\,dx + \int \frac{dx}{2\cos x} - \int \frac{1}{8}\frac{dx}{\cos^3 x} + \cdots\cdots + C.$$

Referring to the examples of number (1324), each term of this series is easily integrated.

APPLICATIONS OF INTEGRAL CALCULUS

QUADRATURE OF CURVES

1325. *General solution of the quadrature of curves.*

Given the equation

$$y = f(x)$$

of a curve C, to find the area included between the ordinates AA' and BB', Y and X being the coördinates of the point A, and Y' and X' those of the point B.

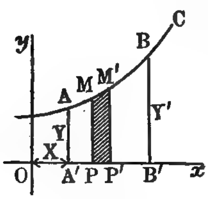

Fig. 387

Considering an element $MPP'M'$ of this area included between the ordinates MP and $M'P'$, y and x being the coördinates of the point M, at the limit those of the point M' will be $y + dy$ and $x + dx$, and the element $MPP'M'$ will be a trapezoid whose area we will designate by dS; then (723)

$$dS = \frac{y + (y + dy)}{2} dx. \tag{1}$$

This being established, we can easily conceive the entire surface $AA'B'B$ as being divided into infinitely small trapezoids; then the total area S will be equal to the sum ΣdS or $\int dS$ of the areas of all the elementary trapezoids, and we have

$$S = \int dS = \int \frac{y + (y + dy)}{2} dx. \tag{2}$$

Since dy in expressions (1) and (2) is negligible at the limit, the first one becomes,

$$dS = y\,dx,$$

and the second,

$$S = \int dS = \int y\,dx.$$

Calculating this integral in terms of x, and integrating between the limits $x = X$ and $x = X'$, we have (1314, 1315),

$$S = \int_X^{X'} y\,dx = \int_X^{X'} f(x)\,dx.$$

The same integral calculated in terms of y between the limits Y and Y', is

$$S = \int_{Y}^{Y'} y\, dx. \tag{3}$$

From the equation of the curve

$$y = f(x),$$

we deduce, dy in terms of x or dx in terms of y; which permits us to calculate the integral (3) in terms of one of the variables x or y.

1326. EXAMPLE 1. *The area of a right triangle.*

Given a straight line OB whose equation is (1117)

$$y = ax, \tag{1}$$

to calculate the area COC' included between the origin O and the ordinate CC'.

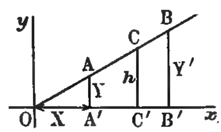

Fig. 388

Let $\quad OC' = b, \;$ and $\; CC' = h.$

The general formula (1325) is

$$S = \int y\, dx.$$

Replacing y by its value in (1), and integrating (1317, 1319),

$$S = \int ax\, dx = \frac{ax^2}{2} + C. \tag{2}$$

To obtain the required area COC', take this integral between the limits $x = 0$ and $x = b$. Since for $x = 0$ and $x = b$ we have respectively,

$$S = 0 + C \text{ and } S = \frac{ab^2}{2} + C,$$

the area COC' is (1315)

$$S = \int_0^b ax\, dx = \frac{ab^2}{2} + C - (0 + C) = \frac{ab^2}{2}. \tag{3}$$

Since for $x = 0$ we have $S = 0$, the relation (2) gives $0 = 0 + C$, therefore, $C = 0$.

The constant being zero, it may be left out of relation (2), which then becomes,

$$S = \int ax\, dx = \frac{ax^2}{2}.$$

This being established, we may put,

$$S = \int_0^b ax\,dx = \frac{ab^2}{2}.$$

In general, when for a determinate value of the variable, the indefinite integral becomes equal to zero, the constant C may be deduced by solving the equation in which the integral is zero. Then the definite integral having 0 and any value of the variable as limits is obtained by substituting the value of the variable at the limit and the value found for the constant, in the indefinite integral.

The preceding example is an application of this rule.

The point C being on the line OB, the values $y = h$ and $x = b$ may be substituted in relation (1); thus,

$$h = ab \text{ and } a = \frac{h}{b}.$$

Substituting this value of a in relation (3), we have the definite value of the required area,

$$S = \frac{hb^2}{2b} = \frac{bh}{2},$$

which is the well-known formula for the area of a triangle $C'OC'$ (718).

The same result is obtained by integrating

$$S = \int y\,dx$$

after having substituted for dx in terms of y. From relation (1) we have,

$$dy = a\,dx \text{ and } dx = \frac{dy}{a},$$

and therefore, $$S = \int \frac{y}{a}\,dy = \frac{y^2}{2a} + C.$$

Since for $y = 0$, $S = 0$,

$$0 = 0 + C \quad \text{or} \quad C = 0,$$

therefore $$S = \int \frac{y}{a}\,dy = \frac{y^2}{2a},$$

and the required area is

$$S = \int_0^h \frac{y}{a}\,dy = \frac{h^2}{2a}. \qquad (2')$$

Substituting the coördinates of the point C in relation (1),

$$h = ab \text{ and } a = \frac{h}{b};$$

now substituting this value in (2), the required area is

$$S = \frac{bh^2}{2h} = \frac{bh}{2}.$$

1327. Example 2. *The area of a trapezoid.*

To obtain the area of the trapezoid $AA'B'B$ (Fig. 378), it suffices to calculate the integral

$$S = \int y\,dx \tag{1325}$$

between the limits $x = X$ and $x = X'$, X and X' being the abscissas at the extreme points A and B. The area of the trapezoid is also equal to the difference between the areas of the triangles BOB' and AOA', that is (1326),

$$S = \int_X^{X'} y\,dx = \int_0^{X'} y\,dx - \int_0^X y\,dx,$$

or

$$S = \frac{aX'^2}{2} - \frac{aX^2}{2} = \frac{a}{2}(X'^2 - X^2) = \frac{a}{2}(X' + X)(X' - X).$$

Since the equation of the line OB,

$$y = ax,$$

gives respectively for the points A and B,

$$Y = aX \text{ and } Y' = aX',$$

by addition we have,

$$Y + Y' = a(X + X') \text{ and } (X + X') = \frac{Y + Y'}{a}.$$

Substituting this value of $X + X'$ in the above formula for S,

$$S = \frac{Y + Y'}{2}(X' - X),$$

which is the same expression given in (723) for the area of a trapezoid having Y and Y' for bases and $X' - X$ for altitude.

1328. Example 3. *Area of an ellipse and of a circle.*

The equation of an ellipse referred to its principal axes is (1131)

$$y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

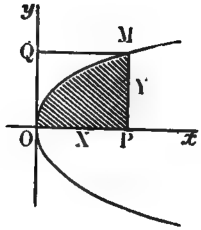

Fig. 389

The general formula for areas (1325),

$$S = \int y\,dx,$$

applied to the ellipse gives (1321, EXAMPLE 5),

$$S = \int \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{b}{a}\,\frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{b}{a}\,\frac{x}{2}\sqrt{a^2 - x^2} + C.$$

Taking this integral for a quarter of an ellipse, that is, between the limits $x = 0$ and $x = a$, for $x = 0$ we have $S = 0$, therefore $C = 0$, and for $x = a$ we have

$$S = \frac{ab}{2}\sin^{-1} 1 = \frac{\pi ab}{4};$$

therefore for a quarter of an ellipse,

$$S = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \frac{\pi ab}{4},$$

and for the total surface (1162),

$$S = \pi ab.$$

When $a = b = r$, the ellipse becomes a circle of radius r, and we have (753, 1162)

$$S = \pi r^2.$$

1329. EXAMPLE 4. *The area of a segment of a parabola.*

The equation of a parabola referred to its vertex being

$$y^2 = 2\,px, \qquad (1)$$

the general formula for areas (1325),

$$S = \int y\,dx,$$

gives

$$S = \int \sqrt{2\,p}\,x^{\frac{1}{2}}\,dx = \frac{\sqrt{2\,p}x^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}(\sqrt{2\,px})\,x + C = \frac{2}{3}xy + C.$$

Designating the coördinates of a point M by Y and X, the area of the segment MOP is obtained by taking the preceding inte-

gral between the limits $x = 0$ and $x = X$. For $x = 0$, $S = 0$, and we have $C = 0$; therefore, the required area is (1221)

$$S = \int_0^X \sqrt{2p}\, x^{\frac{1}{2}} dx = \frac{2}{3} XY.$$

We can integrate $$S = \int y\, dx$$

with respect to the variable y. Thus from relation (1)

$$2y\, dy = 2p\, dx \text{ and } dx = \frac{y}{p} dy.$$

This value of dx substituted in the general formula, gives

$$S = \int \frac{y^2}{p} dy = \frac{y^3}{3p} + C.$$

Taking this integral between the limits $y = 0$ and $y = Y$; since for $y = 0$, S $= 0$ and $C = 0$, the required area is

$$S = \int_0^Y \frac{y^2}{p} dy = \frac{Y^3}{3p},$$

or, since $Y^2 = 2pX$,

$$S = \frac{2pXY}{3p} = \frac{2}{3} XY.$$

1330. EXAMPLE 5. *The area of a sine wave.*

The equation of this curve being

$$y = \sin x,$$

the general formula for areas (1325),

$$S = \int y\, dx,$$

gives (1317) $$S = \int \sin x\, dx = -\cos x + C.$$

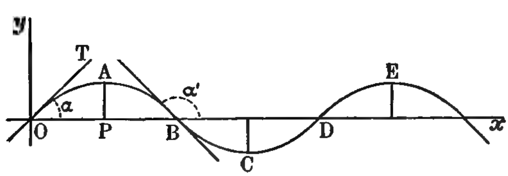

Fig. 390

To obtain the area S of a segment OAP, take this integral between the limits $x = 0$ and $x = OP = \frac{\pi}{2}$, which gives respectively

$$S = -1 + C \text{ and } S = -0 + C.$$

Therefore, neglecting the constant C, the area OAP is

$$S = \int_0^{\frac{\pi}{2}} \sin x\,dx = 0 - (-1) = 1.$$

Following the second method (1326), noting that for $x = 0$, $S = 0$, and that relation (2) becomes

$$0 = -1 + C \text{ and } C = 1.$$

Since for $x = \frac{\pi}{2}$ we have $\cos x = 0$,

$$S = \int_0^{\frac{\pi}{2}} \sin x\,dx = 0 + 1 = 1.$$

The practical interpretation of this result is easy. The equation (1) assumes that the radius R of the arc x is taken as unity, and from this it follows that the area $S = OAP$ is equivalent to that of a square whose side is equal to R. Thus if $R = 3$, $S = 9$.

The area OAB is double that of OAP, and its numerical value is 2, which is obtained by taking the integral (2) between the limits $x = 0$ and $x = OB = \pi$, which gives (since $\cos \pi = -1$ or $= \cos \pi = 1$)

$$S = \int_0^{\pi} \sin x\,dx = 1 + 1 = 2.$$

1331. Example 6. *The area of a logarithmic curve.*

$$y = \log x. \tag{1}$$

Substituting this value of y in the general equation for areas (1325), we have (1322),

$$S = \int y\,dx = \int \log x\,dx = x(\log x - \log e) + C = x \log \frac{x}{e} + C.$$

If the logarithms are taken in the Napierian system (407),

$$\log_e e = 1, \text{ and}$$

$$S = \int \log_e x\,dx = x \log_e x - x + C. \tag{2}$$

Since for $x = 0$, the area S is O, from relation (2) we have

$$0 = 0 + C \quad \text{and} \quad C = 0.$$

The constant C being 0, the relation (2) becomes

$$S = \int \log_e x\,dx = x \log_e x - x. \tag{3}$$

Integrating between the limits $x = 0$ and $x = OA = 1$, the area OAM', which indefinitely approaches the negative y-axis, is obtained; thus,

$$S = \int_0^1 \log_e x\,dx = 0 - 1 = -1.$$

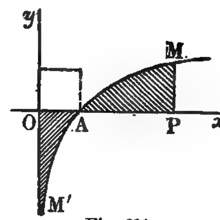

Fig. 391

Thus, neglecting the sign, the area OAM' is equivalent to the area of a square whose side is equal to OA taken as unity. If according to the chosen scale OA be equal to 25 inches, then the area OAM' is equal to -25 square inches.

Integrating the expression (3) between the limits $x = 1 = OA$ and $x = X = OP$, the area AMP is obtained. Since for $x = 1$ and $x = X$, the relation (3) gives respectively

$$S = -1 \text{ and } S = X \log_e X - X,$$

we have for the area AMP,

$$S = \int_1^X \log_e x\,dx = X \log_e X - X + 1.$$

1332. *Measuring areas by approximation.* Let it be required to determine the area of a curve included between the two ordinates AA' and CC'. Draw the ordinate BB' midway between these two extreme ordinates, and assume that the curve which passes through the points ABC, is an arc of a parabola, whose axis is parallel to $A'y$. Then the parabola whose arc passes through A, B, C, is expressed by

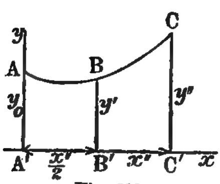

Fig. 392

$$y = a + bx + cx^2. \tag{1}$$

If we take AA' for the axis of y, we have

$$a = y_0,$$

calling y_0 the ordinate at the point A; because for $x = 0$ in equation (1) we have $y_0 = a$, and we may rewrite equation (1),

$$y = y_0 + bx + cx^2, \tag{2}$$

in which b and c are two constant coefficients to be determined.

The general formula for areas (1325) gives for the area $S = AA'C'C$,

$$S = \int_0^{x''} y\,dx = \int_0^{x''} (y_0 + bx + cx^2)\,dx,$$

or (1315, 1318)

$$S = y_0 x'' + \frac{bx''^2}{2} + \frac{cx''^3}{3} = x''\left(y_0 + \frac{bx''}{2} + \frac{cx''^2}{3}\right). \qquad (A)$$

To determine the coefficients b and c, note that formula (2) gives respectively for the points B and C,

$$y' = y_0 + \frac{bx''}{2} + \frac{cx''^2}{3} \text{ or } 4y' = 4y_0 + 2bx'' + cx''^2, \qquad (3)$$

$$y'' = y_0 + bx'' + cx''^2, \qquad (4)$$

and from these last two equations we can determine b and c in terms of known quantities.

But this is not necessary, and the sum within the parentheses in relation (A) can be calculated more simply by eliminating b and c. Thus, adding the relation $y_0 = y_0$, and (3) and (4) together, we have,

$$y_0 + 4y' + y'' = 6y_0 + 3bx'' + 2cx''^2 = 6\left(y_0 + \frac{bx''}{2} + \frac{cx''^2}{3}\right);$$

and
$$y_0 + \frac{bx''}{3} + \frac{cx''^2}{3} = \frac{y_0 + 4y' + y''}{6}.$$

Substituting this value in relation (A),

$$S = \int_0^{x''} y\,dx = \frac{x''}{6}(y_0 + 4y' + y'');$$

and putting
$$A'B' = B'C' = \frac{x''}{2} = \delta, \qquad \frac{x''}{6} = \frac{\delta}{3},$$

we have
$$S = \int_0^{x''} y\,dx = \frac{\delta}{3}(y_0 + 4y' + y''). \qquad (B)$$

1333. *Thomas Simpson's formula.* To calculate the area of a curve included between two ordinates AA' and EE' divide the projection $A'E'$ into an even number n of equal parts, and draw ordinates through the points of division. That done, apply successively the preceding formula (B) to the areas S, S', . . . included between the ordinates AA' and BB', BB' and CC', . . . which gives,

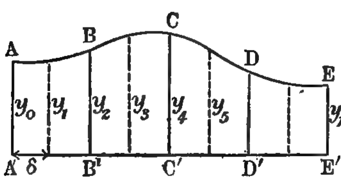

Fig. 393

$$s\ \ = \frac{\delta}{3}(y_0 + 4\,y_1 + y_2),$$
$$s' = \frac{\delta}{3}(y_2 + 4\,y_3 + y_4),$$
$$s'' = \frac{\delta}{3}(y_4 + 4\,y_5 + y_6),$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

Summing all these areas, we obtain the total area $S = s + s' + \ldots$

$$S = \frac{\delta}{3}[y_0 + y_n + 4\,(y_1 + y_3 + \cdots + y_{n-1}) + 2\,(y_2 + y_4 + \cdots + y_{n-2})]. \quad (C)$$

This formula was given in article (1268), where $\frac{E}{n}$ replaces δ.

1334. *The use of Thomas Simpson's formula for finding the approximate value of a finite integral of the form.*

$$\int_{x_0}^{x_n} uz\,dx,$$

when, for determinate values of x, the corresponding values of the other two variables u and z are known. Divide the difference $x_n - x_0$ of the limits into an even number n of equal parts, and putting

$$\frac{x_n - x_0}{n} = \delta \text{ and } uz = y,$$

the given integral becomes (1333)

$$\int_{x_0}^{x_n} y\,dx = \frac{\delta}{3}[y_0 + y_n + 4\,(y_1 + y_3 + \cdots + y_{n-1}) + 2\,(y_2 + y_4 + \cdots + y_{n-2})],$$

or, making

$$y_0 = u_0z_0, \quad y_1 = u_1z_1, \quad y_2 = u_2z_2, \quad \cdots, \quad y_n = u_nz_n,$$

$$\int_{x_0}^{x_n} y\,dx = \frac{\delta}{3}[u_0z_0 + u_nz_n + 4\,(u_1z_1 + u_3z_3 + \cdots) + 2\,(u_2z_2 + u_4z_4 + \cdots)].$$

We would proceed in the same way in calculating the integral

$$\int_{x_0}^{x_n} uvz\,dx.$$

Putting $\qquad \dfrac{x_n - x_0}{n} = \delta$ and $uvz = y,$

and substituting, we have,

$$\int_{x_0}^{x_n} uvz\,dx = \frac{\delta}{3}[u_0v_0z_0 + u_nv_nz_n + 4\,(u_1v_1z_1 + u_3v_3z_3 + \cdots) + 2\,(u_2v_2z_2 + u_4v_4z_4 + \cdots)].$$

1335. *Example of an integration obtained by means of the area of a circle.*

Find the value of the following integral between the limits $x = 0$ and $x = 2\,a$.

$$S = \int_{x=0}^{x=2a} dx\,\sqrt{(2\,a - x)\,x}. \tag{1}$$

$2\,a - x$ and x may be considered as two segments of a diameter $2\,a$ of a circle, referred to this diameter as the x-axis and a tangent as y-axis; such that y being an ordinate of a point in the semi-circumference above the axis, we may write,

$$y = \sqrt{(2\,a - x)}\,x. \tag{2}$$

Substituting this in (1),

$$S = \int_{x=0}^{x=2a} y\,dx = \frac{1}{2}\pi a^2.$$

Since each of the elements $y\,dx$ is included between the ordinates of the circle, their sum or integral is equal to the area of the semicircle of radius a, that is, $\frac{1}{2}\,\pi a^2$. The constant is zero because the value $x = 0$ gives $S = 0$.

Numerical example. Given

$$S = \int \frac{4\,dx}{\pi}\,\sqrt{(1 - x)\,x}.$$

From that which was said above, we have to consider here a circle whose diameter is 1. The quantities $1 - x$ and x are the two segments of this diameter, and the ordinate y of this circle is expressed thus:

$$y = \sqrt{(1 - x)\,x}.$$

The integral of the above expression, neglecting the coefficient $\frac{4}{\pi}$, is expressed by the area of a semicircle whose diameter is 1. Thus,

$$S = \frac{4}{\pi}\int y\,dx = \frac{4}{\pi}\,\frac{\pi \times 1^2}{4} = 1.$$

Remark. The preceding integration, in the form

$$S = \int dx \int \sqrt{(2\,a - x)\,x} = \frac{\pi x^2}{2},$$

is used in finding the area of a cycloid (1336).

1336. *The area of a cycloid* (1243).

Referring to (1297), the equation of the cycloid and its derivative are

$$x = \sin^{-1}\frac{\sqrt{2\,Ry - y^2}}{R} - \sqrt{2\,Ry - y^2}, \tag{1}$$

$$y' = \frac{dy}{dx} = \sqrt{\frac{2\,R - y}{y}}. \tag{2}$$

These two equations, together with the general formula for areas (1325),

$$S = \int_0^{y=2\,R} y\,dx, \tag{3}$$

are used for determining the area of the cycloid. The calculations may be greatly simplified by taking the origin at the vertex B of the curve (Fig. 379, 1243), the x-axis tangent to the curve at B and the y-axis normal $B\,4$ to the curve at that point. In thus changing the origin from A to B the ordinate y becomes $2\,R - y$, and consequently the equation (2) becomes

$$\frac{dy}{dx} = \sqrt{\frac{2\,R - (2\,R - y)}{2\,R - y}}.$$

$$\frac{dy}{dx} = \sqrt{\frac{y}{2\,R - y}}.$$

$$dx = dy\sqrt{\frac{2\,R - y}{y}}. \tag{4}$$

It is easy to recognize that equation (3) in the new system expresses the area ABL included by the curve and the lines BL and AL. Therefore, substituting the above (4) value of dx in (3),

$$S = \int y\,dy\sqrt{\frac{2\,R - y}{y}} = \int dy\,\sqrt{\,2\,R - y)\,y}.$$

Referring to (1335), we may write

$$S = \frac{\pi R^2}{2}$$

Thus the area ABL is equal to half that of the generating circle.

Also, the area of the rectangle $ALB\,4$ is equal to the product of the base πR by the altitude $2\,R$ or $2\,\pi R^2$. Therefore, the area $AB\,4 = \Omega$ of the cycloid included between the curve and its

base is equal to the difference between the two areas calculated above; thus,

$$\Omega = 2\,\pi R^2 - \frac{\pi R^2}{2} = \frac{4\,\pi R^2 - \pi R^2}{2},$$

or

$$\Omega = \frac{3\,\pi R^2}{2},$$

and

$$2\,\Omega = 3\,\pi R^2,$$

that is, the total area of the cycloid is three times that of the generating circle.

THE CUBATURE OF SOLIDS

1337. *General solution of the cubature of solids. The application of the formula of Thomas Simpson to the cubature of any solid.*

Given, a solid bounded by two planes A_0 and A_n perpendicular to the axis Ox. The volume of any element mm' included between two planes parallel to the bounding planes A_0 and A_n, is expressed,

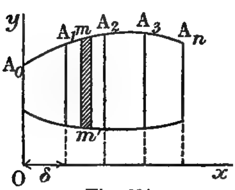

Fig. 394

$$dV = A\,dx,$$

wherein A is a mean section of the element made parallel to the end A_0, and dx is the infinitesimal thickness of the element. Therefore, *the general formula for volumes* is the integral

$$V = \int A\,dx,$$

which in special cases is taken between certain limits x_0 and x_n, which are the abscissas at the points where the planes A_0 and A_n cut the axis Ox.

To perform an approximate integration, divide the distance, $x_n - x_0$, between the bounding planes A_0 and A_n, into an even number n of equal parts δ; through the points of division draw planes parallel to the plane A_0, and find the area of the bases A_0 and A_n and the sections A_1, A_2, A_3 . . .; then applying Thomas Simpson's formula as for areas (1333), we have

$$V = \int A\,dx = \frac{\delta}{3}[A_0 + A_n + 4\,(A_1 + A_3 + \cdots + A_{n-1}) + 2\,(A_2 + A_4 + \cdots + A_{n-2})].$$

It is seen that numerically the volume V is equal to the area of a curve whose ordinates are proportional to the sections A_0, A_1, A_2, ... A_n, and whose abscissas are the same as those of these sections.

RECTIFICATION OF CURVES

1338. *To rectify a curve,* is to find its length expressed in linear units.

Given a curve AB whose equation is

$$y = f(x). \tag{1}$$

y and x being coördinates of the point M, those of the point M', which is infinitely near, are $y + dy$ and $x + dx$; the arc MM' coincides with its chord, and the right triangle $MM'Q$ gives

$$MM' = \sqrt{\overline{M'Q}^2 + \overline{MQ}^2},$$

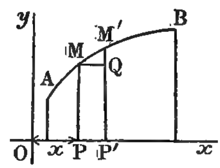

Fig. 395

which is an infinitely short arc rectified; representing it by dL, its differential is,

$$dL = \sqrt{(dy)^2 + (dx)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\sqrt{1 + [f'(x)]^2}.$$

Therefore the length L of a finite arc AB is given by the following integral, taken between the limits a and b of the variable corresponding to the extreme points A and B:

$$L = \int_a^b dL = \int_a^b dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \int_a^b dx\sqrt{1 + [f'(x)]^2}. \tag{2}$$

This is the general formula for the rectification of curves. In application, the derivative $f'(x)$ of the relation (1) is determined and its square substituted in relation (2); then the integral of the resulting expression is equal to the required length L.

Remark. The formula for rectification can also be written in the form

$$L = \int dy\sqrt{1 + \left(\frac{dx}{dy}\right)^2}.$$

Example 1. *Rectification of the parabola.*

Let it be required to rectify the parabola, whose equation is (Fig. 389, 1329)

$$y^2 = 2\,px.$$

We have $\dfrac{dy}{dx} = f'(x) = \dfrac{p}{y}$, then $dx = \dfrac{y}{p}\,dy$ and $[f'(x)]^2 = \dfrac{p^2}{y^2}$.

Substituting these values in formula (2),

$$L = \int_a^b dx\sqrt{1 + [f'(x)]^2} = \int_a^b \frac{y}{p}\,dy\sqrt{1 + \frac{p^2}{y^2}} = \frac{1}{p}\int_a^b dy\,\sqrt{y^2 + p^2}.$$

If the required length is the arc OM included between the vertex O and the point M (Fig. 389), the integral is taken between the limits $a = y = 0$ and $b = y = MP$, that is, between the limits O and $Y = MP$ From (1321, Example 6),

$$\frac{1}{p}\int dy\,\sqrt{y^2 + p^2} = \frac{p}{4} + \frac{y}{2p}\sqrt{y^2 + p^2} + \frac{p}{2\log e}\log\left(y + \sqrt{y^2 + p^2}\right) + C.$$

This expression should become zero for $y = 0$, since the arc is reduced to a point, and we have

$$0 = \frac{p}{4} + \frac{p}{2}\,\frac{\log p}{\log e} + C,\quad \text{whence}\quad C = -\frac{p}{4} - \frac{p}{2}\,\frac{\log p}{\log e}.$$

Substituting this value of C in the preceding integral, we obtain the required length,

$$L = \frac{1}{p}\int_0^Y dy\,\sqrt{y^2 + p^2} = \frac{Y}{2p}\sqrt{Y^2 + p^2} + \frac{p}{2\log e}\log\frac{Y + \sqrt{Y^2 + p^2}}{p}.$$

Example 2. *Rectification of the ellipse.*

This rectification depends upon an integral obtained by a series. Let a and b be the semi-axes of the ellipse, and e the eccentricity (1161).

$$e = \sqrt{\frac{a^2 - b^2}{a^2}}.$$

Then the length of the semi-ellipse is given by the formula

$$L = \pi a\left[1 - \left(\frac{1}{2}e\right)^2 - \frac{1}{3}\left(\frac{1}{2}\cdot\frac{3}{4}e^2\right)^2 - \frac{1}{5}\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}e^3\right)^2 - \cdots\right].$$

For $a = b = r$, this formula gives the value for a semicircle,

$$L = \pi a.$$

Example 3. *Rectification of a logarithmic curve.*

The equation of the curve is (1171)

$$y = \log x. \tag{1}$$

The rectification is given by the integral (**1338**)

$$L = \int dx\,\sqrt{1 + f'(x)^2} + C. \tag{2}$$

From (1) we deduce

$$f'(x) = \frac{\log e}{x};$$

therefore (2) becomes

$$L = \int dx\sqrt{1 + \left(\frac{\log e}{x}\right)^2} = \int \frac{dx}{x}\sqrt{x^2 + (\log e)^2}.$$

Putting $$x^2 + (\log e)^2 = z^2,$$

we have $$x = \sqrt{z^2 - (\log e)^2} \text{ and } z = \sqrt{x^2 + (\log e)^2},$$

$$dx = \frac{z\,dz}{x} = \frac{z\,dz}{\sqrt{z^2 - (\log e)^2}},$$

$$\frac{dx}{x} = \frac{z\,dz}{z^2 - (\log e)^2}.$$

Substituting for $\frac{dx}{x}$ in terms of z in the above integral, we obtain,

$$L = \int \frac{z^2dz}{z^2 - (\log e)^2} = \int \frac{dz}{1 - \left(\frac{\log e}{z}\right)^2}.$$

The value of this integral is (**1323**, Example 5),

$$L = z - \frac{1}{2}\log(z + \log e) + \frac{1}{2}\log(z - \log e).$$

Then substituting for z, we have

$$\left.\begin{aligned} L = &\sqrt{x^2 + (\log e)^2} - \frac{1}{2}\log\left(\sqrt{x^2 + (\log e)^2} + \log e\right) \\ &+ \frac{1}{2}\log\left(\sqrt{x^2 + (\log e)^2} - \log e\right) + C \end{aligned}\right\} \tag{3}$$

The constant is determined by noting (Fig. 391, **1331**), that $x = 1$ corresponds to the point A, since the equation (1) gives $y = \log 1 = 0$; and at this point the length of the corresponding arc is zero. Consequently the constant is determined by making

$$x = 1, \quad L = 0,$$

in formula (3), which will give C.

Replacing the value of C in (3), the length of any arc of the curve corresponding to any value of x starting from A can be obtained. For $x > 1$ the value of L is positive, and for $x < 1$ the value of L is negative.

REMARK. From formula (3), for $x = \infty$, $L = \infty$, which corresponds to the graph of the curve, since from the point A the curve extends to infinity in the direction of the positive y-axis.

For $x = 0$, $L = -\infty$, since the curve extends to infinity in the direction of the negative y-axis. Thus for $x = 0$ the formula (3) gives

$$L = \log e - \frac{1}{2}\log(\log e + \log e) + \frac{1}{2}\log(\log e - \log e) + C.$$

The last term gives $\quad \frac{1}{2}\log(0) = -\infty;$

therefore, $\quad L = -\infty.$

EXAMPLE 4. *Rectification of a cycloid.*

With the aid of the formula (1338),

$$L = \int dy\sqrt{1 + \left(\frac{dx}{dy}\right)^2}, \tag{1}$$

and the derivative of the equation of the cycloid (1297),

$$\frac{dy}{dx} = \sqrt{\frac{2R - y}{y}}, \tag{2}$$

the problem is solved as shown below.

To simplify the calculations the origin is changed to the vertex B (Fig. 349, 1243) of the cycloid (as was done in 1336). Then the ordinate y becomes $(2R - y)$, which, substituted in the derivative (2), gives

$$\frac{dy}{dx} = \sqrt{\frac{y}{2R - y}},$$

thus $\quad \frac{dx}{dy} = \sqrt{\frac{2R - y}{y}}$ and $\left(\frac{dx}{dy}\right)^2 = \frac{2R - y}{y}.$

Substituting this value in (1),

$$L = \int dy\sqrt{1 + \frac{2R - y}{y}} = \int dy\sqrt{\frac{2R}{y}},$$

$$L = \sqrt{2R}\frac{dy}{\sqrt{y}} = \sqrt{2R}\cdot 2\sqrt{y},$$

$$L = 2\sqrt{2Ry} + C \text{ or } L = 2\sqrt{2Ry}.$$

The constant $C = 0$, since the value $y = 0$ corresponds to the vertex B of the curve, the origin of the axes.

For $y = 2R$, we have,

$$L = 2\sqrt{4R^2} = 4R.$$

The total length of the curve,

$$2L = 8R = 4D,$$

that is, the length of the cycloid is equal to four times the diameter of the generating circle. The base of the curve is equal to $2\pi R = 3.1416\,D$.

RECTIFICATION OF CURVES EXPRESSED IN POLAR COÖRDINATES

1339. *General formula for rectification.* Referring to the formula (1338) for the length of the differential arc, and substituting polar coördinates, we have,

$$\rho = F(\omega),$$

$$dL = \sqrt{(d\rho)^2 + (\rho\, d\omega)^2} = d\omega\sqrt{\left(\frac{d\rho}{d\omega}\right)^2 + \rho^2},$$

wherein ρ and ω are the coördinates of the point, and L the length of the arc.

$$L = \int d\omega\sqrt{\left(\frac{d\rho}{d\omega}\right)^2 + \rho^2} + C, \qquad (A)$$

or

$$L = \int d\rho\sqrt{1 + \rho^2\left(\frac{d\omega}{d\rho}\right)^2} + C. \qquad (B)$$

EXAMPLE 1. *Rectification of the logarithmic spiral.*

We have,

$$\left.\begin{aligned}\log\rho &= A\omega\\ \rho &= b^{A\omega}\end{aligned}\right\}(1) \qquad \begin{aligned}&\text{For } \omega = 0 \quad \text{we have } \rho = 1,\\ &\text{For } \omega = -\infty \text{ we have } \rho = 0,\end{aligned}$$

and

$$\frac{d\rho}{d\omega} = Ab^{A\omega}\frac{\log b}{\log e} \quad \text{and} \quad \left(\frac{d\rho}{d\omega}\right)^2 = A^2b^{2A\omega}\left(\frac{\log b}{\log e}\right)^2.$$

These values of ρ^2 and $\left(\frac{d\rho}{d\omega}\right)^2$ substituted in the formula for rectification,

$$L = \int d\omega\sqrt{\left(\frac{d\rho}{d\omega}\right)^2 + \rho^2}, \qquad (1)$$

give $$L = \int d\omega \sqrt{A^2 b^{2A\omega}\left(\frac{\log b}{\log e}\right)^2 + b^{2A\omega}},$$

or $$L = \int d\omega b^{A\omega} \sqrt{\left(\frac{\log b}{\log e}\right)^2 A^2 + 1},$$

$$L = \int \frac{d\omega b^{A\omega}}{\log e} \sqrt{(\log b)^2 A^2 + (\log e)^2}. \tag{2}$$

In order to integrate, put

$$A\omega = x.$$

Differentiating, $$A\,d\omega = dx,$$

$$d\omega = \frac{dx}{A},$$

$$\int d\omega\, b^{A\omega} = \int \frac{dx}{A} b^x = \frac{\log e}{A \log b} b^x = \frac{\log e}{A \log b} b^{A\omega};$$

therefore, relation (2) becomes

$$L = \frac{b^{A\omega}}{A \log b} \sqrt{(\log b)^2 A^2 + (\log e)^2} + C,$$

or, in putting $$H = \frac{\sqrt{(\log b)^2 A^2 + (\log e)^2}}{A \log b},$$

we have for the length of the logarithmic spiral,

$$L = Hb^{A\omega} + C. \tag{3}$$

To determine the constant C, note that for $\omega = 0$, equation (1) gives $\rho = 1$, and $L = 0$; which corresponds to the origin of the spiral situated upon the polar axis. Therefore, C is obtained by substituting $\omega = 0$ and $L = 0$ in formula (3), which gives

$$0 = Hb^0 + C,$$

$$C = -H;$$

therefore relation (3) becomes

$$L = Hb^{A\omega} - H = H(b^{A\omega} - 1).$$

From equation (1), $$\rho = b^{A\omega};$$

therefore L in terms of the radius vector is

$$L = H(\rho - 1). \tag{4}$$

For $\omega = -\infty$, we have $\rho = 0$ and $L = -H$. Therefore, starting from the polar axis which corresponds to $\omega = 0$, the spiral makes an infinite number of turns before arriving at the

pole. The length of this portion of the spiral included between the pole and the origin (for which $\rho = 1$) is negative and has the value $-H$.

REMARK. From relation (4),

$$L + H = H\rho,$$

that is, the length of the logarithmic spiral, measured from the pole to any point on the curve, is proportional to the radius vector which ends at that point. This property, which has long been known, may be used in graphically representing a system of logarithms.*

EXAMPLE 2. *The rectification of the spiral of Archimedes* (1230).

Taking the equation of the curve in the form

$$\rho = K\omega, \tag{1}$$

the formula for rectification is

$$L = \int d\omega \sqrt{\left(\frac{d\rho}{d\omega}\right)^2 + \rho^2}. \tag{2}$$

From (1), $\qquad \dfrac{d\rho}{d\omega} = K,$

therefore relation (2) may be written,

$$L = \int d\omega\ \sqrt{K^2 + K^2\omega^2} = \int d\omega\, K\sqrt{\omega^2 + 1}. \tag{3}$$

* From (5) $\qquad \dfrac{L+H}{H} = \rho.$

Then $\qquad \log \rho = \log\left(\dfrac{L+H}{H}\right),$

or $\qquad \log \rho = A\omega,$

and we may write $\qquad \log \dfrac{S+H}{H} = A\omega.$

Letting H represent a number one (1), the quantity $\frac{S+H}{H}$ will represent a number, N, greater than one. The logarithm of this number is measured by $A\omega$. If we put $\frac{S+H}{H} = 10\,H = 10$ units, and if the base of the logarithmic system is 10 and the angular measure of the logarithm of this base is 2π, we have,

$$1 = \log 10 = A\,2\pi,$$

and $\qquad A = \dfrac{1}{2\pi}.$

Substituting in equation (2),

$$\log \rho = \frac{1}{2\pi}\,\omega,$$

which gives

for	$\omega = 0,$	$\rho = 1,$	$L + H = H,$
	$\omega = 2\pi,$	$\rho = 10,$	$L + H = 10\,H.$

The spiral will have 1 at the origin and 10 at the end, and the points 2, 3 . . . 9, 10, will divide it into equal arcs.

Putting $$\sqrt{\omega^2+1} = z - \omega, \tag{4}$$
we have $$\omega^2 + 1 = z^2 - 2z\omega + \omega^2,$$
$$\omega = \frac{z^2-1}{2z}, \tag{5}$$
$$d\omega = \frac{2z\,2z\,dz - (z^2-1)\,2\,dz}{4z^2},$$
or $$d\omega = \frac{dz}{2} + \frac{dz}{2z^2}. \tag{6}$$
Relation (4) gives
$$\sqrt{\omega^2+1} = z - \omega = z - \frac{z^2-1}{2z} = \frac{z^2+1}{2z}.$$

Substituting these values of $d\omega$ and $\sqrt{\omega^2+1}$ in (3),
$$L = \int K\,d\omega\,(z-\omega) = \int K\,d\omega z - \int K\omega\,d\omega.$$

Now $$\int K d\omega z = \int Kz\left(\frac{dz}{2z^2} + \frac{dz}{2z^2}\right) = K\frac{z^2}{4} + \frac{K\log z}{2\log e},$$
and $$\int K\omega\,d\omega = \frac{K\omega^2}{2}.$$

From (4) we have
$$z = \sqrt{\omega^2+1} + \omega.$$

Now substituting the value of z in the expression for L,
$$L = \frac{K}{4}(\omega^2+1+2\omega\sqrt{\omega^2+1}+\omega^2) + \frac{K}{2\log e}\log(\sqrt{\omega^2+1}+\omega) - \frac{K\omega^2}{2},$$
or
$$L = \frac{K}{4}(2\omega^2+1+2\omega\sqrt{\omega^2+1}) - \frac{K\omega^2}{2} + \frac{K}{2\log e}\log(\sqrt{\omega^2+1}+\omega) + C.$$
For $\omega = 0$, $L = 0$, and
$$0 = \frac{K}{4} + C \quad \text{or} \quad C = -\frac{K}{4}.$$

This value substituted in the above gives the length of the spiral
$$L = \frac{K}{4}\left[(2\omega\sqrt{\omega^2+1}) + \frac{K}{2\log e}\log(\sqrt{\omega^2+1}+\omega)\right]$$

If the equation of the spiral is given in the ordinary form
$$\rho = \frac{a}{2\pi}\,\omega,$$

a being the radius vector corresponding $\omega = 2\pi$, K is replaced by $\frac{a}{2\pi}$ in the above formula; thus,

$$L = \frac{a}{2\pi}\left[\frac{\omega}{2}\sqrt{1+\omega^2} + \frac{1}{2\log e}\log\left(\omega + \sqrt{1+\omega^2}\right)\right].$$

This formula gives the rectification of the spiral of Archimedes taken from the pole.

AREA OF SURFACES OF REVOLUTION

1340. *General formula for the area of surfaces of revolution, and examples.* AB being the meridian of a surface of revolution whose axis is Ox (Fig. 395), an infinitesimal element $MM' = dL$ of this curve coincides with its subtended chord and describes the lateral surface dS of the frustum of a cone; such that designating the coördinates of the point M by y and x, we have (912)

$$dS = 2\pi\left(y + \frac{dy}{2}\right) dL.$$

Neglecting $\frac{dy}{2}$ in comparison with y, and substituting the general expression for dL (1338),

$$dS = 2\pi y\sqrt{(dx)^2 + (dy)^2}.$$

Therefore, the area S generated by the revolution of the curve AB is expressed by the general formula

$$S = 2\pi\int y\sqrt{(dx)^2 + (dy)^2}. \qquad (1)$$

Example 1. *The area of a sphere.*

The origin of the meridian being at the center of the sphere, its equation is (1123)

$$y^2 + x^2 = r^2,$$

and $$2\,y\,dy = -\,2\,x\,dx,\ dy = -\frac{x\,dx}{y},\quad (dy)^2 = \frac{x^2\,(dx)^2}{y^2}.$$

Substituting this value of $(dy)^2$ in the preceding integral (1),

$$S = 2\pi\int y\sqrt{(dx)^2 + \frac{x^2(dx)^2}{y^2}} = 2\pi\int y\sqrt{\frac{y^2+x^2}{y^2}(dx)^2} = 2\pi\int dx\sqrt{y^2+x^2},$$

$$S = 2\pi\int dx\,r = 2\pi r x + C.$$

Taking this integral between the limits $x = 0$ and $x = r$, we obtain the surface of a hemisphere. Since for $x = 0$, $S = 0$, we have $C = 0$, and

$$S = 2\pi \int_0^r dx\, r = 2\pi r x + 0 = 2\pi r^2.$$

Therefore, the total surface of the sphere is equal to $4\pi r^2$ (**917**).

EXAMPLE 2. *The area of a paraboloid of revolution.*

Let
$$y^2 = 2px$$
be the equation of the meridian curve (1197), then

$$\frac{dy}{dx} = \frac{p}{y}, \quad \left(\frac{dy}{dx}\right)^2 = \frac{p^2}{y^2} = \frac{p^2}{2px} = \frac{p}{2x}.$$

Substituting for $\left(\frac{dy}{dx}\right)^2$ in the indefinite integral (1), we obtain

$$S = 2\pi \int y \sqrt{(dx^2) + (dy)^2} = 2\pi \int y\, dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$
$$= 2\pi \int y\, dx \sqrt{1 + \frac{p^2}{y^2}}$$

$$S = 2\pi \int dx \sqrt{y^2 + p^2} = 2\pi \int dx \sqrt{2px + p^2} = 2\pi \sqrt{p} \int dx \sqrt{2x + p}.$$

Putting
$$2x + p = z, \qquad dx = \frac{dz}{2},$$

and
$$S = \pi\sqrt{p} \int z^{\frac{1}{2}} dz = \pi\sqrt{p}\, \frac{z^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}\pi \sqrt{p\,(2x + p)^3} + C. \quad (a)$$

Since for $x = 0$, $S = 0$,

$$0 = \frac{2}{3}\pi p^2 + C \text{ and } C = -\frac{2}{3}\pi p^2.$$

To obtain the surface of a paraboloid included between the vertex and a section whose abscissa is X (Fig. 389), take the preceding integral between the limits $x = 0$ and $x = X$, which is done simply by replacing x by X and C by its value, in expression (a); thus,

$$S = 2\pi\sqrt{p} \int_0^X dx\, \sqrt{2x + p} = \frac{2}{3}\pi \sqrt{p\,(2X + p)^3} - \frac{2}{3}\pi p^2$$
$$= \frac{2}{3}\pi \left[\sqrt{p\,(2X + p)^3} - p^2\right].$$

CUBATURE OF SOLIDS OF REVOLUTION

1341. *General formula for the volume of a solid of revolution.*

Let
$$y = f(x)$$
be the equation of a meridian curve (Fig. 395) of a solid of revolution about the axis Ox. Consider this solid V as being made up of infinitely thin slices included between planes perpendicular to the axis Ox. Since any one of these slices, that generated by $MPP'M'$ for example, at the limit may be considered as the frustum of a cone, the radii of whose bases are $MP = y$ and $M'P' = y + dy$, and whose altitude is $PP' = dx$, the volume dV of this slice is (913),

$$dV = \frac{1}{3}\pi\left[y^2 + (y + dy)^2 + y(y + dy)\right]dx;$$

or, neglecting dy in comparison with y,

$$dV = \frac{1}{3}\pi(y^2 + y^2 + y^2)\,dx = \pi y^2\,dx.$$

Therefore, the volume V corresponding to the meridian AB is expressed by the indefinite integral

$$V = \pi\int y^2dx. \tag{1}$$

1342. Example 1. *The volume of a cone,* generated by a right triangle OBP turning about the axis Ox which coincides with the side OP. The equation of the meridian being (1117)

$$y = ax,$$

substituting this value of y in the general equation (1) of the preceding article this equation becomes,

$$V = \pi\int a^2x^2dx = \frac{\pi a^2x^3}{3} + C = \frac{1}{3}\pi a^2x^2x + C = \frac{1}{3}\pi y^2x + C.$$

Since for $x = 0$, we have $V = 0$,

$$0 = 0 + C \quad \text{and} \quad C = 0.$$

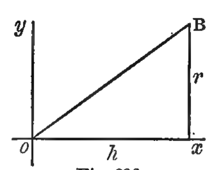

Fig. 396

Taking the integral between the limits $x=0$, which corresponds to $y = 0$, and $x = h$, which corresponds to $y = r$, and since $C = 0$, the required volume is

$$V = \pi\int_0^h a^2x^2\,dx = \frac{1}{3}\pi r^2h. \tag{909}$$

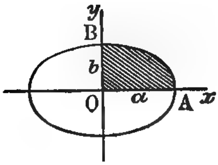

Fig. 397

EXAMPLE 2. *The volume of an ellipsoid of revolution.* The equation of the meridian is (1131)

$$a^2y^2 + b^2x^2 = a^2b^2$$

and
$$y^2 = \frac{b^2}{a^2}(a^2 - x^2).$$

Substituting this value of y^2 in equation (1) of the preceding article,

$$V = \pi\int\frac{b^2}{a^2}(a^2 - x^2)\,dx = \pi\int\frac{b^2a^2}{a^2}dx - \pi\int\frac{b^2}{a^2}x^2dx$$
$$= \pi b^2x - \pi\frac{b^2}{a^2}\frac{x^3}{3} + C.$$

Since for $x = 0$, $V = 0$, and substituting these values in the above integral $C = 0$, taking the integral between the limits $x = 0$ and $x = a$, we obtain for half the volume of the ellipsoid,

$$V = \pi b^2a - \pi\frac{b^2}{a^2}\frac{a^3}{3} = \frac{2}{3}\pi b^2a,$$

and for the whole volume,

$$V = \frac{4}{3}\pi b^2a. \tag{a}$$

If the generating ellipse turned about its minor axis, we would have,

$$V = \frac{4}{3}\pi a^2b, \tag{1166}$$

which result is obtained by substituting b for a and a for b in formula (a), or by taking from the equation of the ellipse

$$a^2x^2 + b^2y^2 = a^2b^2$$

the following value of y^2,

$$y^2 = \frac{a^2}{b^2}(b^2 - x^2),$$

and substituting in the general formula.

CENTER OF GRAVITY

1343. *The moment and center of gravity of a figure.* In order to calculate the center of gravity of a body from its geometrical form, we must assume that the body is composed of strictly homogeneous material.

A figure (line, surface or volume) may be considered as being composed of infinitesimal elements.

The product of one of these elements and its distance from a plane is called *the moment of this element with respect to this plane.* The moments of two elements on opposite sides of the plane have opposite signs. The *moment of a figure or a system of elements* is the algebraic sum of the moments of the different elements which compose the figure or system.

The center of gravity of a system of elements (lines, surfaces, or volumes) is a point, such that, if all the elements were concentrated in it, the product of the sum of all the elements and the distance of the point from a certain plane, would be equal to the algebraic sum of the moments of the different elements with respect to the same plane.

1344. *The center of gravity of a straight line.* First, the center of gravity is on the line, because, if we suppose it to be outside the line and pass a plane through it leaving the line entirely on one side of the plane, the product of the sum of all the elements and the distance of the center of gravity from the plane will be zero, while the moment of the line with respect to the same plane will evidently not be zero.

The center of gravity is at the middle of the line, because, with respect to any plane passing through the middle, the product of the sum of all the elements and the distance from the point to the plane will be zero, and since the middle point divides the line into two symmetrical parts opposite in sign, the moment of the total line is also zero.

REMARK. By an analogous course of reasoning, we have in general:

1st. That all systems of geometrical lines, surfaces or volumes possessing a geometrical center have their center of gravity at the geometrical center.

2d. That any system composed of elements symmetrical in pairs with respect to a line or a plane (836, 839) has its center of gravity on this line or plane.

1345. *Center of gravity of any plane curve AB.* Drawing the coördinate axes Ox and Oy in the plane of the curve, the

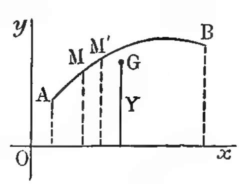

Fig. 398

required center of gravity G will be determined when its coordinates X and Y are known. y being the ordinate of a point M, the moment of the element $MM' = dL$ with respect to Ox is

$$dL\left(y + \frac{dy}{2}\right),$$

or, since $\frac{dy}{2}$ may be neglected in comparison with y, we have

$$y\,dL.$$

The algebraic sum of all the elementary moments, that is, the moment of the curve, is therefore,

$$\sum y\,dL = \int y\,dL,$$

and since this moment is equal to LY, L being the length of the curve, we have,

$$LY = \int y\,dL \text{ and } Y = \frac{\int y\,dL}{L}. \tag{1}$$

With respect to Oy, we have,

$$LX = \int X\,dL \text{ and } X = \frac{\int x\,dL}{L}. \tag{2}$$

REMARK. When the curve is given by its equation,

$$y = f(x).$$

From (1338) we have

$$dL = \sqrt{(dy)^2 + (dx)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

and

$$L = \int dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

and these values are substituted in equations (1) and (2).

When the integrals resulting from these substitutions are too complicated, or the functions (1) and (2) are unknown, an approximate result may be obtained by using Thomas Simpson's formula (1333) for the calculation of the integrals

$$\int y\,dL \quad \text{and} \quad \int x\,dL.$$

To do this, divide the curve into an even number n of equal parts; from the points of division drop perpendiculars upon Ox;

measure these perpendiculars $y_0, y_1, y_2, \ldots y_n$, and making $\frac{L}{n} = \delta$, we have

$$\int y\,dL = \frac{\delta}{3}[y_0 + y_n + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})].$$

1346. *Center of gravity of an arc of a circle.* The moment of the element MM' with respect to the axis OX (Fig. 399), which in this case is taken as the y-axis, is

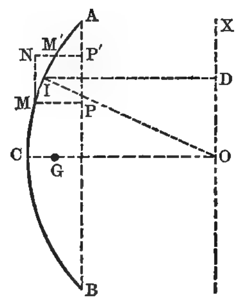

Fig. 399

$$MM' \times ID \text{ or } x\,dL,$$

and the moment of the arc is

$$\Sigma x\,dL = \int x\,dL;$$

but since $$MM' \times ID = PP' \times r,$$

or $$x\,dL = r\,dy,$$

the moment of the arc is also,

$$\Sigma r\,dy = r\Sigma\,dy = rc,$$

wherein c is the chord AB which is equal to $\Sigma\,dy$.

The distance X from the center of gravity G to the center O, designating the length of the arc L by a, is

$$X = \frac{\int x\,dL}{L} = \frac{rc}{a}. \tag{1}$$

The arc being of n degrees, we have (**758**),

$$a = \frac{2\,\pi r n}{360},$$

and $$\sin\frac{n}{2} = \frac{c}{2\,r} \text{ or } c = 2\,r\sin\frac{n}{2}.$$

These values of a and c substituted in relation (1) give

$$X = \frac{360\,r\sin\frac{n}{2}}{\pi n}.$$

For $n = 180°$, for example, we have $\sin\frac{n}{2} = \sin 90° = 1$, and therefore,

$$X = \frac{360\,r}{180\,\pi} = \frac{2\,r}{\pi} = \frac{2\,r}{\frac{22}{7}} = \frac{7}{11}r.$$

Thus the center of gravity of a semicircle is very approximately $\frac{7}{11}$ of a radius from the center.

1347. *Center of gravity of plane surfaces, and in general of any surfaces or solids. General solution.*

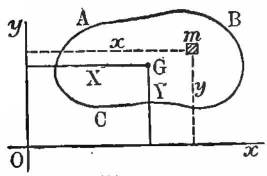

Fig. 400

Let m be an element dS of the surface bounded by any plane curve ABC, and y the distance of this element from the axis Ox, drawn in the plane of this surface. The product $y\,dS$ is the moment of this element m, and the moment of the entire surface is (1343)

$$SY = \Sigma y\,dS = \int y\,dS \quad \text{and} \quad Y = \frac{\int y\,dS}{S}, \tag{1}$$

and with respect to the axis Oy we have,

$$SX = \int x\,dS \quad \text{and} \quad X = \frac{\int x\,dS}{S}. \tag{2}$$

If the surface was not plane, instead of using two axes Ox and Oy in one plane, we would use three planes perpendicular to each other, and for each of these planes we would have a formula analogous to formula (1); which would make it possible to determine the coördinate X, Y, and Z of the center of gravity with respect to the three planes.

For solids we operate in the same manner, using the same formula (1), replacing the elements of surface dS by elements of volume dV.

Whenever integrals (1) and (2) are obtained which are too complicated, the formula of Thomas Simpson may be used (1333). Thus, choosing the axes Ox and Oy tangent to the surface, divide the projection l of the surface on the axis Ox into an even number n of equal parts $\frac{l}{n} = \delta$; through these points of division draw perpendiculars to Ox; measure the portions $y_0, y_1, y_2,$ $\ldots y_n,$ of these perpendiculars intercepted by the curve, and then from (1333),

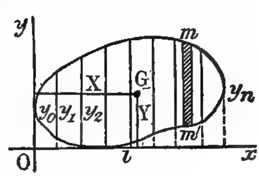

Fig. 401

$$S = \frac{\delta}{3}[y_0 + y_n + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})].$$

Considering an infinitesimal element mm', limited by two parallels to Oy, the surface of this element is

$$dS = y\,dx,$$

taking y as the length mm' intercepted by the curve. Therefore, the moment of this element with respect to Oy is

$$x\,dS = xy\,dx,$$

and that of the total surface

$$SX = \int xy\,dx, \text{ from which } X = \frac{\int xy\,dx}{S}.$$

To calculate $\int xy\,dx$, put

$$xy = y'$$

and then we have approximately,

$$\int y'\,dx = \frac{\delta}{3}[y'_0 + y'_n + 4(y'_1 + y'_3 + \cdots + y'_{n-1}) + 2(y'_2 + y'_4 + \cdots + y'_{n-2})],$$

in which,

$$\begin{aligned} y_0' &= y_0x_0 = y_0 \times 0 = 0, \\ y_1' &= y_1x_1 = y_1\delta, \\ y_2' &= y_2x_2 = 2\,y_2\delta, \\ y_3' &= y_3x_3 = 3\,y_3\delta, \\ &\cdots\cdots \\ y_n' &= y_nx_n = ny_n\delta. \end{aligned}$$

Substituting these values, we have,

$$\int xy\,dx = \frac{\delta^2}{3}\Big\{ny_n + 4[y_1 + 3y_3 + \cdots + (n-1)y_{n-1}] + 2[2y_2 + 4y_4 + \cdots + (n-2)y_{n-2}]\Big\},$$

and

$$X = \frac{\delta\{ny_n + 4[y_1 + 3y_3 + \cdots + (n-1)y_{n-1}] + 2[2y_2 + 4y_4 + \cdots + (n-2)y_{n-2}]\}}{y_0 + y_n + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})}.$$

Operating in the same manner for the axis Oy, the distance Y of the center of gravity from the axis Ox is obtained; but when the elements have been determined as in the above operation, it is simpler to operate as follows. z being the distance from the middle, that is, the center of gravity of the element $mm' = dS$

$= y\,dx$, to the axis Ox, the moment of this element with respect to the axis Ox is

$$z\,dS = zy\,dx,$$

and the moment of the total surface with respect to the same axis is

$$SY = \int zy\,dx, \text{ from which } Y = \frac{\int zy\,dx}{S}.$$

Putting $$zy = y',$$
we have

$$SY = \int y'dx = \frac{\delta}{3}[y_0' + y_n' + 4(y_1' + y_3' + \cdots + y'_{n-1}) + 2(y'_2 + y'_4 + \cdots + y'_{n-2})];$$

in which
$$y_0' = y_0 z_0,$$
$$y_1' = y_1 z_1,$$
$$\cdots\cdots$$
$$y_n' = y_n z_n,$$

$z_0, z_1, z_2, \ldots z_n$ being the distances from the middle points of the heights $y_0, y_1, y_2, \ldots$ or y_n to the axis Ox.

Substituting these values, we have,

$$Y = \frac{y_0 z_0 + y_n z_n + 4(y_1 z_1 + y_3 z_3 + \cdots) + 2(y_2 z_2 + y_4 z_4 + \cdots)}{y_0 + y_n + 4(y_1 + y_3 + \cdots) + 2(y_2 + y_4 + \cdots)}.$$

1348. *Center of gravity of the surface of a triangle.* Through the vertex A draw an axis Ox parallel to the base BC. Then the surface of an infinitesimal element mm', parallel to the base, is

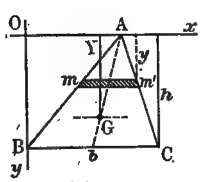

Fig. 402

$$dS = mm' \times dy,$$

and its moment is

$$y\,ds = y \times mm' \times dy.$$

The two similar triangles Amm' and ABC give

$$\frac{mm'}{b} = \frac{y}{h} \text{ and } mm' = \frac{by}{h}.$$

The elementary moment is

$$y\,dS = \frac{by^2}{h}dy,$$

and the total moment

$$SY = \int \frac{b}{h} y^2\,dy = \frac{by^3}{3h} + C.$$

Taking this integral between the limits $y = 0$ and $y = h$, and making

$$\frac{bh}{2} = S,$$

we have for the total moment of the triangle ABC,

$$\frac{bh}{2} Y = \frac{bh^3}{3h}, \text{ and } Y = \frac{2}{3} h.$$

Thus the center of gravity G lies on a line parallel to the base BC at a distance equal to one-third the altitude from it. In the same manner all three sides can be taken as bases, and the three parallels to the three sides intersect in a point G which is meeting-point of the three medians and the center of gravity.

1349. *Center of gravity of a segment of a parabola,* limited by a straight line AB perpendicular to the principal axis Ox, the equation of the parabola being (1197)

$$y^2 = 2\,px.$$

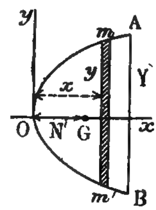

Fig. 403

The center of gravity being on the axis Ox, it is only necessary to determine its abscissa $OG = X'$.

The surface of an element mm' included between two parallels infinitely near each other and parallel to the axis Oy, is

$$dS = mm'\,dx = 2\,y\,dx,$$

and its moment is $$x\,dS = 2\,xy\,dx,$$

and therefore the moment of a parabolic segment is

$$SX' = \int 2\,xy\,dx = \int 2\,x\sqrt{2\,px}\,dx = \int 2\sqrt{2\,p}x^{\frac{3}{2}}\,dx = \frac{4}{5}\sqrt{2\,p}x^{\frac{5}{2}} + C.$$

Designating the coördinates of a point A by X and Y, and taking this integral between the limits $x = 0$ and $x = X$, the constant $C = 0$, and we have,

$$SX' = \frac{4}{5}\sqrt{2\,p}X^{\frac{5}{2}} = \frac{4}{5}\sqrt{2\,pX}X^2 = \frac{4}{5}YX^2.$$

Since in (1329) $$S = \frac{4}{3}YX,$$

we have $$X' = \frac{\frac{4}{5}YX^2}{\frac{4}{3}YX} = \frac{3}{5}X.$$

1350. *Center of gravity of a zone $AA'B'B$.* Since the figure is symmetrical, the center of gravity G lies upon the radius OC perpendicular to the planes AA' and BB' of the bases; and its distance $OG = X$ from the center is all that remains to be determined. Take OC as the x-axis, and let Oy be the trace of a plane perpendicular to Ox.

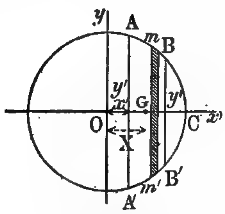

Fig. 404

Reasoning as in the two preceding articles, the surface of an element mm' of a zone included between two planes infinitely near each other and parallel to the plane Oy, is (915)

$$dS = 2\,\pi R\,dx,$$

and its moment with respect to Oy is

$$x\,dS = 2\,\pi Rx\,dx,$$

therefore the moment of the zone is

$$SX = \int 2\,\pi Rx\,dx = \pi Rx^2 + C.$$

Taking this integral between the limits $x = x'$ and $x = x''$, we have

$$SX = \pi R\,(x''^2 - x'^2);$$

and since $$S = 2\,\pi RH = 2\,\pi R\,(x'' - x'),$$

we have $$X = \frac{\pi R\,(x''^2 - x'^2)}{2\,\pi R\,(x'' - x')} = \frac{1}{2}(x'' + x'),$$

which shows that the center of gravity G is at the middle of the height H of the zone.

1351. *The center of gravity of the lateral surface of right cone.* This center of gravity is situated upon the axis OP of the cone, and we have only to determine the value of $OG = X'$ Taking OP as the axis of x, and the moments with respect to a plane Oy passing through the vertex perpendicular to Ox, designating the slant height OA of the cone by l, for the expression of the surface of the element mm' included between two parallel planes perpendicular to the axis Ox, we have (912)

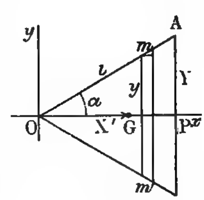

Fig. 405

$$dS = 2\,\pi\left(y + \frac{dy}{2}\right)dl.$$

Neglecting $\frac{dy}{2}$, $$dS = 2\,\pi y\,dl;$$

then the moment of this element with respect to the plane Oy is

$$x\,dS = 2\,\pi y x\,dl.$$

Since we have $\frac{dx}{dl} = \cos\alpha,\quad dl = \frac{dx}{\cos\alpha},$

and since $\frac{y}{x} = \tan\alpha,\quad y = x\tan\alpha.$

Substituting these values, we have,

$$x\,dS = \frac{2\,\pi\tan\alpha}{\cos\alpha}x^2\,dx.$$

The moment of the lateral surface of the cone is, therefore,

$$X'S = \frac{2\,\pi\tan\alpha}{\cos\alpha}\int x^2\,dx = \frac{2\,\pi\tan\alpha}{\cos\alpha}\cdot\frac{x^3}{3} + C.$$

Designating the coördinates of the point A by X and Y, and taking the preceding integral between the limits $x = 0$ and $x = X$; since the constant $C = 0$, we have for the moment of the lateral surface of the given cone,

$$X'S = \frac{2\,\pi\tan\alpha}{\cos\alpha}\frac{X^3}{3}.$$

From (908), $$S = \pi Y l,$$

or, since $Y = X\tan\alpha$ and $l = \frac{X}{\cos\alpha}$,

$$S = \frac{\pi\tan\alpha}{\cos\alpha}X^2,$$

and we have,

$$X' = \frac{\frac{2\,\pi\tan\alpha}{\cos\alpha}\times\frac{X^3}{3}}{\frac{\pi\tan\alpha}{\cos\alpha}X^2} = \frac{2}{3}X.$$

Thus the center of gravity of the lateral surface of a cone is at $\frac{2}{3}$ the altitude as measured from the vertex. This is analogous to the position of the center of gravity of the surface of a triangle (1348).

1352. *The center of gravity of any solid.* Using three refer-

ence planes perpendicular to each other, and operating with each as indicated in (1347), we obtain the three equations,

$$VX = \int x\,dV, \quad \text{and} \quad X = \frac{\int x\,dV}{V},$$

$$VY = \int y\,dV, \quad \text{and} \quad Y = \frac{\int y\,dV}{V},$$

$$VZ = \int z\,dV, \quad \text{and} \quad Z = \frac{\int z\,dV}{V}.$$

In practice, when the integrals cannot be solved or are very complicated, the formula of Thomas Simpson is used (1333).

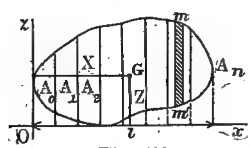

Fig. 406

Thus, three planes perpendicular to each other and tangent to the solid are chosen. Let Ox and Oz be the intersections of two of these planes with that of the paper to determine the distance X of the center of gravity of the solid from the plane Oz.

Draw a plane A_n tangent to the solid and parallel to the plane Oz; divide the portion l intercepted on Ox by the two planes Oz and A_n into an even number n of equal parts $\frac{l}{n} = \delta$; through these points of division draw planes perpendicular to Ox; measure the areas A_0, A_1, A_2, . . . A_n of the sections determined by these planes and by O_2 and A_n; the areas A_0 and A_n may be zero. Then from (1337) we have,

$$V = \frac{\delta}{3}[A_0 + A_n + 4(A_1 + A_3 + \cdots) + 2(A_2 + A_4 + \cdots)].$$

The volume of an element mm' of the solid, determined by two planes infinitely near each other and parallel to the axis Oz, is

$$dV = A\,dx,$$

A being the area of the section mm', and dx the thickness.

The moment of the element mm' with respect to Oz is therefore,

$$x\,dV = Ax\,dx,$$

and that of the total volume,

$$VX = \int Ax\,dx \text{ and } X = \frac{\int Ax\,dx}{V}. \tag{1}$$

To calculate $\int Ax\,dx$, put

$$Ax = y,$$

and we have approximately,

$$\int Axdx = \frac{\delta}{3}[y_0 + y_n + 4(y_1 + y_3 + \cdots) + 2(y_2 + y_4 + \cdots)],$$

in which formula

$$\begin{aligned} y_0 &= A_0x_0 = A_0 \times 0 = 0, \\ y_1 &= A_1x_1 = A_1\delta, \\ y_2 &= A_2x_2 = A_2 2\,\delta, \\ &\ldots\ldots\ldots\ldots \\ y_n &= A_n y_n = A_n n\delta. \end{aligned}$$

Substituting these values, we obtain,

$$\int Axdx = \frac{\delta^2}{3}[nA_n + 4(A_1 + 3\,A_3 + \cdots) + 2(2A_2 + 4\,A_4 + \cdots)].$$

then substituting the value of V in (1), we have

$$X = \frac{\delta[nA_n + 4(A_1 + 3\,A_3 + \cdots) + 2(2\,A_2 + 4\,A_4 + \cdots)]}{A_0 + A_n + 4(A_1 + A_3 + \cdots) + 2(A_2 + A_4 + \cdots)}.$$

In the same way we can find Z and Y, but if the centers of gravity of the sections A_0, A_1, A_2 . . . are easily determined it is convenient to have recourse to the method in (**1347**) for obtaining Y.

1353. *Center of gravity of any pyramid SABC.* Any section of the pyramid made by a plane parallel to the base, has its center of gravity on a straight line Sg which joins the vertex and the center of gravity of the base. From this it follows that any element mm' included between two planes infinitely near each other and parallel to the base has its center of gravity on the line Sb and therefore the center of gravity of the pyramid is also on this line. This established, it remains to find the distance SG.

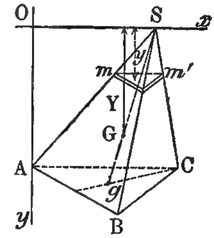

Fig. 407

Through the vertex S draw a plane parallel to the base ABC. Let Ox be the intersection of this plane with that of the paper.

b being the base of the element mm', which at the limit may be supposed to be a prism, its volume is

$$dV = b\,dy,$$

and its moment with respect to the plane Ox is

$$y\,dV = yb\,dy,$$

and therefore the moment of the pyramid is

$$YV = \int yb\,dy. \tag{1}$$

B and H being the base and the altitude of the pyramid, we have (891)

$$V = \frac{1}{3}BH.$$

Furthermore, since

$$\frac{b}{B} = \frac{y^2}{H^2}, \quad b = \frac{B}{H^2}y^2.$$

Substituting these values of V and b in (1), we obtain,

$$\frac{1}{3}BHY = \frac{B}{H^2}\int y^3 dy = \frac{B}{H^2}\frac{y^4}{4} + C.$$

Taking this integral between the limits $y = 0$ and $y = H$, we obtain the moment of the given pyramid,

$$\frac{1}{3}BHY = \frac{B}{H^2}\frac{H^4}{4} = \frac{BH^2}{4},$$

and
$$Y = \frac{3\,BH^2}{4\,BH} = \frac{3}{4}H.$$

Therefore, the center of gravity lies upon the line Sg at a distance $Y = \frac{3}{4}H$ from the plane Ox, and we have

$$SG = \frac{3}{4}Sg.$$

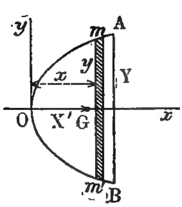

Fig. 408

1354. *Center of gravity of solids of revolution.* The general formulas of (1352) apply also to solids of revolution. But since solids of revolution are symmetrical with respect to the axis of revolution Ox, the center of gravity always lies upon this axis, and we have simply to determine its distance $OG = X'$ from a certain plane perpendicular to Ox, which is expressed by a single equation.

Thus,

$$VX' = \int x\,dV \quad \text{and} \quad X' = \frac{\int x\,dV}{V}.$$

The volume of an element mm' included between two planes infinitely near each other and perpendicular to the plane Ox, being (1341)

$$dV = \pi y^2\,dx,$$

the volume

$$V = \pi \int y^2\,dx.$$

Furthermore, the moment of the element dV being

$$x\,dV = \pi y^2 x\,dx,$$

the total moment of the solid is

$$VX' = \pi \int y^2 x\,dx \quad \text{and} \quad X' = \frac{\pi \int y^2 x\,dx}{\pi \int y^2 dx}. \tag{1}$$

When the value of V is known, it may be substituted in the denominator of (1), leaving the integral in the numerator to be calculated. However, the two integrals are so analogous that the value of one is easily deduced from the value of the other, and it is scarcely worth while to substitute the value V in the denominator.

EXAMPLE 1. *Center of gravity of a paraboloid of revolution.*

The equation of the meridian curve or generatrix OA being (1197)

$$y^2 = 2\,px,$$

substituting this value of y^2 in equation (1), and taking the integrals between the limits $x = 0$ and $x = X$, we have,

$$X' = \frac{2p\int_0^X x^2\,dx}{2p\int_0^X x\,dx} = \frac{\frac{1}{3}X^3}{\frac{1}{2}X^2} = \frac{2}{3}X.$$

EXAMPLE 2. *Center of gravity of a right cone.*

The equation of the generatrix OA being that of a straight line (1117)

$$y = ax,$$

substituting this value of y in equation (1), and taking the integrals between the limits $x = 0$, and $x = X$,

$$X' = \frac{a^2 \int_0^X x^3\,dx}{a^2 \int_0^X x^2\,dx} = \frac{\frac{1}{4} X^4}{\frac{1}{3} X^3} = \frac{3}{4} X,$$

which is the same as obtained in (1353) for the pyramid, and should be compared with that given for the lateral surface of the cone (1351).

EXAMPLE 3. *Center of gravity of a spherical segment $AA'BB'$* (Fig. 404).

The equation of the generatrix AB being (1123)

$$y^2 = r^2 - x^2,$$

substituting in the general equation (1) and taking the integrals between the limits $x = x'$ and $x = x''$,

$$X' = \frac{\int_{x'}^{x''} r^2 x\,dx - \int_{x'}^{x''} x^3\,dx}{\int_{x'}^{x''} r^2\,dx - \int_{x'}^{x''} x^2\,dx} = \frac{r^2\left(\frac{x''^2}{2} - \frac{x'^2}{2}\right) - \frac{x''^4}{4} + \frac{x'^4}{4}}{r^2(x'' - x') - \frac{x''^3}{3} + \frac{x'^3}{3}}$$

$$= \frac{\frac{r^2}{2}(x''^2 - x'^2) - \frac{1}{4}(x''^4 - x'^4)}{r^2(x'' - x') - \frac{1}{3}(x''^3 - x'^3)}.$$

For the hemi-sphere the limits are $x = 0$ and $x = r$, and we have

$$X' = \frac{\frac{1}{2}r^4 - \frac{1}{4}r^4}{r^3 - \frac{1}{3}r^3} = \frac{\frac{1}{4}r^4}{\frac{2}{3}r^3} = \frac{3}{8}r.$$

Thus the center of gravity of a hemi-sphere is at a distance from the center equal to $\frac{3}{8}$ of the radius.

RADIUS OF GYRATION AND MOMENT OF INERTIA.

1356. The product mr^2 of a material element and the square of its distance from the axis of rotation is called the *moment of inertia of the element with respect to that axis,* and the sum Σmr^2

of the moments of inertia of all the material elements of a body with respect to an axis is *the moment of inertia of the body with respect to that axis.*

The radius of gyration is a value R of r such that if the whole mass of the body was concentrated at that distance from the axis of rotation, the moment of inertia and consequently the kinetic energy of the body would remain unchanged for any given angular velocity. Since the bodies are supposed to be homogeneous, we may substitute the volume u of the elements for the mass m, and we have for the moment of inertia,

$$\Sigma ur^2 = R^2 \Sigma u = UR^2 \text{ and } R^2 = \frac{\Sigma ur^2}{U},$$

or

$$R^2 = \frac{\int ur^2}{U},$$

wherein u is the volume of an element, U the total volume of the body, r the distance of an element from the axis of revolution, and R the radius of gyration.

EXAMPLE 1. *Find the radius of gyration of a very small rod, which rotates about an axis Oy, one end of the rod being upon the axis.*

Let $AB = 1$ be the length of the rod, and s the area of its cross-section; then m being an element of the rod, whose length is dl, the volume of this element is

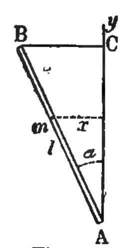

Fig. 409

$$u = s\,dl,$$

and its moment of inertia,

$$ux^2 = sx^2\,dl.$$

Since

$$dl = \frac{dx}{\sin a},$$

the moment of inertia of the element may be written

$$ux^2 = \frac{s}{\sin a}\, x^2 dx.$$

Therefore the general expression for the moment of inertia of the rod is

$$\Sigma ux^2 = UR^2 = \frac{s}{\sin a}\int x^2\,dx = \frac{s}{\sin a}\,\frac{x^3}{3} + C.$$

Taking this integral between the limits $x = 0$ and $x = BC$, the constant $C = 0$, and we obtain for the given rod AB,

$$UR^2 = \frac{s}{\sin \alpha} \frac{\overline{BC}^3}{3};$$

and noting that

$$U = ls = s\frac{BC}{\sin \alpha},$$

$$R^2 = \frac{\frac{s}{\sin \alpha} \frac{\overline{BC}^3}{3}}{\frac{s}{\sin \alpha} BC} = \frac{1}{3}\overline{BC}^2.$$

EXAMPLE 2. *Find the radius of gyration of right circular cylinder turning about its axis.*

Let ρ be the radius of the cylinder and l its length.

The volume of an element included between two cylindrical surfaces having the same axis as the cylinder is

$$u = [\pi (x + dx)^2 - \pi x^2]\, l,$$

wherein u is the volume, x the radius of the inner cylinder, and $x + dx$ that of the outer one.

Simplifying and neglecting the infinitesimal of the second order $\pi (dx)^2 l$, we have,

$$u = 2\,\pi l x\, dx.$$

The moment of inertia of this element is

$$ux^2 = 2\,\pi l x^3\, dx,$$

and therefore the moment of inertia of the cylinder is

$$UR^2 = 2\,\pi l \int x^3 dx = 2\,\pi l \frac{x^4}{4} + C. \qquad (1)$$

Taking this integral between the limits $x = 0$ and $x = \rho$, we have for the given cylinder,

$$UR^2 = \frac{1}{2}\pi l \rho^4.$$

Substituting $\pi\rho^2 l$ for U, we obtain,

$$R^2 = \frac{\pi l \rho^4}{2\,\pi\rho^2 l} = \frac{1}{2}\rho^2.$$

EXAMPLE 3. *Find the radius of gyration of a hollow cylinder, the exterior radius being ρ and the interior ρ'.*

Take the integral (1) of Example 2, between the limits $x = \rho'$ and $x = \rho$, which gives

$$UR^2 = \frac{\pi l}{2}(\rho^4 - \rho'^4),$$

from which

$$U = (\pi\rho^2 - \pi\rho'^2)\,l,$$

$$R^2 = \frac{\pi l\,(\rho^4 - \rho'^4)}{2\,\pi\,(\rho^2 - \rho'^2)\,l} = \frac{1}{2}(\rho^2 + \rho'^2).$$

Example 4. *Find the radius of gyration of right circular cone turning about its axis.*

Let h be the altitude of the cone, and ρ the radius of its base.

Taking the axis of the cone as the x-axis, the volume of an element included between two planes perpendicular to this axis is

$$u = \pi y^2 dx,$$

and its moment of inertia

$$\frac{1}{2}uy^2 = \frac{1}{2}\pi\, y^4 dx.$$

Since

$$\frac{dx}{dy} = \frac{h}{\rho},\quad dx = \frac{h}{\rho}\,dy,$$

and we may write,

$$\frac{1}{2}uy^2 = \frac{\pi h}{2\,\rho}\,y^4 dy.$$

Therefore the general expression for the moment of inertia of a right circular cone is

$$\Sigma\,\frac{1}{2}uy^2 = UR^2 = \frac{\pi h}{2\,\rho}\int y^4 dy = \frac{\pi h}{10\,\rho}\,y^5 + C.$$

Taking this integral between the limits $y = 0$ and $y = \rho$, we obtain for the cone in question,

$$UR^2 = \frac{\pi h}{10}\rho^4;$$

and since

$$U = \frac{1}{3}\pi\rho^2 h,$$

we have

$$R^2 = \frac{3\,\pi h\rho^4}{10\,\pi\rho^2 h} = \frac{3}{10}\rho^2.$$

1357. *Radius of gyration of any geometrical body.* Referring to a system of three coördinate axes; let one of the axes be the axis of rotation O, perpendicular to the plane of the paper; then u being the volume of an element situated at a distance

$$r = \sqrt{x^2 + y^2}$$

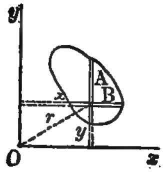

Fig. 410

from the axis, its moment of inertia is

$$ur^2 = ux^2 + uy^2,$$

and therefore the moment of inertia of the body is

$$\Sigma ur^2 = UR^2 = \Sigma ux^2 + \Sigma uy^2. \qquad (1)$$

Each of the two sums Σux^2 and Σuy^2 which make up the value of UR^2 are calculated separately. Considering an infinitely thin slice of the body included between two planes perpendicular to the x-axis, A being the area of the section, the volume of the slice is $A\,dx$, and since each element of the slice gives the same value for ux^2 we have for the whole slice $\Sigma ux^2 = Ax^2dx$, and consequently for the whole body

$$\Sigma ux^2 = \int Ax^2dx.$$

The degree of accuracy of this calculation depends evidently upon the section A, which may be constant or a variable following a certain law with respect to x, or vary in any manner.

Considering the body as composed of infinitely thin slices perpendicular to the y-axis, B being the area of the variable section, we have

$$\Sigma uy^2 = \int By^2dy.$$

Substituting these values in relation (1), we obtain

$$UR^2 = \int Ax^2dx + \int By^2dy, \text{ whence } R^2 = \frac{\int Ax^2dx + \int By^2dy}{U}.$$

Example 1. *Find the radius of gyration of a rectangular parallelopiped turning about one of its edges.*

Let the edge c be the axis of rotation, and a and b coincide with the axes x and y. First the sections A and B are constant, since

$$A = bc \text{ and } B = ac,$$

and we have,

$$UR^2 = bc\int_0^a x^2\,dx + ac\int_0^b y^2dy = bc\frac{a^3}{3} + ac\frac{b^3}{3}.$$

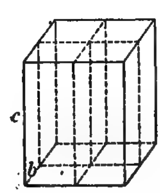

Fig. 411

Since $U = \Sigma u = abc$, we have

$$R^2 = \frac{\frac{1}{3}abc\,(a^2+b^2)}{abc} = \frac{1}{3}(a^2+b^2).$$

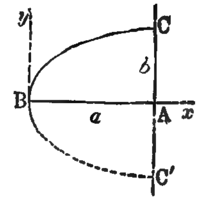

Fig. 412

EXAMPLE 2. *Find the radius of gyration of a right cylinder, whose base ABC is semi-parabolic, revolving about an axis A parallel to the axis of the cylinder.* Using the axes of the parabola Bx and By as coordinate axes, designating AB by a, AC by b, and the distance of an element from the axis of rotation by r, we have the relation

$$r^2 = (a-x)^2 + y^2.$$

From this it follows that

$$\Sigma ur^2 = \Sigma u\,(a-x)^2 + \Sigma uy^2,$$

or

$$UR^2 = \int_0^a A\,(a-x)^2dx + \int_0^b By^2dy.$$

The radius of gyration being independent of the length of the cylinder, we may assume the length to be 1. Therefore, for any section A or B, the equation of BC being $y^2 = 2\,px$, we have

$$A = y = \sqrt{2\,px} \quad\text{and}\quad B = a - x = a - \frac{y^2}{2\,p}.$$

Substituting these values in the above integrals,

$$\int_0^a A\,(a-x)^2dx = \sqrt{2\,p}\int_0^a x^{\frac{1}{2}}(a^2 - 2\,ax + x^2)\,dx$$

$$= \sqrt{2\,p}\left(\frac{2}{3}a^{\frac{7}{2}} - \frac{4}{5}a^{\frac{7}{2}} + \frac{2}{7}a^{\frac{7}{2}}\right) = \frac{16}{105}\sqrt{2\,pa}\,a^3 = \frac{16}{105}ba^3,$$

$$\int_0^b By^2dy = \int_0^b\left(ay^2 - \frac{y^4}{2\,p}\right)dy = \frac{1}{3}ab^3 - \frac{1}{5}\frac{b^5}{2\,p} = \frac{2}{15}ab^3;$$

therefore $$UR^2 = \frac{16}{105}ba^3 + \frac{2}{15}ab^3 = \frac{2}{15}ab\left(\frac{8}{7}a^2 + b^2\right).$$

Since, furthermore, we have

$$\Sigma u \text{ or } U = \int_0^a A\,dx = \sqrt{2\,p}\int_0^a x^{\frac{1}{2}}dx = \frac{2}{3}\sqrt{2\,pa}^{\frac{3}{2}} = \frac{2}{3}ab,$$

then $$R^2 = \frac{1}{5}\left(\frac{8}{7}a^2 + b^2\right).$$

REMARK. When the integrals $\int Ax^2dx$ and $\int By^2dy$ cannot be obtained algebraically, or when they are too complicated, the formula of Thomas Simpson may be used (1333).

Thus, to calculate approximately $\int Ax^2dx$, divide the maximum value of x into an even number n of equal parts $\delta = \frac{l}{n}$; through the points of division and at the extremities of l, draw planes perpendicular to the x-axis; determine the areas A_0, A_1, A_2, . . . A_n of the sections made by the planes, and putting

$$\begin{aligned}
y_0 &= A_0x_0^2 = A_0 \times 0 = 0, \\
y_1 &= A_1x_1^2 = A_1\delta^2, \\
y_2 &= A_2x_2^2 = A_2\, 4\,\delta^2, \\
y_3 &= A_3x_3^2 = A_3\, 9\,\delta^2, \\
&\cdots\cdots\cdots\cdots \\
y_n &= A_nx_n^2 = A_nn^2\delta^2,
\end{aligned}$$

we have approximately,

$$\int Ax^2dx = \frac{\delta}{3}\left[y_n + 4\,(y_1 + y_3 + \cdots + y_{n-1}) + 2\,(y_2 + y_4 + \cdots + y_{n-2})\right]$$
$$= \frac{\delta^3}{3}\left[n^2A_n + 4\,(A_1 + 9\,A_3 + 25A_5 + \cdots) + 2\,(4A_2 + 16A_4 + 36A_6 + \cdots)\right].$$

In the same way $\int By^2dy$ is calculated, and dividing the sum of the results by $U = \Sigma u = \int A\,dx$, which may also be determined by the formula of Thomas Simpson (1337), we obtain R^2 with sufficient approximation for all practical purposes.

MOMENT OF INERTIA OF PLANE SURFACES

1358. *Moment of inertia of plane surfaces* with respect to an axis drawn in the plane of the surface (1356).

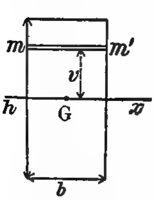

Fig. 413

1st. The section being a rectangle, or in general a parallelogram, whose base is b and altitude h, if the base b is parallel to the neutral line Gx for any element, we have

$$i = b\,dv\,v^2,$$

wherein the moment of inertia is i, the area of the element is $b\,dv$, and its distance from the axis of rotation is v.

Therefore, the moment of inertia I of the section is

$$I = b \int v^2 dv = \frac{bv^3}{3} + C. \qquad (a)$$

Taking the integral between the limits 0 and $\frac{h}{2}$, C being 0 for $v = 0$, we have for the moment of inertia I' of the part above the neutral axis Gx,

$$I' = \frac{b}{3}\left(\frac{h}{2}\right)^3 = \frac{bh^3}{24}.$$

Taking the same integral between the limits $-\frac{h}{2}$ and 0, we have for the moment of inertia I'' of the part below the neutral axis Gx,

$$I'' = -\frac{b}{3}\left(-\frac{h}{2}\right)^3 = \frac{bh^3}{24}.$$

Therefore,

$$I' = I'' \text{ and } I = I' + I'' = 2\frac{bh^3}{24} = \frac{bh^3}{12}.$$

The same value is obtained when the integral is taken directly between the limits $-\frac{h}{2}$ and $\frac{h}{2}$:

$$I = b\int_{-\frac{h}{2}}^{\frac{h}{2}} v^2\, dv = \frac{bh^3}{24} - \frac{b\,(-h)^3}{24} = \frac{bh^3}{12}. \quad (1315).$$

2d. *The section being a hollow rectangle symmetrical about its axis*, the moment of inertia I is the difference between the moments of inertia of two rectangles, one having the dimensions b and h, and the other b' and h'; then from 1st,

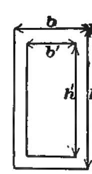

Fig. 414

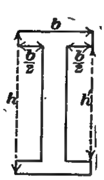

Fig. 415

$$I = \frac{bh^3}{12} - \frac{b'h'^3}{12} = \frac{bh^3 - b'h'^3}{12}.$$

If $b' = b$, that is, if the web which joins the heads can be neglected, we have simply

$$I = \frac{b\,(h^3 - h'^3)}{12}.$$

3d. *Moment of inertia of a parallelogram ABCD with respect to one of its diagonals AC taken as axis.*

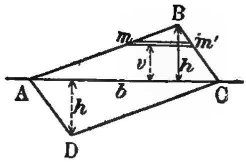

Fig. 416

Calling I' the moment of inertia of the triangle ABC with respect to its base $AC = b$, and noting that

$$mm' : b = (h - v) : h,$$

we have $mm' = \dfrac{b\,(h - v)}{h} = b - \dfrac{b}{h}v,$

and therefore $$mm'\,dv = b\,dv - \frac{b}{h}v\,dv,$$

and $$I' = b\int_0^h v^2\,dv - \frac{b}{h}\int_0^h v^3\,dv = \frac{bh^3}{3} - \frac{bh^3}{4} = \frac{bh^3}{12}.$$

For the parallelogram $ABCD$ (1st),

$$I = 2\,I' = \frac{bh^3}{6}.$$

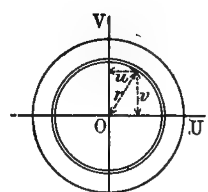

Fig. 417

4th. *The moment of inertia of a circle* being the same for the axes OV and OU, we have

$$I = \int v^2\,d\omega = \int u^2\,d\omega \text{ or } I = \frac{1}{2}\int (v^2 + u^2)\,d\omega,$$

wherein $d\omega$ is the area of an element.

Making $v^2 + u^2 = r^2$ (733), and taking the element concentric to the circle, we have,

$$d\omega = 2\,\pi r\,dr,$$

and then $$I = \frac{1}{2}\int 2\,\pi r^3\,dr.$$

Taking this integral between the limits 0 and the exterior radius R,

$$I = \frac{\pi R^4}{4}.$$

5th. *For a hollow circular section*, whose exterior and interior radii are respectively R and R' (2d and 4th), we have,

$$I = \frac{\pi R^4}{4} - \frac{\pi R'^4}{4} = \frac{\pi}{4}(R^4 - R'^4).$$

6th. *Moment of inertia of an elliptical section* having $2\,a$ for its major axis and $2\,b$ for its minor axis.

Describing a circle upon the major axis as diameter, the elements mm' and nn', taken at the same distance v from the axis, one in the circle and the other in the ellipse (1142), give

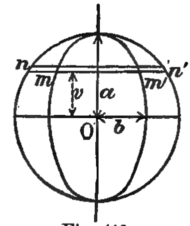

Fig. 418

$$mm' : nn' = b : a \text{ and } mm' = \frac{b}{a} nn',$$

and we have $d\omega = \frac{b}{a} \times nn' \times dv.$

and therefore $\quad I = \frac{b}{a} \int v^2 \times nn' \times dv.$

But from (4th): $\quad \int v^2 \times nn' \times dv = \frac{\pi a^4}{4};$

therefore for the ellipse $\quad I = \frac{\pi}{4} ba^3.$

7th. *For a hollow elliptical section,* $2\,a$ and $2\,b$ being the axes of the exterior ellipse and $2\,a'$ and $2\,b'$ the axes of the interior ellipse, we have (2d and 6th),

$$I = \frac{\pi}{4} ba^3 - \frac{\pi}{4} b'a'^3 = \frac{\pi}{4}(ba^3 - b'a'^3).$$

8th. *A triangular section ABC,* one side AC of which is parallel to the axis Gx.

The preceding examples show that when a figure is symmetrical with respect to the axis of moments passing through the center of gravity, or simply with respect to the center of gravity, it suffices to find the moment of inertia of the surface situated on one side of the axis and multiply it by two to obtain the moment of the entire section.

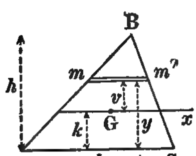

Fig. 419

In certain cases, as in that of a triangle, for example, it may be convenient to first take the moment of inertia I' with respect to an axis AC parallel to the axis Gx which passes through the center of gravity, and from that deduce the moment of inertia I with respect to the latter axis Gx.

First of all, the general relation which exists between I and I' must be determined. Designating the variable distances of any element mm' from the axes AC and Gx respectively by y and v, and the constant distance between these axes by k, we have for any element

$$y^2 = (v \pm k)^2 = v^2 + k^2 \pm 2\,kv,$$

and therefore,

$$\int y^2\,d\omega = \int v^2\,d\omega + \int k^2\,d\omega \pm \int 2\,kv\,d\omega.$$

Noting that $\int k^2 d\omega = k^2\Omega$, representing the area of the section $\int d\omega$ by Ω, and that $\int 2\,kv\,d\omega = 0$, and since $\int v\,d\omega$ is the moment of the section with respect to the axis passing through the center of gravity, we have

$$\int y^2\,d\omega = \int v^2\,d\omega + k^2\Omega.$$

Let $\qquad I' = I + k^2\Omega,$ then $I = I' - k^2\Omega.$

For the triangle we have

$$I' = \int y^2\,d\omega = \frac{bh^3}{12}\,(3d),\quad k^2 = \frac{h^2}{9}\quad \Omega = \frac{bh}{2}\,(682);$$

therefore $\qquad I = \frac{bh^3}{12} - \frac{bh^3}{18} = \frac{bh^3}{36}.$

9th. *The moment of inertia of any plane surface with respect to any axis Ox situated in the same plane.*

l being the greatest dimension of the surface perpendicular to the axis Ox, k the shortest distance from the axis to the surface, u the variable length of the elements mm' included between parallels to Ox, we have,

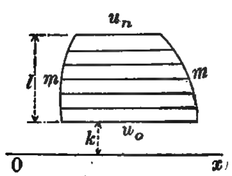

Fig. 420

$$I = \int_k^{k+l} v^2 u\,dv.$$

To obtain the approximate value of this integral, divide l into an even number n of equal parts $\frac{l}{n} = \delta$; through the extremities of l and the points of division draw parallels to the axis Ox, thus dividing the surface into n bands of equal height $\frac{l}{n} = \delta$; then calling the successive chords thus obtained, u_0, u_1, u_2, u_3, . . . u_n, from the formula of Simpson we have (1333),

$$I = \frac{\delta}{3}[k^2u_0 + 4(k+\delta)^2u_1 + 2(k+2\,\delta)^2u_2 + 4(k+3\,\delta)^2u_3 + \cdots + (k+l)^2u_n].$$

When Ox coincides with u_0, it suffices to make $k = 0$ in the above expression, and if Ox passes through the center of gravity of the surface, k is made equal to zero and the moment of inertia of each part calculated separately; then the sum of the two results gives the moment of inertia of the entire surface.

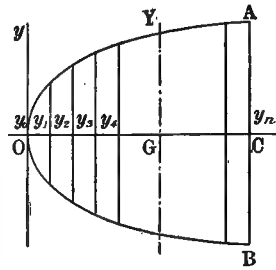

Fig. 421

1359. *Calculation of the moment of inertia of a plane surface with respect to an axis passing through its center of gravity.* (Contributed by M. Le Brun.)

The solution of this problem generally involves that of two others; namely:

1. The determination of the area of the surface;
2. The determination of the center of gravity of the surface.

These three calculations are represented by the formulas:

$$\Omega = \int_0^n d\omega, \tag{1}$$

$$\Omega V = \int_0^n v\, d\omega, \tag{2}$$

$$I + \Omega V^2 = \int_0^n v^2\, d\omega. \tag{3}$$

When the integrations are difficult, the formula of Thomas Simpson (1268) is used. Let

Ω be the area of the given surface AOB;

v be the distance from the center of gravity of the element $d\omega$ to the axis Oy parallel to the required axis GY;

V be the distance from the center of gravity G to the axis Oy;

I be the moment of inertia of the surface Ω with respect to the axis GY (1358);

n be the even number of divisions of OC;

$\dfrac{OC}{n} = \delta$ be the distance between two successive divisions;

$y_0, y_1, y_2, \ldots y_n$ be the ordinates drawn through the points of division.

The value Ω is given by the approximate formula (1333),

$$\Omega = \frac{\delta}{3}[y_0 + y_n + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]. \tag{1'}$$

This formula gives the sum of the surfaces ω', ω'', ω'' . . . (Fig. 422), included between the ordinates y_0 and y_2, y_2 and y_4 . . ., that is, between the successive even ordinates.

The position of the center of gravity is often very difficult to determine; however, it is always near the middle ordinate, which it approaches as δ is indefinitely decreased. To simplify the calculations, the following hypothesis, which is very near the truth, will be adopted.

Giving the values which were found for s (1268, Figs. 358 and 359), to ω', ω'', ω''' . . ., and noting that $v' = \delta$, $v'' = 3\,\delta$, $v''' = 5\,\delta$ . . ., we may put,

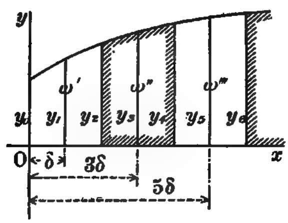

Fig. 422

$$\omega'v' = \frac{o}{3}(y_0 + 4\,y_1 + y_2)\,\delta = \frac{\delta^2}{3}(y_0 + 4\,y_1 + y_2),$$

$$\omega''v'' = \frac{\delta}{3}(y_2 + 4\,y_3 + y_4)\,3\,\delta = \frac{\delta^2}{3}(3\,y_2 + 4 \times 3\,y_3 + 3\,y_4),$$

$$\omega'''v''' = \frac{\delta}{3}(y_4 + 4\,y_5 + y_6)\,5\,\delta = \frac{\delta^2}{3}(5\,y_4 + 4 \times 5\,y_5 + 5\,y_6),$$

. . . .

$$\omega v = \frac{\delta}{3}(y_{n-2} + 4\,y_{n-1} + y_n)\,(n-1)\delta$$
$$= \frac{\delta^2}{3}[(n-1)\,y_{n-2} + 4\,(n-1)\,y_{n-1} + (n-1)\,y_n].$$

Adding these equations, for formula (2) we obtain

$$\Sigma\omega\,v = \Omega\,V = \frac{\delta^2}{3}\{y_0 + (n-1)\,y_n + 4\,[y_1 + 3\,y_3 + 5\,y_5 + \cdots + (n-1)y_{n-1}]$$
$$+ 2\,[2\,y_2 + 4\,y_4 + 6\,y_6 + \cdots + (n-2)\,y_{n-2}]\}. \qquad (2')$$

To calculate the formula (3), each element ω of the surface must be multiplied by v^2; thus,

$$\omega' v'^2 = \frac{\delta}{3}(y_0 + 4y_1 + y_2)\,\delta^2 = \frac{\delta^3}{3}(y_0 + 4y_1 + y_2),$$

$$\omega'' v''^2 = \frac{\delta}{3}(y_2 + 4y_3 + y_4)\,3^2\delta^2 = \frac{\delta^3}{3}(3^2 y_2 + 4 \times 3^2 y_3 + 3^2 y_4),$$

$$\omega''' v'''^2 = \frac{\delta}{3}(y_4 + 4y_5 + y_6)\,5^2\delta^2 = \frac{\delta^3}{3}(5^2 y_4 + 4 \times 5^2 y_5 + 5^2 y_6),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\omega v^2 = \frac{\delta}{3}(y_{n-2} + 4y_{n-1} + y_n)(n-1)^2\delta^2$$

$$= \frac{\delta^3}{3}[(n-1)^2 y_{n-2} + 4(n-1)^2 y_{n-1} + (n-1)^2 y_n].$$

Adding these equations, and taking $\frac{\delta^3}{3}$ as a common factor, it is seen that the coefficient of the first ordinate is unity, and that of the last is the square of its index less one; that the odd ordinates are multiplied by 4 and the square of their indices; and finally, that the even ordinates are multiplied by the sum of the square of their index k plus 1 $(k+1)^2$ and the square of the same index minus 1 $(k-1)^2$; thus, for the even ordinate y_k, we have

$$[(k-1)^2 + (k+1)^2]y_k = 2(k^2+1)\,y_k,$$

that is, that each even ordinate is multiplied by 2 and 1 plus the square of its index k.

Then the formula (3) becomes

$$\Sigma\omega v^2 = I + V^2\Omega = \frac{\delta^3}{3}[y_0 + (n-1)^2 y_n + 4[y_1 + 9y_3 + 25y_5 + \cdots$$
$$+ (n-1)^2 y_{n-1}] + 2\{(2^2+1)y_2 + (4^2+1)y_4 + (6^2+1)y_6 + \cdots$$
$$+ [(n-2)^2+1]\,y_{n-2}\}]. \tag{3'}$$

The auxiliary axis should be taken tangent to the surface when possible; if the surface has no axis of symmetry, its center of gravity is calculated by determining its distance from a second axis perpendicular to the first, thus determining its coördinates.

The computations of the elements of the formulas (1′), (2′) and (3′) may be tabulated as follows: column (5) refers to even ordinates of formula (3′).

k (1)	y (2)	ky (3)	k^2y (4)	k^2y+y (5)
0	y_0	“	“	“
1	y_1	y_1	y_1	“
2	y_2	$2\,y_2$	$4\,y_2$	$4\,y_2+y_2$
3	y_3	$3\,y_3$	$9\,y_3$	“
4	y_4	$4\,y_4$	$16\,y_4$	$16\,y_4+y_4$
5	y_5	$5\,y_5$	$25\,y_5$	“
6	y_6	$6\,y_6$	$36\,y_6$	$36\,y_6+y_6$
7	y_7	$7\,y_7$	$49\,y_7$	“
8	y_8	$8\,y_8$	$64\,y_8$	$64\,y_8+y_8$
. . .	. . .	. . . . . .	. . . . . .	. . . .
$n-2$	y_{n-2}	$(n-2)\,y_{n-2}$	$(n-2)^2y_{n-2}$	$(n-2)^2y_{n-2}+y_{n-2}$
$n-1$	y_{n-1}	$(n-1)\,y_{n-1}$	$(n-1)^2y_{n-1}$	“
n	y_n	“	“	“

Column (3) is obtained by multiplying the figures in column (2) by those in column (1).

Column (4), by multiplying (3) by (1).

Column (5), by adding (4) and (2).